Improved Sensitivity Estimate for the H^2 Estimation Problem*

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Abstract. The paper deals with the local sensitivity analysis of the discrete-time infinite-horizon H^2 estimation problem. An improved, nonlinear sensitivity estimate is derived which is less conservative than the existing, condition number based sensitivity estimates.

1 Introduction

In the last four decades the H^2 (Wiener-Kalman) estimators have been widely used in numerous applications in signal processing and control. However, the computational and robustness aspects of the H^2 estimation problem have not been studied in a sufficient extent and the efficient and reliable H^2 estimator design and implementation is still an open problem.

In this paper we study the local sensitivity of the discrete-time infinite-horizon H^2 estimation problem [1] relative to perturbations in the data. Using the nonlinear perturbation analysis technique proposed in [2, 3] and further developed in [4, 5], we derive a new, nonlinear local perturbation bound for the solution of the matrix Riccati equation that determines the sensitivity of the H^2 estimation problem. The new sensitivity estimate is a first order homogeneous function of the data perturbations and is less conservative than the existing, condition number based linear sensitivity estimates [6] - [12].

The following notations are used later on: $\mathcal{R}$ is the field of real numbers; $\mathcal{R}^{m\times n}$ is the space of $m \times n$ matrices over $\mathcal{R}$; $A^T \in \mathcal{R}^{n\times m}$ is the transpose of $A = [a_{ij}] \in \mathcal{R}^{m\times n}$; I_n is the identity $n \times n$ matrix; $\mathrm{vec}(A) \in \mathcal{R}^{mn}$ is the column-wise vector representation of the matrix $A \in \mathcal{R}^{m\times n}$; $\Pi_{n^2} \in \mathcal{R}^{n^2\times n^2}$ is the vec-permutation matrix, so that $\mathrm{vec}(A^T) = \Pi \mathrm{vec}(A)$ for all $A \in \mathcal{R}^{n\times n}$; $\|\cdot\|_2$ and $\|\cdot\|_F$ are the spectral and Frobenius norms in $\mathcal{R}^{m\times n}$, while $\|\cdot\|$ is the induced operator norm or an unspecified matrix norm. The Kronecker product of the matrices A, B is denoted by $A \otimes B$ and the symbol := stands for "equal by definition".

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2 Problem Statement

Consider the linear discrete-time model

$$\begin{aligned} x_{i+1} &= Fx_i + Gu_i \\ y_i &= Hx_i + v_i, \quad -\infty < i < \infty \\ s_i &= Lx_i \end{aligned} \tag{1}$$

where $F \in \mathcal{R}^{n.n}$, $G \in \mathcal{R}^{n.m}$, $H \in \mathcal{R}^{p.n}$, and $L \in \mathcal{R}^{q.n}$ are known constant matrices, x_i is the state, y_i is the measured output, s_i is the desired signal, and $\{u_i\}$, $\{v_i\}$ are zero-mean white-noise processes with variance matrices $Q \geq 0$ and $R > 0$, respectively. It is assumed that the pair (F, H) is detectable and the pair $(F, GQ^{1/2})$ – stabilizable.

Given the observations $\{y_j, j \leq i\}$, the H^2 estimation problem [1] consists in finding a linear estimation strategy $\hat{s}_{i|i} = \mathcal{F}(y_0, y, y_1, \ldots, y_j)$ that minimizes the expected filtered error energy, i.e.,

$$\min_{\mathcal{F}} E \sum_{j=0}^{i} (s_j - \hat{s}_{j|j})^T (s_j - \hat{s}_{j|j}).$$

As it is well known [1], in the infinite-horizon case the H^2 estimation problem is formulated as

$$\min_{\text{causal}\, K(z)} \| [(L(zI - F)^{-1}G - K(z)H(zI - F)^{-1}G)Q^{1/2} \quad - K(z)R^{1/2}] \|_2$$

and its solution is given by

$$\begin{aligned} K(z) &= LPH^T(R + HP_0H^T)^{-1} \\ &\quad + L(I - PH^T(R + HP_0H^T)^{-1}H)(zI - F_p)^{-1}K_p \end{aligned}$$

where $K_p = FP_0H^TR_e^{-1}$, $R_e = R + HP_0H^T$, and $P_0 \geq 0$ is the unique stabilizing ($F_p = F - K_pH$ stable) solution of the matrix Riccati equation

$$FPF^T - P + GQG^T - K_pR_eK_p^T = 0. \tag{2}$$

A state-space model for $K(z)$ can be given by

$$\begin{aligned} \hat{x}_{i+1} &= F\hat{x}_i + K_p(y_i - H\hat{x}_i) \\ \hat{s}_{i|i} &= L\hat{x}_i + LPH^TR_e^{-1}(y_i - H\hat{x}_i). \end{aligned}$$

In the sequel we shall write equation (2) in the equivalent form

$$\bar{F}(P, D)PF^T - P + GQG^T = 0 \tag{3}$$

where $\bar{F}(P, D) = F - FPH^TR_e^{-1}(P, D)H$ and $D = (F, H)$.

Assuming that the matrices F, G, H in (1) are subject to perturbations ΔF, ΔG, ΔH, we obtain the perturbed equation

$$\bar{F}(P, D + \Delta D)P(F + \Delta F)^T - P + (G + \Delta G)Q(G + \Delta G)^T = 0 \quad (4)$$

where $\Delta D = (\Delta F, \Delta H)$,

$$\bar{F}(P, D+\Delta D) = (F+\Delta F) - (F+\Delta F)P(H+\Delta H)^T R_e^{-1}(P, D+\Delta D)(H+\Delta H)$$

and
$$R_e(P, D + \Delta D) = R_e(P, D) + \Delta H P H^T + H P \Delta H^T + \Delta H P \Delta H^T.$$

Since the Fréchet derivative of the left-hand side of (3) in P at $P = P_0$ is invertible, the perturbed equation (4) has a unique solution $P = P_0 + \Delta P$ in the neighborhood of P_0.

Denote by $\Delta_M = \|\Delta M\|_F$ the absolute perturbation of a matrix M and let $\Delta := [\Delta_F, \Delta_G, \Delta_H]^T \in \mathcal{R}_+^3$.

The sensitivity analysis of the Riccati equation (3) consists in finding estimate for the absolute perturbation $\Delta_P := \|\Delta P\|_{\mathrm{F}}$ in the solution P as a function of the absolute perturbations Δ_F, Δ_G, Δ_H in the coefficient matrices F, G, H.

A number of linear local bounds for Δ_P have been derived in [6] - [12], based on the condition numbers of (3). For instance, in [9] a perturbation bound of the type

$$\Delta_P \leq K_F \Delta_F + K_G \Delta_G + K_H \Delta_H + \mathrm{O}(\|\Delta\|^2), \ \Delta \to 0 \quad (5)$$

has been obtained, where K_F, K_G, K_H are the condition numbers of (3) relative to perturbations in the matrices F, G and H, respectively. However, it is possible to obtain nonlinear local sensitivity estimates for (3) that are less conservative than the condition numbers based sensitivity estimates.

The problem considered in this paper is to find a first order homogeneous local perturbation bound

$$\Delta_P \leq f(\Delta) + \mathrm{O}(\|\Delta\|^2), \ \Delta \to 0, \quad (6)$$

which is tighter than the condition number based perturbation bounds.

3 Main Result

Denote by $\Phi(P, D)$ the left-hand side of the Riccati equation (3) and let P_0 be the positive definite or semi-definite solution of (3). Then $\Phi(P_0, D) = 0$.

Setting $P = P_0 + \Delta P$, the perturbed equation (4) may be written as

$$\Phi(P_0 + \Delta P, D + \Delta D) = \quad (7)$$

$$\Phi(P_0, D) + \Phi_P(\Delta P) + \Phi_F(\Delta F) + \Phi_G(\Delta G) + \Phi_H(\Delta H) + S(\Delta P, \Delta D) = 0$$

where $\Phi_P(.)$, $\Phi_F(.)$, $\Phi_G(.)$, $\Phi_H(.)$ are the Fréchet derivatives of $\Phi(P, D)$ in the corresponding matrix arguments, evaluated for $P = P_0$, and $S(\Delta P, \Delta D)$ contains the second and higher order terms in ΔP, ΔD.

The calculation of the Fréchet derivatives of $\Phi(P, D)$ gives

$$\Phi_P(Z) = \bar{F}_0 Z \bar{F}_0^T - Z \tag{8}$$

$$\Phi_F(Z) = \bar{F}_0 P_0 Z + Z^T P_0 \bar{F}_0^T$$

$$\Phi_G(Z) = GQZ + Z^T Q G^T$$

$$\Phi_H(Z) = -\bar{F}_0 P_0 Z R_0^{-1} H P_0 F^T - F P_0 H^T R_0^{-1} Z^T P_0 \bar{F}_0^T$$

where$\bar{F}_0 = \bar{F}(P_0, D), \quad R_0 = R_e(P_0, D)$.

Denoting $M_P \in \mathcal{R}^{n^2 \times n^2}$, $M_F \in \mathcal{R}^{n^2 \times n^2}$, $M_G \in \mathcal{R}^{n^2 \times n^2}$, $M_H \in \mathcal{R}^{n^2 \times n^2}$ the matrix representations of the operators $\Phi_P(.)$, $\Phi_F(.)$, $\Phi_G(.)$, $\Phi_H(.)$, we have

$$M_P = \bar{F}_0 \otimes \bar{F}_0 - I_{n^2}$$

$$M_F = I_n \otimes \bar{F}_0 P_0 + (\bar{F}_0 P_0 \otimes I_n)\Pi \tag{9}$$

$$M_G = I_n \otimes GQ + (GQ \otimes I_n)\Pi$$

$$M_H = -F P_0 H^T R_0^{-1} \otimes \bar{F}_0 P_0 - (\bar{F}_0 P_0 \otimes F P_0 H^T R_0^{-1})\Pi$$

where $\Pi \in \mathcal{R}^{n^2 \times n^2}$ is the permutation matrix such that $\text{vec}(M^T) = \Pi \text{vec}(M)$ for each $M \in \mathcal{R}^{n \times n}$.

The operator equation (7) may be written in a vector form as

$$\begin{aligned} \text{vec}(\Delta P) = {} & N_1 \text{vec}(\Delta F) + N_2 \text{vec}(\Delta G) \\ & + N_3 \text{vec}(\Delta H) - M_P^{-1} \text{vec}(S(\Delta P, \Delta D)) \end{aligned} \tag{10}$$

where

$$N_1 = -M_P^{-1} M_F,\ N_2 = -M_P^{-1} M_G,\ N_3 = -M_P^{-1} M_H.$$

It is easy to show that the condition number based perturbation bound (5) is a corollary of (10). Indeed, it follows from (10)

$$\begin{aligned} \|\text{vec}(\Delta P)\|_2 \le {} & \|N_1\|_2 \|\text{vec}(\Delta F)\|_2 + \|N_2\|_2 \|\text{vec}(\Delta G)\|_2 \\ & + \|N_3\|_2 \|\text{vec}(\Delta H)\|_2 + \text{O}(\|\Delta\|^2) \end{aligned}$$

and having in mind that

$$\|\text{vec}(\Delta M)\|_2 = \|\Delta M\|_F = \Delta_M$$

and denoting

$$K_F = \|N_1\|_2,\ K_G = \|N_2\|_2,\ K_H = \|N_3\|_2$$

we obtain

$$\Delta_P \le K_F \Delta_F + K_G \Delta_G + K_H \Delta_H + \text{O}(\|\Delta\|^2). \tag{11}$$

Relation (10) also gives

$$\Delta_P \le \|N\|_2 \|\Delta\|_2 + \mathrm{O}(\|\Delta\|^2) \tag{12}$$

where $N = [N_1, N_2, N_3]$.

Note that the bounds in (11) and (12) are alternative, i.e. which one is less depends on the particular value of Δ.

There is also a third bound, which is always less than or equal to the bound in (11). We have

$$\Delta_P \le \sqrt{\Delta^\top U(N) \Delta} + \mathrm{O}(\|\Delta\|^2) \tag{13}$$

where $U(N)$ is the 3×3 matrix with elements

$$u_{ij}(N) = \|N_i^\top N_j\|_2.$$

Since $\left\|N_i^\top N_j\right\|_2 \le \|N_i\|_2 \|N_j\|_2$ we get

$$\sqrt{\Delta^\top U(N) \Delta} \le \|N_1\|_2 \Delta_F + \|N_2\|_2 \Delta_G + \|N_3\|_2 \Delta_H.$$

Hence we have the overall estimate

$$\Delta_P \le f(\Delta, N) + \mathrm{O}(\|\Delta\|^2), \ \Delta \to 0 \tag{14}$$

where

$$f(\Delta, N) = \min\{\|N\|_2 \|\Delta\|_2, \sqrt{\Delta^\top U(N) \Delta}\} \tag{15}$$

is a non-linear, first order homogeneous and piece-wise real analytic function in Δ.

4 Numerical Example

Consider a third order model of type (1) with matrices

$$F = VF^*V, \ G = VG^*, \ H = H^*V$$

where $V = I_3 - 2vv^T/3, \ v = [1, 1, 1]^T$ and

$$F^* = \mathrm{diag}(0, 0.1, 0), \ G^* = \mathrm{diag}(2, 1, 0.1)$$

$$H^* = \mathrm{diag}(1, 0.5, 10).$$

The perturbations in the data are taken as

$$\Delta F = V \Delta F^* V, \ \Delta G = V \Delta G^*, \ \Delta H = \Delta H^* V$$

where

$$\Delta F^* = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -9 \\ 0 & -9 & 5 \end{bmatrix} \times 10^{-i},$$

$$\Delta G^* = \begin{bmatrix} 10 & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & 10 \end{bmatrix} \times 10^{-i-1},$$

$$\Delta H^* = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 10 \end{bmatrix} \times 10^{-i-1}$$

for $i = 10, 9, \ldots, 3$.

The absolute perturbations Δ_P in the solution of the Riccati equation are estimated by the linear bound (11) and the nonlinear homogeneous bound (14). The results obtained for $Q = R = I$ and different values of i are shown in Table 1. The actual relative changes in the solution are close to the quantities predicted by the local sensitivity analysis.

Table 1.

i	Δ_P	Est. (11)	Est. (14)
10	5.1×10^{-10}	9.8×10^{-9}	7.7×10^{-10}
9	5.1×10^{-9}	9.8×10^{-8}	7.7×10^{-9}
8	5.1×10^{-8}	9.8×10^{-7}	7.7×10^{-8}
7	5.1×10^{-7}	9.8×10^{-6}	7.7×10^{-7}
6	5.1×10^{-6}	9.8×10^{-5}	7.7×10^{-6}
5	5.1×10^{-5}	9.8×10^{-4}	7.7×10^{-5}
4	5.1×10^{-4}	9.8×10^{-3}	7.7×10^{-4}
3	5.1×10^{-3}	9.8×10^{-2}	7.7×10^{-3}

5 Conclusion

In this paper the local sensitivity of the discrete-time infinite-horizon H^2 estimation problem has been studied. A new, nonlinear local perturbation bound has been obtained for the solution of the Riccati equation that determines the sensitivity of the problem. The new local sensitivity estimate is a first order homogeneous function of the data perturbations and is tighter than the condition number based sensitivity estimates.

References

1. B. Hassibi, A. H. Sayed, T. Kailath: *Indefinite-Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theories*. SIAM, Philadelphia, 1999
2. M. M. Konstantinov, P. Hr. Petkov, N. D. Christov, D. W. Gu, V. Mehrmann: Sensitivity of Lyapunov equations. In: N. Mastorakis (Ed.): *Advances in Intelligent Systems and Computer Science*. WSES Press, N.Y., 1999, 289–292
3. M. M. Konstantinov, P. Hr. Petkov, D. W. Gu: Improved perturbation bounds for general quadratic matrix equations. *Numer. Func. Anal. and Optimiz.* **20** (1999) 717–736
4. N. D. Christov, S. Lesecq, M. M. Konstantinov, P. Hr. Petkov, A. Barraud: New perturbation bounds for Sylvester equations. In: *Proc. 39th IEEE Conf. on Decision and Control*, Sydney, 12-15 December 2000, 4233–4234
5. N. D. Christov, M. Najim, E. Grivel,D. Henry: On the local sensitivity of the discrete-time H^∞ estimation problem. In: *Proc. 15th IFAC World Congress*, Barcelona, 21-26 July 2002, paper T-Tu-A01/1091
6. M. M. Konstantinov, P. Hr. Petkov, N. D. Christov: Perturbation analysis of the continuous and discrete matrix Riccati equations. In: *Proc. 1986 American Control Conf.*, Seattle, 18-20 June 1986, vol. 1, 636-639
7. P. Gahinet, A. J. Laub: Computable bounds for the sensitivity of the algebraic Riccati equation. *SIAM J. Contr. Optim.* **28** (1990) 1461–1480
8. P. Hr. Petkov, N. D. Christov, M. M. Konstantinov: *Computational Methods for Linear Control Systems*. Prentice Hall, N.Y., 1991
9. M. M. Konstantinov, P. Hr. Petkov, N. D. Christov: Perturbation analysis of the discrete Riccati equation. *Kybernetika* **29** (1993) 18–29
10. A. R. Ghavimi, A. J. Laub: Backward error, sensitivity and refinement of computed solutions of algebraic Riccati equations. *Numer. Lin. Alg. Appl.* **2** (1995) 29–49
11. J.-G. Sun: Perturbation theory for algebraic Riccati equations. *SIAM J. Matrix Anal. Appl.* **19** (1998) 39–65
12. J.-G. Sun: Condition numbers of algebraic Riccati equations in the Frobenius norm. *Lin. Alg. Appl.* **350** (2002) 237–261