Symbolic Analysis of Economical Models with Mathematica

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Abstract. Functional equations is a very powerful technique to obtain
consistent models in Economics. Unfortunately, there is only a few com-
puter tools for solving functional equations. In this paper a *Mathematica*
package, `FSolve`, is applied to the symbolic analysis of some economical
models for price and advertising policies. The procedure is as follows:
firstly, we make some assumptions about the functional structure of the
functions describing the models. Such assumptions are given in terms
of functional equations that account for some economical properties.
Then, the package is applied to compute the solutions of these equations
and check for inconsistencies. The interesting cases of the monopoly and
duopoly models are discussed.

1 Introduction

One of the most appealing applications of the computer algebra systems (CAS) is
the symbolic analysis of economical models [8, 9]. By this we mean to determine
the symbolic solutions of the equations governing those models, which must be
based on reasonable properties. The starting point of this process is to establish
the conditions - which must be meaningful from the viewpoint of economical
sciences - that those functions must satisfy necessarily. Then, the derived equa-
tions must be solved. However, it is quite hard in general to determine which
are the conditions characterizing the problem and thus, some assumptions are
usually taken. For instance, the functional structure of the functions describing
the models is sometimes arbitrarily chosen, based either on convenience or on
empirical considerations. Very often, this fact leads to absurd solutions and/or
inconsistencies.

Our experience is that the functional equations is an optimal technique to
overcome these undesirable situations. They provide powerful and consistent
methods to describe the common sense properties of the economical functions
and, simultaneously, the mathematical tools for solving the resulting equations
[1, 2, 3, 4, 5]. The drawback of this picture is that most of this work must be per-
formed by hand, as there is is only a few computer tools for solving functional
equations. One remarkable exception is a *Mathematica* package, `FSolve`, devel-
oped by one of the authors and described in [6]. In this paper, the package will
be successfully applied to the analysis of some monopoly and duopoly models
for price and advertising policies.

The structure of this paper is as follows: in Section 2 we describe briefly the package `FSolve`. Then, in Section 3 we analyze some economical models for price and advertising policies. The procedure is as follows: firstly, we make some assumptions about the functional structure of the functions describing the models. Such assumptions are given in terms of functional equations that account for some economical properties. Then, the package is applied to compute the solutions of these equations and check for inconsistencies. The interesting cases of the monopoly and duopoly models are discussed.

2   The Package FSolve

In this section the *Mathematica* package `FSolve` is briefly described. The package has been implemented a few years ago in *Mathematica 3.0* [6] but updated versions for *Mathematica 4.0, 4.1, 4.2* and *5.0* are also available. We start our discussion by loading the package:

`In[1]:=<<FunctionalEquations'FSolve'`

which includes the command

$$\texttt{FSolve}[eqn, \{functions\}, \{variables\}, options]$$

where *eqn* denotes the functional equation to be solved, $\{functions\}$ is the list of unknown functions, $\{variables\}$ is the list of variables and *options* allows the users to consider different domains for the variables (see Table 1) and classes of feasible functions (see Table 2).

For instance, we can calculate the solution of the functional equation $f(x + y) = g(x) + h(y)$ where $x, y \in \mathbb{R}$ and f, g, h are continuous functions as:

`In[2]:= FSolve[f[x+y]==g[x]+h[y],{f,g,h},{x,y},Domain->Real,`
`             Class->Continuous]`

$Out[2] := \begin{array}{l} \{f(x) \rightarrow C(1)\,x + C(2) + C(3), g(x) \rightarrow C(1)\,x + C(2), \\ h(x) \rightarrow C(1)\,x + C(3)\} \end{array}$

where $C(1)$, $C(2)$ and $C(3)$ are arbitrary constants. Note that the general solution can depend on one or more arbitrary constants and even on arbitrary functions (see $Out(3)$ and $Out(4)$ for two examples). Note also that a single equation can determine several unknown functions (such as f, g and h in this example). The reader is referred to [7] for a general introduction to the theory of functional equations and their applications.

3   Price and Advertising Policies

In this section the package `FSolve` is applied to analyze some economical models for price and advertising policies. In particular, we focus on the problem of modeling the sales $S(p, v)$ of a single-product firm such that they depend on the price p of its product and on the advertising expenditure v. Because of limitations of space, we will restrict our discussion to the cases of monopoly and duopoly models.

Table 1. List of all valid domains used in the `FSolve` package

Domains	Package names	Definition
$\mathbb{R}$	Arbitrary	$\mathbb{R}$
$\mathbb{R}$	Real	$\mathbb{R}$
$\mathbb{I}$	Irrational	$\mathbb{R} - \mathbb{Q}$
$\mathbb{Q}$	Rational	$\mathbb{Q}$
$\mathbb{Z}$	Integer	$\mathbb{Z}$
$\mathbb{R} - \{0\}$	NonZeroReal	$\mathbb{R} - \{0\}$
$\mathbb{Q} - \{0\}$	NonZeroRational	$\mathbb{Q} - \{0\}$
$\mathbb{Z} - \{0\}$	NonZeroInteger	$\mathbb{Z} - \{0\}$
$\mathbb{R}_+$	RealPositiveZero	$\{x \in \mathbb{R}/x \geq 0\}$
$\mathbb{Q}_+$	RationalPositiveZero	$\{x \in \mathbb{Q}/x \geq 0\}$
$\mathbb{Z}_+$	IntegerPositiveZero	$\{x \in \mathbb{Z}/x \geq 0\}$
$\mathbb{R}_{++}$	RealPositive	$\{x \in \mathbb{R}/x > 0\}$
$\mathbb{I}_{++}$	IrrationalPositive	$\{x \in \mathbb{R} - \mathbb{Q}/x > 0\}$
$\mathbb{Q}_{++}$	RationalPositive	$\{x \in \mathbb{Q}/x > 0\}$
$\mathbb{Z}_{++}$	IntegerPositive	$\{x \in \mathbb{Z}/x > 0\}$
$\mathbb{R}_-$	RealNegativeZero	$\{x \in \mathbb{R}/x \leq 0\}$
$\mathbb{Q}_-$	RationalNegativeZero	$\{x \in \mathbb{Q}/x \leq 0\}$
$\mathbb{Z}_-$	IntegerNegativeZero	$\{x \in \mathbb{Z}/x \leq 0\}$
$\mathbb{R}_{--}$	RealNegative	$\{x \in \mathbb{R}/x < 0\}$
$\mathbb{I}_{--}$	IrrationalNegative	$\{x \in \mathbb{R} - \mathbb{Q}/x < 0\}$
$\mathbb{Q}_{--}$	RationalNegative	$\{x \in \mathbb{Q}/x < 0\}$
$\mathbb{Z}_{--}$	IntegerNegative	$\{x \in \mathbb{Z}/x < 0\}$

Table 2. Classes of valid functions used in the `FSolve` package

Name	Classes of functions
Differentiable	Differentiable
Invertible	Invertible
Increasing	Increasing
Decreasing	Decreasing
Continuous	Continuous
ContinuousPoint	Continuous at a point
Monotonous	Monotonous
Arbitrary	Arbitrary

3.1 The Monopoly Model

Let us assume a firm such that the sales S of a single product depend on the unitary price p and on the advertising expenditure v, that is, $S = S(p, v)$. The function S cannot be arbitrary, but it must satisfy the following properties:

(M1) The $S(p, v)$ function is continuous in both arguments.

(M2) $\forall v$, the $S(p, v)$ function, considered as a function of p only, must be convex from below and decreasing. This implies that, for the same advertising

expenses, any increment in the unit price of the product leads to a reduction in sales, and that its derivative decreases with p.

(M3) $\forall\,p$, the $S(p,v)$ function, considered as a function of v only, must be concave from below and increasing. This implies that, for the same unit price, an increment in the advertising expenses leads to an increment in sales.

(M4) A multiplicative change in the advertising expenditure leads to an additive change in sales, that is,

$$S(p, v\,w) = S(p, v) + T(p, w), \tag{1}$$

where $p \geq 0$, $v \geq 0$, $w \geq 0$, $T(p, 1) = 0$ and $T(p, w)$ is increasing with w.

(M5) The sales due to an increment q in price are equal to the previous sales times a real number, which depends on q and v, that is,

$$S(p + q, v) = S(p, v)\,R(q, v) \tag{2}$$

where $p \geq 0$, $p + q \geq 0$, $v \geq 0$, $R(0, v) = 1$ and $R(q, v)$ is decreasing in q.

Eq. (1) can be solved by using the package `FSolve` as follows:

```
In[3]:= em1=FSolve[S[p,v*w]==S[p,v]+T[p,w],{S,T},{p,v,w},
        Domain->RealPositiveZero,Class->Continuous]
```

$Out[3] := \{S(p, v) \rightarrow Arb1(p)\,Log(v) + Arb2(p), T(p, w) \rightarrow Arb1(p)\,Log(w)\}$

where $Arb1(p)$ and $Arb2(p)$ denote two arbitrary functions depending on the variable p. Similarly, we can solve eq. (2) as:

```
In[4]:= em2=FSolve[S[p+q,v]==S[p,v]*R[q,v],{S,R},{p,q,v},
        Domain->RealPositiveZero,Class->Continuous]
```

$Out[4] := \left\{S(p, v) \rightarrow Arb3(v)\,e^{p\,Arb4(v)}, R(q, v) \rightarrow e^{q\,Arb4(v)}\right\}$

where $Arb3(v)$ and $Arb4(v)$ denote two arbitrary functions depending on the variable v. Once we have solved functional equations (1) and (2) separately, the general solution of the system (1)-(2) is given by:

```
In[5]:= FSolve[Equal @@ (S[p,v] /. First[#]& /@ {em1,em2}),
                {Arb1,Arb2,Arb3,Arb4},{p,v}];
In[6]:= S[p,v] /. %
```

$Out[6] := (C(1) + C(2)\,Log[v])\,Exp[-C(3)\,p]$

where $C(1)$, $C(2)$ and $C(3)$ are arbitrary constants. Note that in *Out[6]* we have no longer arbitrary functions, but arbitrary constants. This means that the parametric model is completely specified and that we can estimate its parameters $C(1)$, $C(2)$ and $C(3)$ using empirical data.

The obtained solution shows a logarithmic increment of sales with advertising expenditures and an exponential decrease with price, in agreement with assumptions (M4) and (M5). One justification of this model of sales is the so-called Weber-Fechner law, that states that the stimuli of the intensity of perception is a linear function of the logarithm of the intensity of the stimulus.

The previous model was proposed by Eichhorn in 1978 [5]. It can be argued, however, that the function R should depend on the price p, instead of v. Thus, we can replace (M5) by:

(M6) The sales due to an increment q in price are equal to the previous sales times a real number, which depends on q and p, that is,

$$S(p+q, v) = S(p, v)\, R(q, p) \tag{3}$$

where $p \geq 0$, $p + q \geq 0$, $v \geq 0$, $R(0, p) = 1$ and $R(q, p)$ is decreasing in q.

The general solution of (3) can be obtained by using the package `FSolve` as:

```
In[7]:= em3=FSolve[S[p+q,v]==S[p,v]*R[q,p],{S,R},{p,q,v},
        Domain->RealPositiveZero,Class->Continuous]
```

$$Out[7] := \left\{ S(p, v) \rightarrow Arb5(p)\, Arb6(v),\ R(q, p) \rightarrow \frac{Arb5(p + q)}{Arb5(p)} \right\}$$

Alternatively, we can assume a multiplicative, instead of an additive, change in the price p and we can question whether or not choosing between one of these assumptions influences the resulting model. In other words, we can assume:

(M7) The sales due to a multiplicative change (w times) in the price are equal to the previous sales times a real number, which depends on w and p, that is,

$$S(p\, w, v) = S(p, v)\, R(w, p) \tag{4}$$

where $p \geq 0$, $w \geq 0$, $v \geq 0$, $R(1, p) = 1$ and $R(w, p)$ is decreasing in w.

```
In[8]:= FSolve[S[p*w,v]==S[p,v]*R[w,p],{S,R},{p,v,w},
        Domain->RealPositiveZero,Class->Continuous]
```

$$Out[8] := \left\{ S(p, v) \rightarrow Arb5(p)\, Arb6(v),\ R(w, p) \rightarrow \frac{Arb5(p\, w)}{Arb5(p)} \right\}$$

Note that the S functions in *Out[7]* and *Out[8]* are identical. Thus, equations (3) and (4) are equivalent. Consequently, the above mentioned two assumptions (M6) and (M7) lead to the same model.

Now, the solution of the system (1)-(3) can be obtained as:

```
In[9]:= FSolve[Equal @@ (S[p,v] /. First[#]& /@ {em1,em3}),
              {Arb1,Arb2,Arb5,Arb6},{p,v}]
```

$$Out[9] := \left\{ Arb2(p) \rightarrow \frac{C(4)}{C(3)}\, Arb1(p),\, Arb5(p) \rightarrow \frac{-Arb1(p)}{C(3)}, \right.$$
$$\left. Arb6(p) \rightarrow -C(3)\, Log(p) - C(4) \right\}$$

which leads to the model:

```
In[10]:= S[p,v] /. %
```

$$Out[10] := Arb1(p) \left[Log(v) + \frac{C(4)}{C(3)} \right]$$

where the function $Arb1(p)$ and the constants $C(3)$ and $C(4)$ are arbitrary. For this solution to satisfy assumptions (M2) and (M3) above, $Arb1(p)$ must be

convex from below and decreasing. Note that $Log(v) + \dfrac{C(4)}{C(3)}$ is increasing. We also remark that model in $Out[10]$ is more general than model in $Out[6]$. In fact, the resulting model is not completely specified because it depends on arbitrary functions. This means that new requirements might be established.

3.2   The Duopoly Model

Assume now that we have two different firms that compete in the market. Assume also that the sales S of the product by firm 1 depend on the unit prices p and q and on the advertising expenditures u and v, of the two firms, that is, $S = S(p, q, u, v)$. The function $S(p, q, u, v)$ must satisfy the following properties:

(D1) The $S(p, q, u, v)$ function is continuous in all arguments.
(D2) $S(p, q, u, v)$ is increasing in q and u.
(D3) $S(p, q, u, v)$ is decreasing in p and v.
(D4) A multiplicative change in the advertising expenditure of firm 1 leads to an additive change in sales, that is,

$$S(p, q, u\,w, v) = S(p, q, u, v) + T(p, q, w, v) \tag{5}$$

(D5) The sales due to an increment r in price of firm 1 are equal to the previous sales times a real number, which depends on r and p, that is,

$$S(p + r, q, u, v) = S(p, q, u, v)\, R(r, p, q, v) \tag{6}$$

where $p \geq 0$, $p + r \geq 0$, $v \geq 0$ and $R(0, p, q, v) = 1$.

The general solution of the system (5)-(6) is given by the following sequence of calculations: firstly, we compute the functions S, T and R of the previous equations, and then we apply the outputs to calculate the functional structure of function S.

```
In[11]:= ed1=FSolve[S[p,q,u*w,v]==S[p,q,u,v]+T[p,q,w,v],{S,T},
                {p,q,u,v,w},Domain->RealPositiveZero,
                Class->Continuous];

In[12]:= ed2=FSolve[S[p+r,q,u,v]==S[p,q,u,v]+R[r,p,q,v],{S,R},
                {p,q,r,u,v},Domain->RealPositiveZero,
                Class->Continuous];

In[13]:= FSolve[Equal @@ (S[p,q,u,v] /. First[#]& /@ {ed1,ed2}),
                {Arb1,Arb2,Arb3,Arb4},{p,q,u,v}];

In[14]:= S[p,q,u,v] /. %
```

$Out[14] := Arb1(p, q, v)\, [Log(u) + Arb2(q, v)]$

where $Arb1(p, q, v)$ and $Arb2(q, v)$ are arbitrary functions. In addition we can consider the following assumption:

(D6) The total sales of both firms is a constant K, that is,

$$S(p, q, u, v) + S(q, p, v, u) = K \tag{7}$$

which, using the previous output, leads to

```
In[15]:= FSolve[((S[p,q,u,v]+S[q,p,v,u]) /. %)==K,{Arb1,Arb2},
         {p,q,u,v},Domain->RealPositiveZero,Class->Continuous]
         // FSimplify;
In[16]:= S[p,q,u,v] /. %
```

$$Out[16] := \frac{1}{Arb7(p) + Arb7(q)} \left[Log\left(\frac{u}{v}\right) + K\,Arb7(q) \right]$$

where $Arb7(p)$ is an arbitrary but increasing function of p. The physical interpretation of this model is as follows: if the advertisement expenditures of both firms are coincident, the sales are proportional to the ratios $\dfrac{Arb7(q)}{Arb7(p) + Arb7(q)}$ and $\dfrac{Arb7(p)}{Arb7(p) + Arb7(q)}$ for firms 1 and 2, respectively. On the other hand, the advertisement expenditures influence sales directly proportional to the logarithm of the ratio $\dfrac{u}{v}$ and inversely proportional to $Arb7(p) + Arb7(q)$.

We can now consider two additional assumptions:

(D7) The sales $S(p + r, q + s, u, v)$ of firm 1 due to increments r and s in the prices of firms 1 and 2, respectively, are the initial sales $S(p, q, u, v)$ of firm 1 times two factors which consider the associated reduction and increments due to these two changes, that is,

$$S(p + r, q + s, u, v) = S(p, q, u, v)\,U(r, p, q)\,V(s, p, q) \tag{8}$$

(D8) The sales $S(p, q, u + r, v + s)$ of firm 1 due to increments r and s in the advertisement expenditures of firms 1 and 2, respectively, are the initial sales $S(p, q, u, v)$ of firm 1 times two factors which consider the associated increments and decrements due to these two changes, that is,

$$S(p, q, u + r, v + s) = S(p, q, u, v)\,U(r, u, v)\,V(s, u, v) \tag{9}$$

Combining now (D7) and (D8) and solving the system of equations (8)-(9), we get:

```
In[17]:= FSolve[S[p+r,q+s,u,v]==S[p,q,u,v]*U[r,p,q]*V[s,p,q],
                {S,U,V},{p,q,r,s,u,v},Domain->RealPositiveZero,
                Class->Continuous];

In[18]:= FSolve[S[p,q,u+r,v+s]==S[p,q,u,v]*U[r,u,v]*V[s,u,v],
                {S,U,V},{p,q,r,s,u,v},Domain->RealPositiveZero,
                Class->Continuous];

In[19]:= FSolve[Equal @@ (S[p,q,u,v] /. First[#]& /@ {%%,%}),
                {Arb1,Arb2,Arb3,Arb6,Arb7,Arb8},{p,q,u,v}];

In[20]:= S[p,q,u,v] /. %
```

$$Out[20] := Arb1(p)\,Arb2(q)\,Arb6(u)\,Arb7(v)$$

where the functions $Arb1(p)$ and $Arb7(v)$ are decreasing and the functions $Arb2(q)$ and $Arb6(u)$ are increasing, but otherwise arbitrary. The physical interpretation of this model is that all the factors (prices and advertisement expenditures) act independently and contribute to the total sales of firm 1 as a factor which is less than 1 and decreasing for p and v and greater than 1 and increasing for q and u.

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