No-bend Orthogonal Drawings of Series-Parallel Graphs
(Extended Abstract)

Md. Saidur Rahman[1], Noritsugu Egi[2], and Takao Nishizeki[2]

[1] Department of Computer Science and Engineering,
Bangladesh University of Engineering and Technology (BUET),
Dhaka 1000, Bangladesh
saidurrahman@cse.buet.ac.bd

[2] Graduate School of Information Sciences, Tohoku University, Aoba-yama 05,
Sendai 980-8579, Japan
egi@nishizeki.ecei.tohoku.ac.jp, nishi@ecei.tohoku.ac.jp

Abstract. In a no-bend orthogonal drawing of a plane graph, each vertex is drawn as a point and each edge is drawn as a single horizontal or vertical line segment. A planar graph is said to have a no-bend orthogonal drawing if at least one of its plane embeddings has a no-bend orthogonal drawing. Every series-parallel graph is planar. In this paper we give a linear-time algorithm to examine whether a series-parallel graph G of the maximum degree three has a no-bend orthogonal drawing and to find one if G has.

Keywords: Planar Graph, Algorithm, Graph Drawing, Orthogonal Drawing, Bend, SPQ tree.

1 Introduction

An *orthogonal drawing* of a planar graph G is a drawing of G such that each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end [NR04, RN02, RNN99, T87]. A *bend* is a point where an edge changes its direction in a drawing. If G has a vertex of degree five or more, then G has no orthogonal drawing. On the other hand, if G has no vertex of degree five or more, that is, the maximum degree Δ of G is at most four, then G has an orthogonal drawing, but may need bends. Minimization of the number of bends in an orthogonal drawing is a challenging problem. A *bend-minimum* orthogonal drawing of a planar graph G has the minimum number of bends among all possible planar orthogonal drawings of G. The problem of finding a bend-minimum orthogonal drawing is one of the most famous problems in the graph drawing literature [BEGKLM04] and has been studied both in the fixed embedding setting [RN02, RNN03, RNN99, T87] and in the variable embedding setting [DLV98, GT01]. Some plane graphs with fixed embeddings have an orthogonal drawing without bends, in which each edge is drawn by a

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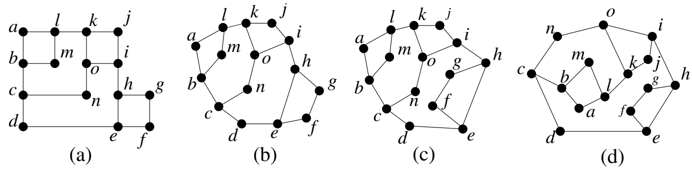

Fig. 1. (a) A no-bend drawing, and (b)–(d) three embeddings of the same planar graph

single horizontal or vertical line segment [RNN03]. We call such a drawing a *no-bend drawing* of a plane graph. Figure 1(a) depicts a no-bend drawing of the plane graph in Fig. 1(b). As a result in the fixed embedding, Rahman *et al.* [RNN03] obtained a necessary and sufficient condition for a plane graph G of $\Delta \leq 3$ to have a no-bend drawing, and gave a linear-time algorithm to find a no-bend drawing if G has.

We say that *a planar graph G has a no-bend drawing* if at least one of the plane embedding of G has a no-bend drawing. Figures 1(b), (c) and (d) depict three of all plane embeddings of the same planar graph G. Among them only the embedding in Fig. 1(b) has a no-bend drawing as illustrated in Fig. 1(a). Thus the *planar* graph G has a no-bend drawing. It is an NP-complete problem to examine whether a planar graph G of $\Delta \leq 4$ has a no-bend drawing in the variable embedding setting [GT01]. However, for a planar graph G of $\Delta \leq 3$, Di Battista *et al.* [DLV98] gave an $O(n^5 \log n)$ time algorithm to find a bend-minimum orthogonal drawing of G. Every series-parallel graph is a planar graph, and their algorithm takes time $O(n^3)$ for a series-parallel graph with $\Delta \leq 3$. Thus, by their algorithm one can examine in time $O(n^3)$ whether a series-parallel graph with $\Delta \leq 3$ has a no-bend drawing. As another result in the variable embedding, Rahman *et al.* [REN05] gave a linear time algorithm to examine whether a subdivision G of a planar triconnected cubic graph has a no-bend drawing, and to find a no-bend drawing of G if G has.

In this paper we study the problem of no-bend orthogonal drawings of series-parallel graphs with $\Delta \leq 3$ in the variable embedding setting, and give a linear algorithm to find a no-bend orthogonal drawing if G has.

The rest of the paper is organized as follows. Section 2 describes some definitions and presents preliminary results. Section 3 presents our algorithm to find a no-bend drawing of a biconnected series-parallel graph G if G has. Finally Section 4 is a conclusion.

2 Preliminaries

In this section we give some definitions and present preliminary results.

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. The *degree* $d(v)$ of a vertex v is the number of edges incident to v in G. We denote the maximum degree of graph G by $\Delta(G)$ or simply by Δ. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. We say that G is *k-connected* if $\kappa(G) \geq k$.

A graph $G = (V, E)$ is called a *series-parallel graph* (with source s and sink t) if either G consist of a pair of vertices connected by a single edge, or there exist two series-parallel graphs $G_i = (V_i, E_i), i = 1, 2$, with source s_i and sink t_i such that $V = V_1 \cup V_2, E = E_1 \cup E_2$, and either $s = s_1$, $t_1 = s_2$ and $t = t_2$ or $s = s_1 = s_2$ and $t = t_1 = t_2$.

A pair $\{u, v\}$ of vertices of a connected graph G is a *split pair* if there exist two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ satisfying the following two conditions: 1. $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{u, v\}$; and 2. $E = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, $|E_1| \geq 1$, $|E_2| \geq 1$. Thus every pair of adjacent vertices is a split pair. A *split component* of a split pair $\{u, v\}$ is either an edge (u, v) or a maximal connected subgraph H of G such that $\{u, v\}$ is not a split pair of H. A split pair $\{u, v\}$ of G is called a *maximal split pair* with respect to a *reference split pair* $\{s, t\}$ if, for any other split pair $\{u', v'\}$, vertices s, t, u and v are in the same split component of $\{u', v'\}$.

Let G be a biconnected series-parallel graph. Let (s, t) be an edge of G. The SPQ-tree $\mathcal{T}$ of G with respect to a *reference edge* $e = (s, t)$ describes a recursive decomposition of G induced by its split pairs [GL99]. Tree $\mathcal{T}$ is a rooted ordered tree whose nodes are of three types: S, P and Q. Each node x of $\mathcal{T}$ corresponds to a subgraph of G, called its *pertinent graph* G_x. Each node x of $\mathcal{T}$ has an associated biconnected multigraph, called the *skeleton* of x and denoted by *skeleton(x)*. Tree $\mathcal{T}$ is recursively defined as follows.

• *Trivial Case*: In this case, G consists of exactly two parallel edges e and e' joining s and t. $\mathcal{T}$ consists of a single Q-node x. The skeleton of x is G itself. The pertinent graph G_x consists of only the edge e'.

• *Parallel Case*: In this case, the split pair $\{s, t\}$ has three or more split components $G_0, G_1, \cdots, G_k, k \geq 2$, and G_0 consists of only a reference edge $e = (s, t)$. The root of $\mathcal{T}$ is a P-node x. The *skeleton*(x) consists of $k + 1$ parallel edges $e_0, e_1, \cdots, e_k$ joining s and t. The pertinent graph $G_x = G_1 \cup G_2 \cup \cdots \cup G_k$ is a union of $G_1, G_2, \cdots, G_k$. (The *skeleton* of P-node p_2 in Fig. 2 consists of three parallel edges joining vertices e and g. Figure 2(e) depicts the pertinent graph of p_2.)

• *Series Case*: In this case the split pair $\{s, t\}$ has exactly two split components, and one of them consists of the reference edge e. One may assume that the other split component has cut-vertices $c_1, c_2, \cdots, c_{k-1}$, $k \geq 2$, that partition the component into its blocks $G_1, G_2, \cdots, G_k$ in this order from s to t. Then the root of $\mathcal{T}$ is an S-node x. The skeleton of x is a cycle $e_0, e_1, \cdots, e_k$ where $e_0 = e$, $c_0 = s$, $c_k = t$, and e_i joins c_{i-1} and c_i, $1 \leq i \leq k$. The pertinent graph G_x of node x is a union of $G_1, G_2, \cdots, G_k$. (The *skeleton* of S-node s_2 in Fig. 2 is the cycle c, d, e, g, h, a, c. Figure 2(d) depicts the pertinent graph G_{s_2} of s_2.)

In all cases above, we call the edge e the *reference edge* of node x. Except for the trivial case, node x of $\mathcal{T}$ has children $x_1, x_2, \cdots, x_k$ in this order; x_i is the root of the SPQ-tree of graph $G_i \cup e_i$ with respect to the reference edge $e_i, 1 \leq i \leq k$. We call edge e_i *the reference edge of node* x_i, and call the endpoints of edge e_i the *poles* of node x_i. The tree obtained so far has a Q-node associated with each

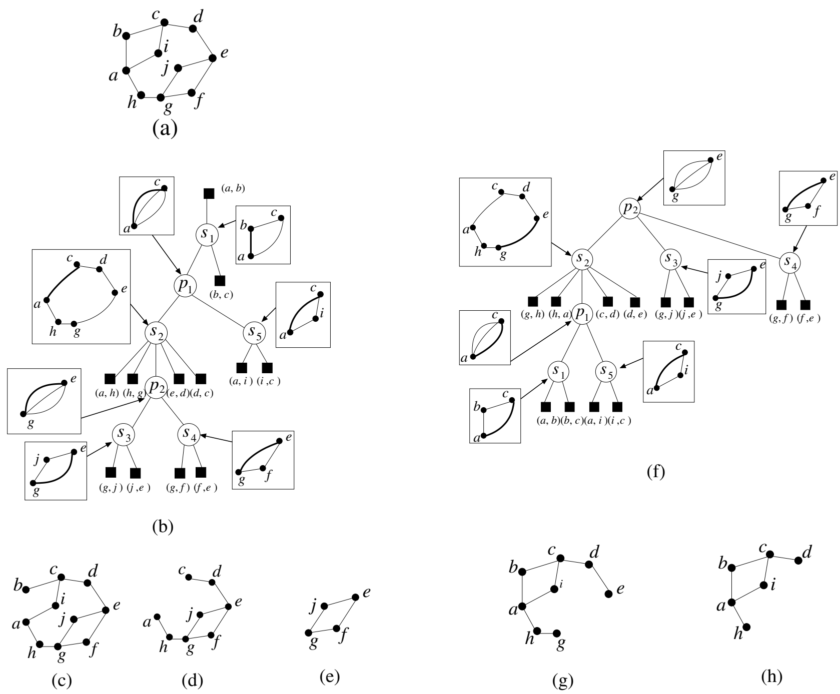

Fig. 2. (a)A biconnected series-parallel graph G with $\Delta = 3$, (b) SPQ-tree $\mathcal{T}$ of G with respect to reference edge (a, b), and skeletons of P- and S-nodes, (c) the pertinent graph G_{s_1} of S-node s_1, (d) the pertinent graph G_{s_2} of S-node s_2, (e) the pertinent graph G_{p_2} of P-node p_2, (f) SPQ-tree $\mathcal{T}$ of G with P-node p_2 as the root, (g) the pertinent graph of S-node s_2, and (h) the core graph of s_2

edge of G, except the reference edge e. We complete the SPQ-tree $\mathcal{T}$ by adding a Q-node, representing the reference edge e, and making it the parent of x so that it becomes the root of $\mathcal{T}$. An example of the SPQ-tree of a biconnected series-parallel graph in Fig. 2(a) is illustrated in Fig. 2(b), where the edge drawn by a thick line in each skeleton is the reference edge of the skeleton.

The SPQ-tree $\mathcal{T}$ defined above is a special case of an "SPQR-tree" [DT96, GL99] where there is no R-node and the root of the tree is a Q-node corresponding to the reference edge e. One can easily modify $\mathcal{T}$ to an SPQ-tree $\mathcal{T}'$ with an arbitrary P-node as the root as illustrated in Fig. 2(f).

In the remainder of this paper, we thus consider a SPQ-tree $\mathcal{T}$ with a P-node as the root. If $\Delta = 2$, then a biconnected series-parallel graph G is a cycle, and a cycle G has a no-bend drawing if and only if G has four or more vertices. One may thus assume that $\Delta \geq 3$, and that the root P-node of $\mathcal{T}$ has three or more children. Then the pertinent graph G_x of each node x is the subgraph of G induced by the edges corresponding to all descendant Q-node of x. The following facts can be easily derived from the fact that each vertex of G has degree at most three and G has no multiple edges.

Fact 1. *Let (s,t) be the reference edge of an S-node x of $\mathcal{T}$, and let $x_1, x_2, \cdots, x_k$ be the children of x in this order from s to t. Then (i) each child x_i of x is either a P-node or a Q-node; (ii) both x_1 and x_k are Q-nodes; and (iii) x_{i-1} and x_{i+1} must be Q-nodes if x_i is a P-node where $2 \leq i \leq k-1$.*

Fact 2. *Each non-root P-node of $\mathcal{T}$ has exactly two children, and either both of the two children are S-nodes or one of them is an S-node and the other is a Q-node.* ■

Let x be an S-node of $\mathcal{T}$, and let u and v be the poles of the pertinent graph of x. Let $x_1, x_2, \cdots, x_k$ be the children of x in this order from u to v. From Fact 1, x_1 and x_k are Q-nodes. Thus x_1 and x_k correspond to edges (u, u') and (v', v) of G, respectively. Then the *core graph* for x is a graph obtained from the pertinent graph of x by deleting vertices u and v. (Figure 2(g) illustrates a pertinent graph of S-node s_1 for $\mathcal{T}$ in Fig. 2(f), and Fig. 2(h) illusrates a core graph for s_1.) Vertices u' and v' are called the *poles* of the core graph for x, and edges (u, u') and (v', v) are called *hands* of the core graph for x. (In Figs. 2(g) and (h) the poles of the core graph of S-node s_1 are vertices d and h.) For a P- or Q-node x in $\mathcal{T}$, we define the *core graph* for x as the pertinent graph of x, and the poles of the core graph for x is the same as the poles of the pertinent graph of x. The core graph of a P- or Q-node has no hand.

A drawing of a planar graph G is called an *orthogonal drawing* of G if each vertex is mapped to a point, each edge is drawn as a sequence of alternate horizontal and vertical line segments, and any two edges do not cross except at their common end. We call an orthogonal drawing D of G a *no-bend drawing* if D has no bend, that is, each edge is drawn as a single horizontal or vertical line segment. A *polar drawing* of a series-parallel graph G is a no-bend drawing of G in which the two poles u and v of G are drawn on the outer face F_o of the drawing.

We call a polar drawing D of a series-parallel graph G a *diagonal drawing* if D intersects neither the first quadrant with the origin at pole u nor the third quadrant with the origin at pole v after rotating the drawing and renaming the poles if necessary, as illustrated in Fig. 3(a). Throughout the paper a quadrant is considered to be a closed plane region. Both a drawing of a single vertex as a point and a drawing of a single edge as a straight line-segment are diagonal drawings.

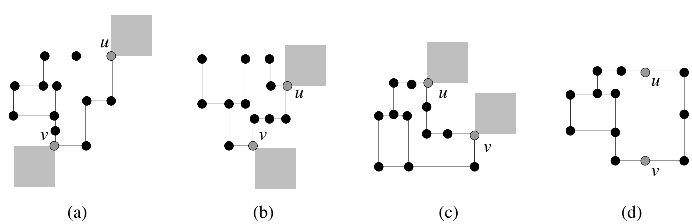

Fig. 3. Polar drawings of a graph G with poles u and v: (a) a diagonal drawing, (b) a side-on drawing, (c) an L-shape drawing, (d) another polar drawing

We call a polar drawing D of G a *side-on drawing* if D intersects neither the first quadrant with the origin at u nor the fourth quadrant with the origin at v after rotating the drawing and renaming the poles if necessary, as illustrated in Fig. 3(b). A drawing of a single vertex as a point is regarded not to be a side-on drawing, while a drawing of a single edge as a straight line-segment is a side-on drawing.

A polar drawing D is called an *L-shape drawing* if D intersects neither the first quadrant with the origin at u nor the first quadrant with the origin at v after rotating the drawing and renaming the poles if necessary, as illustrated in Fig. 3(c). A drawing of a single vertex as a point is regarded not to be an L-shape drawing. A drawing of a single edge as a straight line-segment is not an L-shape drawing.

We say that a polar drawing is *good* if it is a diagonal, side-on or L-shape drawing. Not every polar drawing D is good. For example, the polar drawing in Fig. 3(d) is not good, because it is not a diagonal, side-on drawing or L-shape drawing.

In the next section we give an algorithm for constructing no-bend drawing of a biconnected series-parallel graph G with $\Delta = 3$.

Our idea is as follows. Let $\mathcal{T}$ be an SPQ-tree of G. The core graph of each leaf-node of $\mathcal{T}$ consists of a single edge. For each leaf-node of $\mathcal{T}$ we first draw the core graph by a line segment as a diagonal or side-on drawing. Then, in bottom up fashion, we find a diagonal drawing, a side-on drawing, and an L-shape drawing of the core graph for each internal node x of $\mathcal{T}$ by merging the drawings corresponding to the children of x if they exist. The drawing of the graph corresponding to the root-node of $\mathcal{T}$ yields a no-bend drawing of G if G has a polar drawing with the split pair, corresponding to the root P-node, as the poles. Our algorithm eventually chooses an appropriate SPQ-tree $\mathcal{T}$ of G such that the drawing of a plane graph corresponding to the root-node of $\mathcal{T}$ yields a no-bend drawing of G if G has. (See Fig. 8 for illustration.)

As we see later, we construct a no-bend drawing of the core graph for a node x in $\mathcal{T}$ by merging the no-bend drawings of the core graphs for the childrens of x; the no-bend drawing of the core graph for each children of x must be a polar drawing with the two poles of the core graph. A side-on drawing is found more suitable for merging than a diagonal drawing, and an L-shape drawing is found more suitable for merging than a side-on drawing. Intuitively, to connect the two poles by a sequence of horizontal and vertical line segments, at least three turns are required for a diagonal drawing, at least two turns are required for a side-on drawing and only one turn is required for an L-shape drawing. A graph may have a diagonal drawing although it has no side-on or L-shape drawing and a graph may have a side-on drawing although it has no L-shape drawing. We call a polar drawing D of a core graph $H(x)$ for a node x in $\mathcal{T}$ a *desirable* drawing if one of the following (a), (b) and (c) holds: (a) D is an L-shape drawing; (b) D is a side-on drawing, and $H(x)$ has no L-shape drawing; (c) D is a diagonal drawing, and $H(x)$ has neither an L-shape drawing nor a side-on drawing. Throughout the paper we denote by $D(x)$ a *desirable* drawing of the core graph $H(x)$ for a node x in $\mathcal{T}$.

3 No-bend Drawings of Biconnected Series-Parallel Graphs

In this section we give an algorithm to construct a no-bend orthogonal drawing of a biconnected series-parallel graph G whenever G has.

If G is a cycle, then it is easy to find a no-bend drawing of G; G has a no-bend drawing if and only if G has four or more vertices. We thus assume that G is not a cycle.

Let $\mathcal{T}$ be an SPQ-tree of G whose root is a P-node x_p having three children. (See Fig. 2(f).) We now have the following lemma.

Lemma 3. *Let G be a series-parallel graph with $\Delta \leq 3$, let $\mathcal{T}$ be an SPQ-tree with a P-node x_p as the root, and let x be a non-root node in $\mathcal{T}$. If the core graph $H(x)$ of x has a no-bend drawing, then the following (a) and (b) hold: (a) $H(x)$ has a side-on or diagonal drawing, and hence $H(x)$ has a desirable drawing $D(x)$; and (b) if a desirable drawing of $H(x)$ is a diagonal drawing, then every no-bend drawing of $H(x)$ is a diagonal drawing for the poles of $H(x)$.*

Proof. We will prove the claim by induction based on $\mathcal{T}$.

We first assume that x is a leaf-node, that is, a Q-node. In this case $H(x)$ consists of a single edge $e = (u, v)$, and u and v are the poles of $H(x)$. We thus draw e as a single vertical line segment, which is a side-on drawing $D(x)$ of $H(x)$. Since $H(x)$ has no L-shape drawing, $D(x)$ is a desirable drawing. Thus (a) and (b) hold.

We next assume that x is an inner node other than the root x_p and that $H(x)$ has a no-bend drawing. Let u and v are the poles of $H(x)$. Let $x_1, x_2, \cdots, x_k$ ($k \geq 2$) be the children of x in this order from u to v. Since $H(x)$ has a no-bend drawing, each $H(x_i)$ has a no-bend drawing. Thus we suppose inductively that (a) and (b) hold for each child of x. We now have two cases to consider.

Case 1: x is an S-node.

Suppose that x has exactly two children. Then $H(x)$ consists of a single vertex. We draw $H(x)$ as a point. Then the diagonal drawing is a desirable drawing $D(x)$. Thus (a) and (b) hold.

We thus assume that x has exactly k children and $k \geq 3$. Then $H(x) = H(x_2) \cup H(x_3) \cup \cdots \cup H(x_{k-1})$, where $H(x_i)$ is the core graph of x_i. The hypothesis implies that, for each i, $2 \leq i \leq k-1$, (a) and (b) hold for the core graph $H(x_i)$. We now have the following four subcases to consider.

Case 1(a): $k = 3$.

In this case $H(x) = H(x_2)$, hence (a) and (b) hold for $H(x)$.

Case 1(b): $k = 4$.

In this case $H(x) = H(x_2) \cup H(x_3)$. Fact 1(iii) implies that either both x_2 and x_3 are Q-nodes or one of them is a P-node and the other one is a Q-node.

If x_2 and x_3 are Q-nodes, then we can construct both an L-shape drawing and a side-on drawing of $H(x)$, as illustrated in Figs. 4(a) and 5(a). Thus a desirable drawing of $H(x)$ is an L-shape drawing, and hence (a) and (b) hold. We thus assume that one of them, say x_2, is a P-node and the other is a Q-node.

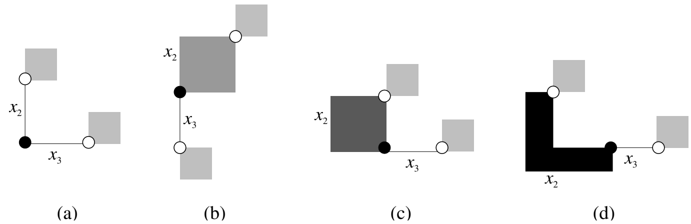

Fig. 4. Desirable drawings of the core graph for S-nodes with four children

We first consider the case where a desirable drawing $D(x_2)$ of $H(x_2)$ is a diagonal drawing. In this case we can construct a side-on drawing $D(x)$ of $H(x)$ as illustrated in Fig. 4(b). Since the desirable drawing of $H(x_2)$ is a diagonal drawing, $H(x_2)$ has neither an L-shape drawing nor a side-on drawing, and hence clearly $H(x)$ has no L-shape drawing. Therefore the side-on drawing $D(x)$ of $H(x)$ is a desirable drawing. Hence (a) and (b) hold.

We next consider the case where the desirable drawing $D(x_2)$ of $H(x_2)$ is a side-on drawing. Then we can construct both an L-shape drawing $D(x)$ and a side-on drawing of $H(x)$ as illustrated in Figs. 4(c) and 5(c). Hence (a) and (b) hold.

We finally consider the case where the desirable drawing $D(x_2)$ of $H(x_2)$ is an L-shape drawing. Then we can construct an L-shape drawing $D(x)$ of $H(x)$ as illustrated in Fig. 4(d). $H(x_2)$ has a side-on or diagonal drawing. From it one can easily construct a side-on drawing of $H(x)$ as illustrated in Figs. 5(b) and (c). Therefore (a) and (b) hold.

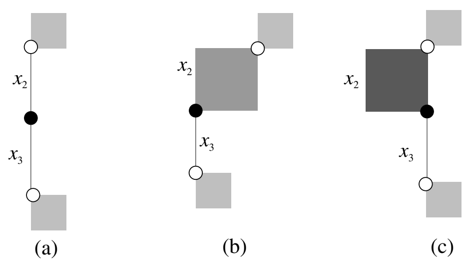

Fig. 5. Side-on drawings of the core graph for S-nodes with four children

Case 1(c): $k = 5$.

In this case, $H = H(x_2) \cup H(x_3) \cup H(x_4)$. Fact 1(iii) implies that at least one of x_2, x_3 and x_4 is a Q-node. In this case we can construct a no-bend drawing of $H(x)$ such that (a) and (b) hold. The details are omitted in this extended abstract.

Case 1(d): $k \geq 6$.

In this case $H = H(x_2) \cup H(x_3) \cup \cdots \cup H(x_{k-1}), k \geq 6$. Fact 1(iii) implies that there are two or more Q-nodes among $x_2, x_3, \cdots x_{k-1}$. Therefore we can easily construct both an L-shape drawing and a side-on drawing D of $H(x)$, and hence (a) and (b) hold.

Case 2: x is a P-node.

In this case $k = 2$ and x has exactly two children x_1 and x_2. Then the hypothesis implies that, for $i = 1, 2$, (a) and (b) hold for $H(x_i)$. By Fact 2 either both x_1 and x_2 are S-nodes or one of x_1 and x_2 is an S-node and the other is a Q-node. We first assume that one of x_1 and x_2, say x_1, is a Q-node, then we have the following two subcases.

Case 2(a): The desirable drawing $D(x_2)$ of $H(x_2)$ is a diagonal drawing.

In this case $H(x_2)$ has neither an L-shape drawing nor a side-on drawing. Furthermore, every no-bend drawing of $H(x_2)$ is a diagonal drawing by induction hypothesis. Then $D(x_1), D(x_2)$ and the drawings of hands of $H(x_2)$ cannot be merged without bends as illustrated in Fig. 6(a). Therefore $H(x)$ does not have a no-bend drawing, contrary to the assumption that $H(x)$ has a no-bend drawing. Therefore this case does not occur.

Case 2(b): The desirable drawing $D(x_2)$ of $H(x_2)$ is a side-on or L-shape drawing.

In this case we can construct a no-bend drawing $D(x)$ of $H(x)$ such that (a) and (b) hold as illustrated in Figs. 6(b)–(i). $\mathcal{Q.E.D.}$

We call the algorithm described in the proof of Lemma 3 for finding a desirable drawing $D(x)$ of $H(x)$ Algorithm **Desirable-Drawing** whenever $H(x)$ has a no-bend drawing. Clearly Algorithm **Desirable-Drawing** takes linear-time.

In the rest of the section we give Algorithm **Biconnected-Draw** for finding a no-bend drawing of G whenever G has. Remember that the root node x_p in $\mathcal{T}$

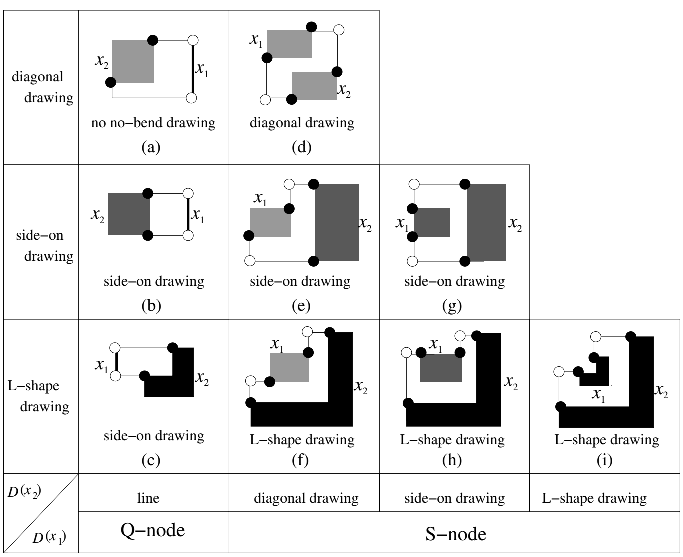

Fig. 6. Drawings of $H(x)$ for a P-node $x \neq x_p$

has three children as depicted in Fig. 2(f). Let x_1, x_2 and x_3 be the three children of x_p in $\mathcal{T}$. If G has a no-bend drawing, then $H(x_i), 1 \leq i \leq 3$, has a no-bend drawing. For $1 \leq i \leq 3$, we find a desirable drawing $D(x_i)$ of $H(x_i)$ by Algorithm **Desirable-Drawing**. If G has a polar drawing for the poles corresponding to x_p, then we now find a no-bend drawing of $G = H(x_p)$ by merging the drawings of $D(x_1), D(x_2), D(x_3)$ and the drawings of their hands. Otherwise, we find appropriate poles for which G has a no-bend polar drawing. Since G is a simple graph, at most one of x_1, x_2 and x_3 is a Q-node. We now have the following two cases to consider.

Case 1: one of them, say x_3, is a Q-node.

In this case only x_3 is a Q-node. If at least one of $D(x_1)$ and $D(x_2)$ is a diagonal drawing, Then G does not have a no-bend drawing as illustrated in Fig. 7(a)-(c). Otherwise, G has a no-bend drawing as illustrated in Fig. 7(d)-(f). The details are omitted.

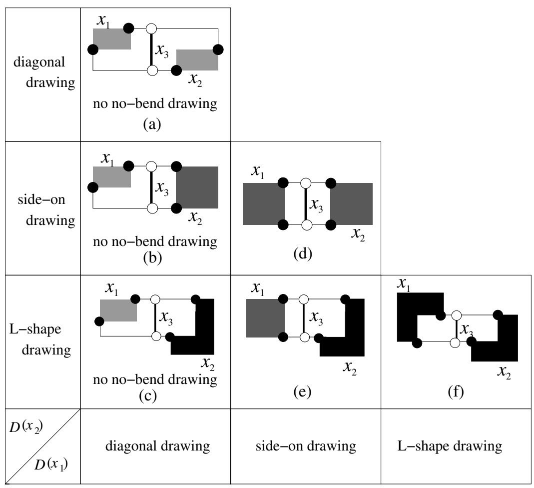

Fig. 7. Illustration for Case 1 of Algorithm **Biconnected-Draw**

Case 2: all of x_1, x_2 and x_3 are S-nodes.

If at most one of $D(x_1), D(x_2)$ and $D(x_3)$ is a diagonal drawing, then we can easily construct a no-bend drawing of G. If all of $D(x_1), D(x_2)$ and $D(x_3)$ are diagonal drawings, then one can easily observe that G does not have a no-bend drawing.

We thus consider the case where exactly two of $D(x_1), D(x_2)$ and $D(x_3)$ are diagonal drawings. If two of $D(x_1), D(x_2)$ and $D(x_3)$ are diagonal drawings and the other is an L-shape drawing, then clearly we can construct a no-bend drawing of G. We may thus assume that two of $D(x_1), D(x_2)$ and $D(x_3)$ are diagonal drawings and the other is a side-on drawing.

We may assume without loss of generality that $D(x_1)$ and $D(x_2)$ are diagonal drawings and $D(x_3)$ is a side-on drawing. By Lemma 3(b) every no-bend drawing of each of $H(x_1)$ and $H(x_2)$ is a diagonal drawing. By merging $D(x_1)$ and $D(x_2)$

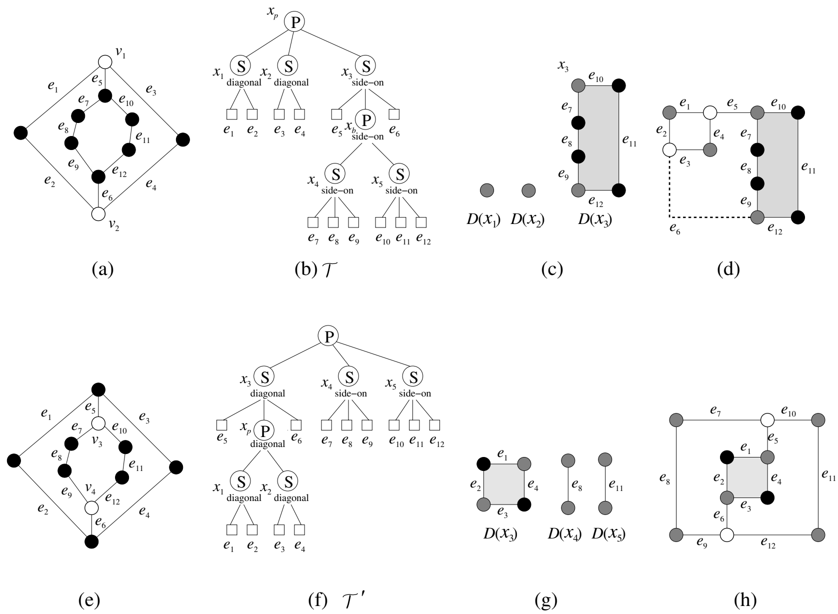

Fig. 8. (a)—(d) A no-bend drawing of G cannot be found using tree $\mathcal{T}$, and (e)—(h) a no-bend drawing of G can be found using tree $\mathcal{T}'$

we can obtain only a diagonal drawing D'. Since $D(x_3)$ is a side-on drawing, D' and $D(x_3)$ cannot be merged to produce a no-bend drawing of G. However, we can construct a no-bend drawing of G if $H(x_3)$ has another appropriate no-bend drawing.

We give an illustrative example in Figure 8 and omit the details of the proof. G has no polar drawing with the poles corresponding to x_p as illustrated in Fig. 8(d). However, G may have a no-bend drawing when one considers some other split pair as poles. We therefore consider an SPQ-tree $\mathcal{T}'$ of G with x_b as the root, as illustrated in Fig. 8(f), where x_3, x_4 and x_5 are the children of x_b. Each of $D(x_4)$ and $D(x_5)$ remains same as one obtained for the SPQ-tree $\mathcal{T}$. Considering $\mathcal{T}'$, $D(x_3)$ is a diagonal drawing D'. We can thus find a no-bend drawing of G by recursively applying Algorithm **Biconnected-Draw** regarding $D(x_3), D(x_4)$ and $D(x_5)$ as new $D(x_1), D(x_2)$ and $D(x_3)$, respectively. (Figure 8(h) shows that G has a no-bend polar drawing with the poles corresponding to root x_b.) If we cannot draw a no-bend orthogonal drawing of G by repeating the operation above, then G does not have a no-bend drawing.

Thus Algorithm **Biconnected-Draw** finds a no-bend drawing of G if G has. One can efficiently implement Algorithm **Biconnected-Draw** so that it takes time $O(n)$. The details are omitted in this extended abstract.

Theorem 1. *Let G be a biconnected series-parallel graph of the maximum degree three. Then Algorithm* **Biconnected-Draw** *finds a no-bend drawing of G in time $O(n)$ whenever G has, where n is the number of vertices of G.*

4 Conclusions

In this paper, we gave a linear-time algorithm to find a no-bend drawing of a biconnected series-parallel graph G of maximum degree at most three. We also gave an algorithm to find a no-bend drawing of a series-parallel graph G which is not always biconnected. However, the algorithm is omitted in this extended abstract due to page limitation. It is left as a future work to find a bend-minimum drawing of series-parallel graphs and to find a linear-time algorithm for a larger class of planar graphs.

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