Computation of the Adjoint Matrix

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Abstract. The best method for computing the adjoint matrix of an order n matrix in an arbitrary commutative ring requires $O(n^{\beta+1/3} \log n \log\log n)$ operations, provided that the complexity of the algorithm for multiplying two matrices is $\gamma n^\beta + o(n^\beta)$. For a commutative domain – and under the same assumptions – the complexity of the best method is $6\gamma n^\beta/(2^\beta - 2) + o(n^\beta)$. In the present work a new method is presented for the computation of the adjoint matrix in a commutative domain. Despite the fact that the number of operations required is now 1.5 times more, than that of the best method, this new method permits a better parallelization of the computational process and may be successfully employed for computations in parallel computational systems.

1 Statement of the Problem

The adjoint matrix is a transposed matrix of algebraic complements. If the determinant of the matrix is nonzero, then the inverse matrix may be computed as the adjoint matrix divided by the determinant. The adjoint matrix of a given matrix A will be denoted by A^*: $A^* = \det(A)A^{-1}$.

The best method for computing the adjoint matrix of an order n matrix in an arbitrary commutative ring requires $O(n^{\beta+1/3} \log n \log\log n)$ operations (see [1] and [2]). For a commutative domain the complexity of the best method is $6\gamma n^\beta/(2^\beta - 2) + o(n^\beta)$ (see [3]). It is asssumed that the complexity of the algorithm for multiplying two matrices is $\gamma n^\beta + o(n^\beta)$.

In a commutative domain the algorithm is based on applications of determinant identities [3]. It generalizes in a commutative domain the following formula for the inverse matrix $\mathcal{A}^{-1}$:

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (D - BA^{-1}C)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & I \end{pmatrix},$$

where $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ – is an invertible matrix with invertible block A.

In the present work a new method is proposed for the computation of the adjoint matrix in a commutative domain. Despite the fact that the number of

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operations required is now 1.5 times more, than that of the algorithm described in [1], this new method permits a better parallelization of the computational process.

This new method generalizes in a commutative domain the following factorization of the inverse matrix $\mathcal{A}^{-1}$:

$$\mathcal{A}^{-1} = \begin{pmatrix} I & -A^{-1}C \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (B^{-1}D - A^{-1}C)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}.$$

In the next section we present without proofs the determinant identity of column replacement, which is used as the basis of the proposed method for computing the adjoint matrix, along with several additional theorems, which are fundamental for the new method. In the final section the algorithm and a small example are presented for the computation of the adjoint matrix.

2 Identity of Column Replacement

Let B be a matrix of order n and assume two different columns are fixed. We denote by $B_{\{x,y\}}$ the matrix which is obtained from B after replacing the two fixed columns by the columns x and y, respectively.

Theorem 1. *(Identity of column replacement.)*
For every matrix $B \in R^{n\times n}$ and columns $a, b, c, d \in R^n$ the following identity holds

$$\det B_{\{ab\}} \det B_{\{cd\}} = \begin{vmatrix} \det B_{\{ad\}} & \det B_{\{db\}} \\ \det B_{\{ac\}} & \det B_{\{cb\}} \end{vmatrix} \tag{1}$$

For example, for the matrix of order 2 the identity of column replacement is as follows

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} \begin{vmatrix} x & u \\ y & v \end{vmatrix} = \begin{vmatrix} a & u \\ b & v \end{vmatrix} \begin{vmatrix} x & c \\ y & d \end{vmatrix} - \begin{vmatrix} a & x \\ b & y \end{vmatrix} \begin{vmatrix} u & c \\ v & d \end{vmatrix}.$$

Theorem 2. *Let R be a commutative domain, $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ a matrix of order $2n$ over R, A, B, C, D square blocks, $\alpha = \det A \neq 0$, $\beta = \det B \neq 0$, $F = \alpha B^* D - \beta A^* C$.*

Then every minor of order k of the matrix F ($k \leq n$) is divisible by $(\alpha\beta)^{k-1}$ and the following identity holds

$$\det F = (\alpha\beta)^{n-1} \det \mathcal{A}. \tag{3}$$

Theorem 3. *Let R be a commutative domain, $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ a matrix of order $2n$ over R, A, B, C, D square blocks, $\alpha = \det A \neq 0$, $\beta = \det B \neq 0$ and $F = \alpha B^* D - \beta A^* C$.*

Then $(\alpha\beta)^{-n+2} F^* \in R^n$ and

$$\mathcal{A}^* = \begin{pmatrix} \alpha^{-1}|\mathcal{A}|I & -\alpha^{-2}\beta^{-1}A^*C \\ 0 & \alpha^{-1}\beta^{-1}I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & (\alpha\beta)^{-n+2}F^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta I & \alpha I \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & B^* \end{pmatrix}. \tag{4}$$

Theorem 4. *Let R be a commutative domain, $0 \neq \gamma \in R$, $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ a matrix of order $2n$ ($n \geq 2$) over R, such that every minor of order k is divisible by γ^{k-1}, A, B, C, D square blocks, $\alpha = \gamma^{1-n} \det A \neq 0$, $\beta = \gamma^{1-n} \det B \neq 0$, $\mathbf{A}^* = \gamma^{2-n} A^*$, $\mathbf{B}^* = \gamma^{2-n} B^*$ and $F = (\alpha\gamma^{-1}\mathbf{B}^* D - \beta\gamma^{-1}\mathbf{A}^* C)$.*

Then,

$$\gamma^{2-2n}\mathcal{A}^* = \begin{pmatrix} \alpha^{-1}\gamma^{1-2n}|\mathcal{A}|I & -(\alpha^2\beta\gamma)^{-1}\mathbf{A}^* C \\ 0 & (\alpha\beta\gamma)^{-1} I \end{pmatrix} \times$$

$$\begin{pmatrix} I & 0 \\ 0 & (\alpha\beta)^{-n+2} F^* \end{pmatrix} \begin{pmatrix} I & 0 \\ -\beta I & \alpha I \end{pmatrix} \begin{pmatrix} \mathbf{A}^* & 0 \\ 0 & \mathbf{B}^* \end{pmatrix}. \tag{5}$$

Here $\gamma^{2-2n}\mathcal{A}^*$ and the last three factors on the right-hand side of (5) are matrices over R.

3 The Algorithm

Using the theorems we proved about the factorization of the adjoint matrix we now introduce the algorithm for computing it along with the determinant of a given matrix.

Let R be a commutative domain, $0 \neq \gamma \in R$, $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ a matrix of order $2n = 2^N$ over R, such that every minor of order k is divisible by γ^{k-1}. Moreover, we assume that all minors, on which a division is performed during the computation of the adjoint matrix, are non-zero.

The inputs to the algorithm are the matrix $\mathcal{A}$ and the number $\gamma = 1$.

The outputs from the algorithm are $\gamma^{1-2n}|\mathcal{A}|$ and $\gamma^{2-2n}\mathcal{A}^*$. Note here that the determinant of the matrix has been divided by γ^{2n-1}, and that the adjoint matrix has been divided by γ^{2n-2}.

Algorithm ParAdjD

{ $\gamma^{1-2n}|\mathcal{A}|, \gamma^{2-2n}\mathcal{A}^*$ **}=ParAdjD(**$\mathcal{A}, \gamma$**)**

Input: $\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, and γ. $A, B, C, D \in R^{n \times n}$, $\gamma \in R$.
Output: $\{\gamma^{1-2n}|\mathcal{A}|,\ \gamma^{2-2n}\mathcal{A}^*\ \}$.

1. If the matrix $\mathcal{A}$ is of order two, then
output:

$$\left\{\gamma^{-1}(AD - BC),\ \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\right\}.$$

otherwise, proceed to the next point.
2. Concurrently compute
{ $\alpha, \mathbf{A}^*$**}=ParAdjD(**A, γ**)** and **{** $\beta, \mathbf{B}^*$**}=ParAdjD(**B, γ**)**.

3. Concurrently compute
   $N = \gamma^{-1}\mathbf{B}^* D$ and $M = \gamma^{-1}\mathbf{A}^* C$, and then
   $F = \alpha N - \beta M$.
4. Compute
   { $\varphi,$ $\mathbf{F}^*$**}=ParAdjD(**$F, \alpha\beta$**)**.
5. Concurrently compute
   $\varphi' = \gamma^{-1}\varphi$, $H = \alpha^{-1}\gamma^{-1}\mathbf{F}^*\mathbf{A}^*$ and $L = \beta^{-1}\gamma^{-1}\mathbf{F}^*\mathbf{B}^*$.
6. Concurrently compute
   $H' = \alpha^{-1}(\varphi'\mathbf{A}^* + MH)$ and $L' = -\alpha^{-1}ML$.

Output:

$$\left\{\varphi', \begin{pmatrix} H' & L' \\ -H & L \end{pmatrix}\right\}.$$

3.1 Example

Input:

$$\mathcal{A} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} 0 & 2 & -2 & 2 \\ 1 & -3 & 1 & -2 \\ 3 & 0 & -3 & 0 \\ -1 & 3 & -1 & 1 \end{pmatrix}, \ \gamma = 1.$$

Output:

$$\left\{\varphi', \begin{pmatrix} H' & L' \\ -H & L \end{pmatrix}\right\} = \left\{6, \begin{pmatrix} -9 & -12 & 4 & -6 \\ -6 & -6 & 2 & 0 \\ -9 & -12 & 2 & -6 \\ 0 & -6 & 0 & -6 \end{pmatrix}\right\}.$$

References

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