Error Estimate on Non-bandlimited Random Signals by Local Averages

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Abstract. We show that a non-bandlimited weak sense stationary stochastic process can be approximated by its local averages near the sampling points, and explicit error bounds are given.

It is well known that the Shannon sampling theorem plays an important role in signal processing. It states that if a function f is band-limited to $[-\Omega, \Omega]$, i.e., $f \in \mathbb{R}$ and supp $\hat{f} \subset [-\Omega, \Omega]$, where

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-it\omega} dt$$

is the Fourier transform of f, then f can be recovered from its sampled values at instances $k\pi/\Omega$. Specifically,

$$f(t) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k\pi}{\Omega}\right) \operatorname{sinc}(\Omega t - k\pi), \tag{1}$$

where $\operatorname{sinc} t = \sin t/t$.

Since signals are often of random characters, random signals play an important role in signal processing, especially in the study of sampling theorems. But in many situations the assumption of band-limitation is not fulfilled exactly, or the correct bandwidth is unknown. For this purpose one usually uses non-bandlimited stochastic processes which are stationary in the weak sense as a model. We will give some new results on this topic in this paper.

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Before stating the results, let us introduce some notations. $L^p(\mathbb{R})$ is the space of all measurable functions on $\mathbb{R}$ for which $\|f\|_p < +\infty$, where

$$\|f\|_p := \left(\int_{-\infty}^{+\infty} |f(u)|^p du\right)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|f\|_\infty := \operatorname*{ess\,sup}_{u\in\mathbb{R}} |f(u)|, \qquad p = \infty.$$

$B_{\Omega,p}$ is the set of all entire functions f of exponential type with type at most Ω that belong to $L^2(\mathbb{R})$ when restricted to the real line [11]. By the Paley-Wiener Theorem, a square integrable function f is band-limited to $[-\Omega, \Omega]$ if and only if $f \in B_{\Omega,2}$.

Given a probability space $(\mathcal{W}, \mathcal{A}, \mathcal{P})$ [5], a real-valued stochastic process $X(t) := X(t, \omega)$ defined on $\mathbb{R} \times \mathcal{W}$ is said to be stationary in weak sense if $E[X(t)^2] < \infty$, $\forall t \in \mathbb{R}$, and the autocorrelation function

$$R_X(t, t+\tau) := \int_{\mathcal{W}} X(t,\omega)X(t+\tau,\omega)dP(\omega)$$

is independent of $t \in \mathbb{R}$, i.e., $R_X(t, t+\tau) = R_X(\tau)$.

A weak sense stationary process $X(t)$ is said to be bandlimited to an interval $[-\Omega, \Omega]$ if R_X belongs to $B_{\Omega,p}$ for some $1 \le p \le \infty$, we note that $X(t) \in \mathcal{L}$.

Noting that any function $R_X \in B_{\Omega,p}$ is infinitely differentiable, so the process $X(t)$ belongs to Lipschitz class

$$\mathcal{L}ip_L\alpha := \{X \in \mathcal{L}; \omega(\mathcal{L}, X, \eta) \le L\eta^\alpha\} \quad (0 < \alpha \le 1),$$

where $\omega(\mathcal{L}, X, \eta) := \sup_{|h|<\eta} \|X(t+h) - X(t)\|_{\mathcal{L}}$ is the modulus of continuity in mean square, $L > 0$ is the Lipschitz constant, and $\|X(t+h) - X(t)\|_{\mathcal{L}} = \sqrt{E[(X(t+h) - X(t))^2]}$.

The convolution of two functions $f, g \in L^1$ is defined by $f * g(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(t-u)du$. The convolution of a process $X \in \mathcal{L}$ and a function $g \in L^1$ is defined similarly by

$$X * g(t, \omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(u,\omega)g(t-u)du.$$

It is not difficult to check that the autocorrelation function of $X * g$ is

$$R_{X*g} = R_X * g * g(\tau).$$

In 1981, Splettstösser proved the following result.

Proposition 1. ([7, Theorem 2.2]) *If the autocorrelation function of the weak sense stationary stochastic process $X(t,\omega)$ belongs to $B_{\Omega,p}$ for some $1 \le p \le 2$ and $\Omega > 0$, then*

$$\lim_{N\to\infty} E\left(\left|X(t,\omega) - \sum_{k=-N}^{N} X\left(\frac{k\pi}{\Omega}, \omega\right) \operatorname{sinc}(\Omega t - k\pi)\right|^2\right) = 0. \tag{2}$$

Proposition 2. ([7, Corollary 2.3]) *If the autocorrelation function R_X of the weak sense stationary stochastic process $X(t,\omega)$ belongs to $B_{\Omega,p}$ for some $1 \le p \le \infty$, where $\Omega > 0$, and satisfies*

$$|R_X(t)| = O(|t|^{-\gamma}), (|t| \to \infty) \tag{3}$$

for some $\gamma > 0$. Then the sampling expansion (2) holds.

Proposition 3. ([7, Theorem 3.1]) *If the weak sense stationary stochastic process $X(t,\omega)$ is r times differentiable (in mean square sense) for some positive integer r and with $X^{(r)} \in \mathcal{L}ip_L\alpha$ for some $\alpha \in (0,1]$, and R_X satisfies (3) for $\gamma \in (0,1]$, then for $\Omega \to \infty$*

$$\lim_{N\to\infty} E\left(\left|X(t,\omega) - \sum_{k=-N}^{N} X\left(\frac{k\pi}{\Omega},\omega\right)\operatorname{sinc}(\Omega t - k\pi)\right|^2\right) = \mathcal{O}[(\frac{\Omega}{\pi})^{-2r-2\alpha}\ln^2(\frac{\Omega}{\pi})]. \tag{4}$$

For physical reasons, e.g., the inertia of the measurement apparatus, measured sampled values obtained in practice may not be values of $f(t)$ precisely at times t_k, but only local average of $f(t)$ near t_k. Specifically, measured sampled values are

$$\langle f, u_k \rangle = \int f(t)u_k(t)dt \tag{5}$$

for some collection of averaging functions $u_k(t), k \in \mathbb{Z}$, which satisfy the following properties,

$$\operatorname{supp} u_k \subset [t_k - \frac{\sigma}{2}, t_k + \frac{\sigma}{2}], \quad u_k(t) \ge 0, \quad \text{and} \quad \int u_k(t)dt = 1. \tag{6}$$

The local averaging method in sampling was first studied by Gröchenig[4] in 1992. Butzer and Lei [2] also gave some interesting results on non-necessarily bandlimited functions in 1998. Recently Sun and Zhou [9, 10] extend some classical results on irregular sampling to local average cases. They all assume that the time intervals for averaging are symmetric. But in applications, it might not be the case. More specifically, if we want to measure the values of $f(t)$ at $k\pi/\Omega$, the measurement apparatus in fact gives a weighted average over a time interval $[k\pi/\Omega - \sigma_k', k\pi/\Omega + \sigma_k'']$, where σ_k', σ_k'' are positive numbers. We assume that $\sigma/4 \le \sigma_k', \sigma_k'' \le \sigma/2$ and that the weight functions u_k are continuous, i.e,

$$\operatorname{supp} u_k \subset [t_k - \sigma_k', t_k + \sigma_k''], \quad u_k(t) \ge 0, \quad \text{and} \quad \int u_k(t)dt = 1. \tag{7}$$

In this paper, the results on approximation of non-bandlimited weak sense stationary stochastic process by local averages near the sampling points will be given. By the property of the weak sense stationary stochastic process, the assumption (3) can be replaced by

$$R_X(t) \le R_X(0)(1 + |t|)^{-\gamma} \quad \text{for} \quad \gamma \in (0,1]. \tag{8}$$

The following is our results, which can be proved by the Proposition of Butzer[1], Splettstösser [8], Li and Wu [5, page 291], Hausdorff-Young inequality [6, page176].

Theorem 1. *If the weak sense stationary stochastic process* $X(t,\omega)$ *is* r *times differentiable (in mean square sense) for some positive integer* r, $X^{(r)} \in \mathcal{L}ip_L\alpha$ *for some* $\alpha \in (0,1]$, *and* R_X *satisfies (8). Then for* $\Omega \geq \max\{\pi e^{1/(\gamma/2+r+\alpha)}, 30\pi\}$, $\delta \leq 1/\Omega$ *we have*

$$\lim_{N\to\infty} E\left[\left|X(t,\omega) - \sum_{k=-N}^{N} \int_{k\pi/\Omega-\sigma'_k}^{k\pi/\Omega+\sigma''_k} u_k(t)X(t,\omega)dt \cdot \mathrm{sinc}(\Omega t - k\pi)\right|^2\right]$$
$$\leq \left(270.16L^2 3^{2\gamma}\left(\frac{2}{\pi}\right)^{2r+2\alpha} + 153.44R_X(0)\right)\left(1+\frac{2(r+\alpha)}{\gamma}\right)^2 \cdot$$
$$(\frac{\Omega}{\pi})^{-2r-2\alpha}\ln^2(\frac{\Omega}{\pi}). \tag{9}$$

where $\{u_k(t)\}$ *is a sequence of continuous weight functions defined by (7).*

References

1. Butzer, P.L., Splettstösser, W. and Stens R. L., The sampling theorem and linear prediction in signal analysis, Jber. d. Dt. Math.-Verein., **90** (1988) 1-70.
2. Butzer, P.L., Lei, J., Errors in truncated sampling series with measured sampled values for non-necessarily bandlimited functions, Funct.Approx., **26** (1988) 25-39.
3. Ditzian, Z. and Totik, V., Moduli of smoothness, Springer-Verlag, 1987.
4. Gröchenig, K., Reconstruction algorithms in irregular sampling, Math. Comput., **59** (1992) 181-194.
5. Li, Z. and Wu, R., A course of studies on stochastic processes, High Education Press, 1987(in chinese).
6. Pinsky, M. A., Introduction to Fourier analysis and wavelets, Wadsworth Group. Brooks/Cole. 2002.
7. Splettstösser, W., sampling series approximation of continuous weak sense stationary processes, Information and Control **50** (1981) 228-241.
8. Splettstösser, W., Stens, R. L. and Wilmes, G., on the approximation of the interpolating series of G. Valiron , Funct. Approx. Comment. Math. **11** (1981) 39-56.
9. Sun, W. and Zhou, X., Reconstruction of bandlimited functions from local averages, Constr. Approx., **18** (2002) 205-222.
10. Sun, W. and Zhou, X., Reconstruction of bandlimited signals from local averages, IEEE Trans. Inform. Theory, **48** (2002) 2955-2963.
11. Zayed,A.I. and Butzer,P.L., Lagrange interpolation and sampling theorems, in “Nonuniform Sampling, Theory and Practice”, Marvasti,F., Ed., Kluwer Academic, 2001, pp. 123–168.