Bar k-Visibility Graphs: Bounds on the Number of Edges, Chromatic Number, and Thickness

Alice M. Dean[1], William Evans[2], Ellen Gethner[3], Joshua D. Laison[4], Mohammad Ali Safari[5], and William T. Trotter[6]

[1] Department of Mathematics and Computer Science, Skidmore College
adean@skidmore.edu
[2] Department of Computer Science, University of British Columbia
will@cs.ubc.ca
[3] Department of Computer Science and Engineering,
University of Colorado at Denver
ellen.gethner@cudenver.edu
[4] Department of Mathematics and Computer Science, Colorado College
jlaison@coloradocollege.edu
[5] Department of Computer Science, University of British Columbia
safari@cs.ubc.ca
[6] Department of Mathematics, Georgia Institute of Technology
trotter@math.gatech.edu

Abstract. Let S be a set of horizontal line segments, or bars, in the plane. We say that G is a bar visibility graph, and S its bar visibility representation, if there exists a one-to-one correspondence between vertices of G and bars in S, such that there is an edge between two vertices in G if and only if there exists an unobstructed vertical line of sight between their corresponding bars. If bars are allowed to see through each other, the graphs representable in this way are precisely the interval graphs. We consider representations in which bars are allowed to see through at most k other bars. Since all bar visibility graphs are planar, we seek measurements of closeness to planarity for bar k-visibility graphs. We obtain an upper bound on the number of edges in a bar k-visibility graph. As a consequence, we obtain an upper bound of 12 on the chromatic number of bar 1-visibility graphs, and a tight upper bound of 8 on the size of the largest complete bar 1-visibility graph. We conjecture that bar 1-visibility graphs have thickness at most 2.

1 Introduction

Recent attention has been drawn to a variety of generalizations of bar visibility graphs [2, 3, 6, 5, 7, 8, 11, 12, 14, 15]. In this note, we report on a new generalization of bar visibility graphs called *bar k-visibility graphs*, and discuss some of their properties; complete details can be found in [4]. In what follows, we use the standard graph theory terminology found in [9, 17].

Let S be a set of disjoint horizontal line segments, or *bars*, in the plane. We say that a graph G is a *bar visibility graph*, and S a *bar visibility representation*

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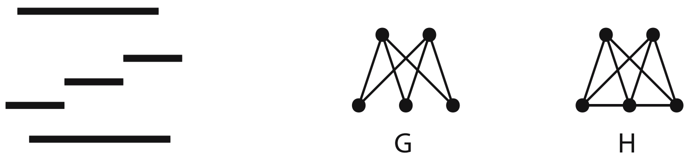

Fig. 1. The bar visibility representation shown is an ε-visibility representation of G and a strong visibility representation of H

of G, if there exists a one-to-one correspondence between vertices of G and bars in S, such that there is an edge between two vertices x and y in G if and only if there exists a vertical line segment L, called a *line of sight*, whose endpoints are contained in X and Y, respectively, and which does not intersect any other bar in S. [1, 12, 13, 18].

If each line of sight is required to be a rectangle of positive width, then S is an *ε-visibility representation* of G, and when each line of sight is a line segment, then S is a *strong visibility representation* of G [16]. In general, these definitions are not equivalent; $K_{2,3}$ admits an ε-visibility representation but not a strong visibility representation, as shown in Figure 1.

Given a set of bars S in the plane, suppose that an endpoint of a bar B and an endpoint of a bar C in S have the same x-coordinate. We elongate one of these two bars so that their endpoints have distinct x-coordinates. If S is a strong visibility representation of a graph G, then we may perform this elongation so that S is still a strong visibility representation of G. If S is an ε-visibility representation of G, then we may perform this elongation so that S is an ε-visibility representation of a new graph H with $G \subseteq H$. Since we are interested in the maximum number of edges obtainable in a representation, we may consider the graph H instead of the graph G. Repeating this process yields a set of bars with pairwise distinct endpoint x-coordinates. For the remainder of this paper, we assume that all bar visibility representations are of this form.

If a set of bars S has all endpoint x-coordinates distinct, the graphs G and H that have S as a strong bar visibility representation and an ε-visibility representation, respectively, are isomorphic. Hence without loss of generality, for the remainder of the paper, all bar visibility representations are strong bar visibility representations.

By contrast, suppose that S is a set of closed intervals on the real line. The graph G is called an *interval graph* and S an *interval representation* of G if there exists a one-to-one correspondence between vertices of G and intervals in S, such that x and y are adjacent in G if and only if their corresponding intervals intersect. Suppose we call a set S of horizontal bars in the plane an *x-ray-visibility representation* if we allow sight lines to intersect arbitrarily many bars in S. Then we can easily transform an x-ray-visibility representation into an interval representation by vertically translating the bars in S, and vice-versa. Therefore G is an x-ray-visibility graph if and only if G is an interval graph.

Motivated by this correspondence, we define a *bar k-visibility graph* to be a graph with a bar visibility representation in which a sight line between bars

X and Y intersects at most k additional bars. As a first step on the road to a characterization of bar k-visibility graphs, since all bar visibility graphs are planar, we seek measurements of closeness to planarity for bar k-visibility graphs.

2 An Edge Bound for Bar 1-Visibility Graphs

Suppose G is a graph with n vertices, and S is a bar 1-visibility representation of G. Since we consider S to be a strong visibility representation of G, without loss of generality, we may assume that all endpoints of all bars in S have distinct x-coordinates, and all bars in S have distinct y-coordinates.

It will be convenient to use four different labeling systems for the bars in S. Label the bars $1_l, 2_l, \ldots, n_l$ in increasing order of the x-coordinate of their left endpoint. Label them $1_r, 2_r, \ldots, n_r$ in decreasing order of the x-coordinate of their right endpoint. Label them $1_b, 2_b, \ldots, n_b$ in increasing order of their y-coordinate. Finally, label them $1_t, 2_t, \ldots, n_t$ in decreasing order of their y-coordinate. So the bar 1_l has leftmost left endpoint, the bar 1_r has rightmost right endpoint, the bar $1_b = n_t$ is bottommost in the representation, and the bar $1_t = n_b$ is topmost in the representation. We use this notation for the remainder of the paper.

Remark 1. Suppose S is a bar k-visibility representation of a graph G with n vertices. We elongate the top and bottom bars of S to obtain a new bar k-visibility representation S' of a new graph G', with the additional property that $1_t = 1_r = 1_l$ and $1_b = 2_r = 2_l$ in S'. The graph G' has n vertices and contains G as a subgraph. We may therefore assume that every edge-maximal bar k-visibility graph has such a bar k-visibility representation.

Lemma 1. *If G is a bar 1-visibility graph with $n \geq 4$ vertices, then G has at most $6n - 17$ edges.*

Proof. Suppose G is a graph with n vertices, and S is a bar 1-visibility representation of G. We define the following correspondence between bars in S and edges of G. Let U be the bar in S associated with vertex u. For every edge $\{u, v\}$ in G, let $\ell(\{u, v\})$ be the vertical line segment from a point in U to a point in V whose x-coordinate is the infimum of x coordinates of lines of sight between U and V. An edge $\{u, v\}$ is called a *left edge* of U (respectively V) if $\ell(\{u, v\})$ contains the left endpoint of U (respectively V). If $\ell(\{u, v\})$ contains neither U nor V's left endpoint then it must contain the right endpoint of some bar B (that blocks the 1-visibility of U from V from that point on). In this case, we call $\{u, v\}$ a *right edge of* B. Note that the right edges of B are not incident to the vertex b of G corresponding to the bar B. Each bar B can have at most 4 left edges (two to bars above B in S and two to bars below B in S) and at most 2 right edges, as shown in Figure 2.

Counting both left and right edges, each bar in S is associated with at most 6 edges. So there are at most $6n$ edges in G. However, the bars 1_l, 2_l, 3_l, and 4_l have at most 0, 1, 2, and 3 left edges, respectively. Similarly, the bars 1_r, 2_r, 3_r,

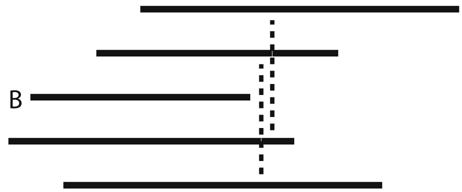

Fig. 2. The two right edges associated to bar B

and 4_r have at most 0, 0, 0, and 1 right edges, respectively. Therefore there are at most $4n - 10$ left edges and at most $2n - 7$ right edges, for a total of at most $6n - 17$ edges in G. □

Theorem 1. *If G is a bar 1-visibility graph with $n \geq 5$ vertices, then G has at most $6n - 20$ edges.*

Proof. We improve the bound given in Lemma 1 by using a slightly more sophisticated technique. We follow the notation of Lemma 1.

By Remark 1, the edge $\{1_t, 1_b\}$ will always be a left edge. Since the edge associated with the right endpoint of the bar 4_r can only be this edge, the bar 4_r must have 0 right edges. So there are at most $2n - 8$ right edges in G, and $6n - 18$ edges in total. If G has exactly $6n - 18$ edges, then bars 1_l, 2_l, 3_l, and 4_l must have at least 0, 1, 2, and 2 left edges, respectively.

Suppose that bar 4_l has only two left edges. Then it does not have a line of sight to bar 3_l, which can happen only if 3_l ends before 4_l begins. Then $3_l = n_r$, and 3_l has 0 right edges. Therefore G has at most $6n - 20$ edges. The only remaining possibility is that bar 4_l has exactly three left edges.

If S had at most $4n - 12$ left edges, then S would have at most $6n - 20$ edges in total. The remaining possibilities are that S has either $4n - 11$ or $4n - 10$ left edges. Since 1_l, 2_l, 3_l, and 4_l have exactly 0, 1, 2, and 3 left edges, respectively, all other bars in S must have exactly four left edges, except perhaps for one bar i_l, which may have three left edges. By the same argument, since 1_r, 2_r, 3_r, and 4_r have no right edges, every additional bar must have exactly two right edges, except one additional bar, which may have only one.

Consider the four edges $e_1 = \{1_t, 1_b\}$, $e_2 = \{1_t, 2_b\}$, $e_3 = \{2_t, 1_b\}$, and $e_4 = \{2_t, 2_b\}$. If $i_l = 2_b$, then the edges e_1 and e_3 are left edges, but the edges e_2 and e_4 may not be. If $i_l = 2_t$, then the edges e_1 and e_2 are left edges, but the edges e_3 and e_4 may not be. If i_l is neither of these bars, then all four of these edges are left edges.

Since the bars 2_t and 2_b have at most one right edge each, one of them must be bar 3_r or bar 4_r. Without loss of generality, assume that bar 2_t is either bar 3_r or bar 4_r. So in the order of the bars 1_r through 5_r given by increasing y-coordinate, the bar 5_r must appear either second or third. Figure 3 shows the four possibilities that may occur.

In each of the four cases shown, and for each of the three possibilities for the bar i_l, one can check that 5_r has at most one right edge. So the remaining bars must all have exactly two right edges. Therefore the bars 2_t and 2_b must be two

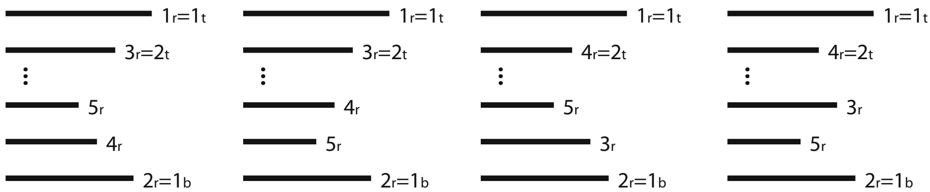

Fig. 3. The four possible arrangements of bars 1_r, 2_r, 3_r, 4_l, and 5_r

of the three bars 3_r, 4_r, and 5_r. But this implies that any right edge associated to 5_r must be between a pair of the bars 1_t, 2_t, 1_b, and 2_b. Therefore 5_r must have no right edges, and G has at most $6n - 20$ edges in total. □

Corollary 2. *The graph K_9 is not a bar 1-visibility graph.*

Proof. Any bar 1-visibility graph with 9 vertices has at most 34 edges, whereas K_9 has 36 edges. □

Corollary 3. *If G is a bar 1-visibility graph, then $\chi(G) \leq 12$.*

Proof. We proceed by induction. Assume that all bar 1-visibility graphs with $n - 1$ vertices have $\chi \leq 12$, and suppose that G is a bar 1-visibility graph with n vertices. By Theorem 1, $\sum_{v \in V(G)} \deg(v) < 12n$, so the average degree of a vertex in G is strictly less than 12. Then there must exist a vertex v in G of degree at most 11. We consider the graph $G - v$. Although this graph may not be a bar 1-visibility graph, it is a subgraph of the graph G' with bar 1-visibility representation obtained from a representation of G by deleting the bar corresponding to v. Therefore the edge bound in Theorem 1 still applies to H. By the induction hypothesis, we may color the vertices of H with 12 colors, replace v, and color v with a color not used on its neighbors. □

Corollary 4. *There are thickness-2 graphs with n vertices that are not bar 1-visibility graphs for all $n \geq 15$.*

Proof. Note that there are no thickness-2 graphs with n vertices and more than $6n - 12$ edges, since if G has thickness 2 then G is the union of two planar graphs, each of which have at most $3n - 6$ edges. Consider the graph $G = C_3 \boxtimes C_5$ formed by replacing each vertex in C_5 with C_3 and taking the join of neighboring C_3's. G has 15 vertices and $6 \cdot 15 - 12 = 78$ edges. Since G is the union of the two planar graphs shown in Figure 4, G has thickness 2.

Let $G_{15} = G$ and suppose L_1 and L_2 are the two plane layers of G_{15}. Let $\{a, b, c\}$ be a face in L_1 and $\{d, e, f\}$ be a face in L_2 such that $\{a, b, c\} \cap \{d, e, f\} = \varnothing$. Add a new vertex v to G_{15} adjacent to $\{a, b, c\}$ in L_1 and $\{d, e, f\}$ in L_2; define the new graph to be G_{16}. The graph G_{16} has 16 vertices and $6 \cdot 16 - 12$ edges, and thickness 2. Following the same procedure, inductively we construct an infinite family of graphs G_n such that for all $n \geq 15$, G_n has n vertices and $6n - 12$ edges, and thickness 2. Therefore none of these graphs can be a bar 1-visibility graph by Theorem 1. □

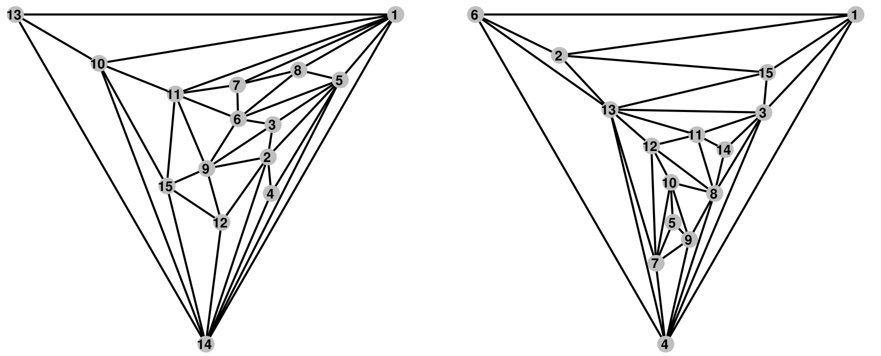

Fig. 4. Two planar graphs whose union is not a bar 1-visibility graph

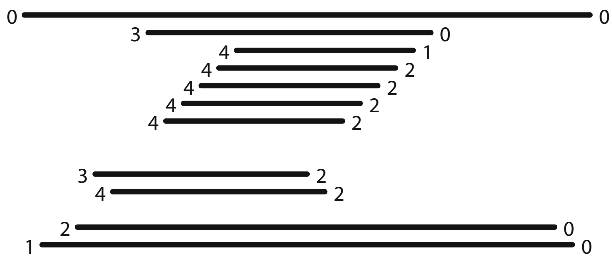

Fig. 5. A bar 1-visibility representation with $6n - 20$ edges

Note that the graphs $\{G_n\}$ given in the proof of Corollary 4 have the largest possible number of edges while having thickness 2.

Theorem 5. *There exist bar 1-visibility graphs with* $6n - 20$ *edges,* $n \geq 5$.

Proof. The graph with representation shown in Figure 5 is a bar 1-visibility graph with $6n - 20$ edges. For ease of counting, the left and right endpoints of bars in this representation are labeled with the number of left and right edges associated to each bar. Note that this representation has $4n - 11$ left edges and $2n - 9$ right edges. Although $n = 11$ in this representation, more bars can easily be deleted to create a representation with as few as 5 bars, or added to create a representation with arbitrarily many bars. For the values $n = 5$ through 8, this representation yields a complete graph. □

Corollary 6. *The graph* K_8 *is a bar 1-visibility graph.*

Proof. Take only eight bars in the representation shown in Figure 5. □

By Corollary 6, if G is a bar 1-visibility graph, then $\chi(G)$ may be 8. No bar 1-visibility graph is known with chromatic number 9. The standard example of a graph with chromatic number 9 but clique number smaller than 9 is the Sulanke graph $K_6 \vee C_5$ [17], which is not a bar 1-visibility graph since it has 11 vertices and 50 edges.

3 Edge Bounds on Bar k-Visibility Graphs

The following theorem generalizes Lemma 1 for $k > 1$. The proof is entirely analogous to the proof of Lemma 1, and can be found in [4].

Theorem 7. *If G is a bar k-visibility graph with $n \geq 2k+2$ vertices, then G has at most $(k+1)(3n - \frac{7}{2}k - 5)$ edges.*

Theorem 8. *There exist bar k-visibility graphs with n vertices and $(k+1)(3n-4k-6)$ edges for $k \geq 0$ and $n \geq 3k+3$.*

Proof. Figure 6 shows a bar k-visibility representation of a graph with n vertices and $(k+1)(3n-4k-6)$ edges. As in Figure 5, the left and right endpoints of bars in this representation are labeled with the number of left and right edges associated to each bar. Although $n = 4k+4$ in this representation, more bars can easily be deleted to create a representation with as few as $3k+3$ bars, or added to create a representation with arbitrarily many bars. □

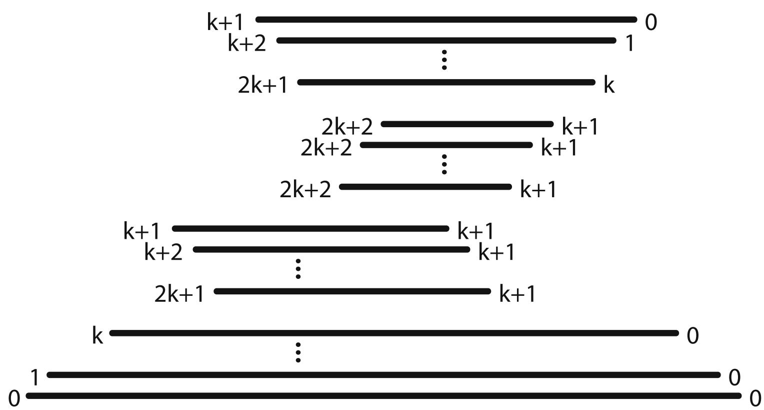

Fig. 6. A bar k-visibility graph with n vertices and $(k+1)(3n-4k-6)$ edges

Note that Theorem 8 gives the largest number of edges in a bar k-visibility graph for $k = 0, 1$. We believe that this is the case for larger k as well. We state this as a conjecture.

Conjecture 1. If G is a bar k-visibility graph, then G has at most $(k+1)(3n-4k-6)$ edges.

The following theorem is a corollary of Theorem 7.

Theorem 9. *K_{5k+5} is not a bar k-visibility graph.*

Proof. By way of contradiction, suppose that G is a graph with $n = 5k+5$ vertices. Then by Theorem 7, G has at most $(k+1)(3(5k+5) - \frac{7}{2}k - 5) = \frac{23}{2}k^2 + \frac{43}{2}k + 10$ edges. However, K_{5k+5} has $\binom{5k+5}{2} = \frac{25}{2}k^2 + \frac{45}{2}k + 10$ edges. □

Note that if Conjecture 1 is true, we immediately obtain the following conjecture as a corollary.

Conjecture 2. K_{4k+4} is the largest complete bar k-visibility graph.

Proof (Assuming Conjecture 1). Figure 6 shows a bar k-visibility representation of K_{4k+4}. Conversely, suppose that G is a graph with $n = 4k+5$ vertices. Then by Conjecture 1, G has at most $(k+1)(3(4k+5) - 4k - 6) = 8k^2 + 17k + 9$ edges. However, K_{4k+5} has $\binom{4k+5}{2} = 8k^2 + 18k + 10$ edges. □

Conjecture 1 is not required to prove Conjecture 2 when $k = 0$ or 1; we have already proved these cases in the previous section. Note also that the graph K_{4k+4} exactly achieves the bound given by Conjecture 1. So if this conjecture is correct, the family of complete graphs K_{4k+4} is an example of a family of edge-maximal bar k-visibility graphs.

4 Thickness of Bar *k*-Visibility Graphs

By Corollary 6, K_8 is a bar 1-visibility graph, and thus there are non-planar bar 1-visibility graphs. Motivated by the fact that all bar 0-visibility graphs are planar [10], we are interested in measuring the closeness to planarity of bar 1-visibility graphs. The *thickness* $\Theta(G)$ of a graph G is the minimum number of planar graphs whose union is G. K_8 has thickness 2 [12], so there exist bar 1-visibility graphs with thickness 2. Conversely, the following theorem from [4] gives an upper bound for the thickness of a bar 1-visibility graph.

Suppose G is a bar 1-visibility graph, and S is a bar 1-visibility representation of G. We define the *underlying bar visibility graph* G_0 of S to be the graph with bar visibility representation S. The following theorem relates the thickness of G to the chromatic number of G_0.

Theorem 10. *If G is a bar 1-visibility graph and G_0 an underlying bar visibility graph of G, then $\Theta(G) \leq \chi(G_0)$. In particular, the thickness of any bar 1-visibility graph is at most four.*

We conjecture that bar 1-visibility graphs have thickness no greater than 2. More generally, we know that the thickness of a bar k-visibility graph is bounded by some function of k [4]. The smallest such function of k is still open.

5 Future Work

We close with a list of open problems inspired by the results of this note.

1. What is the largest number of edges in a bar 2-visibility graph with n vertices?
2. What is the largest number of edges in a bar k-visibility graph with n vertices?

3. Are there bar 1-visibility graphs with thickness 3?
4. More generally, what is the largest thickness of a bar k-visibility graph? Is it $k+1$?
5. Are there bar 1-visibility graphs with chromatic number 9?
6. More generally, what is the largest chromatic number of a bar k-visibility graph?
7. What is the largest crossing number of a bar k-visibility graph?
8. What is the largest genus of a bar k-visibility graph?
9. What is a complete characterization of bar k-visibility graphs?
10. Is there an efficient recognition algorithm for bar k-visibility graphs?
11. *Rectangle visibility graphs* are defined in [7, 8, 15]. Generalize the results of this note to rectangle visibility graphs.
12. *Arc-* and *circle-visibility graphs* are defined in [11]. Generalize the results of this note to arc- and circle-visibility graphs.

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