Convex Drawings of Plane Graphs
of Minimum Outer Apices

Kazuyuki Miura[1], Machiko Azuma[2], and Takao Nishizeki[2]

[1] Faculty of Symbiotic Systems Science,
Fukushima University, Fukushima 960-1296, Japan
miura@sss.fukushima-u.ac.jp
[2] Graduate School of Information Sciences,
Tohoku University, Sendai 980-8579, Japan
azuma@nishizeki.ecei.tohoku.ac.jp, nishi@ecei.tohoku.ac.jp

Abstract. In a convex drawing of a plane graph G, every facial cycle of G is drawn as a convex polygon. A polygon for the outer facial cycle is called an outer convex polygon. A necessary and sufficient condition for a plane graph G to have a convex drawing is known. However, it has not been known how many apices of an outer convex polygon are necessary for G to have a convex drawing. In this paper, we show that the minimum number of apices of an outer convex polygon necessary for G to have a convex drawing is, in effect, equal to the number of leaves in a triconnected component decomposition tree of a new graph constructed from G, and that a convex drawing of G having the minimum number of apices can be found in linear time.

1 Introduction

Recently automatic aesthetic drawing of graphs has created intense interest due to their broad applications, and as a consequence, a number of drawing methods have come out [1, 2, 3, 4, 8]. The most typical drawing of a plane graph G is a *straight line drawing* in which all vertices of G are drawn as points and all edges are drawn as straight line segments without any edge-intersection. A straight line drawing of G is called a *convex drawing* if every facial cycle is drawn as a convex polygon, as illustrated in Fig. 1 [1, 9].

In a convex drawing of a plane graph G, the outer facial cycle $F_o(G)$ of G must be drawn as a convex polygon. A polygonal drawing F_o^* of $F_o(G)$, called an *outer convex polygon*, plays a crucial role in finding a convex drawing of G. The plane graph G in Fig. 2(a) admits a convex drawing if an outer convex polygon F_o^* has four or more apices as illustrated in Fig. 2(b), where apices are drawn as white circles. However, if F_o^* has only three apices, that is, F_o^* is a triangle, then G does not admit a convex drawing as illustrated in Fig. 2(c).

A necessary and sufficient condition for a plane graph G to have a convex drawing is known [1, 9]. A linear-time algorithm is also known for finding a convex drawing of G if G satisfies the condition [1, 7]. We recently give a necessary and sufficient condition for a plane graph G to have a convex drawing such that
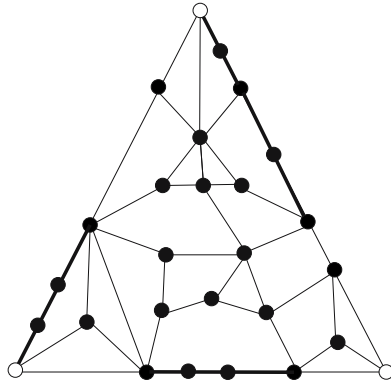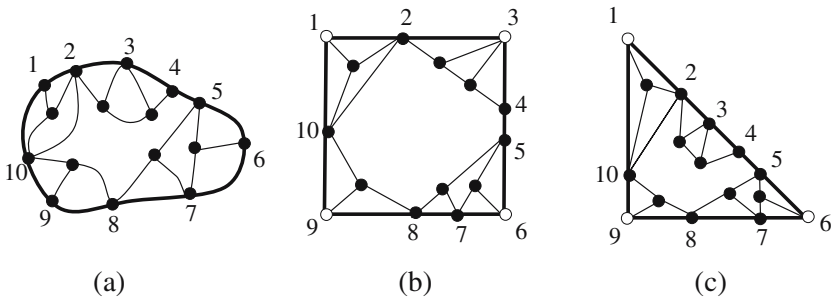
Fig. 1. Convex drawing of a plane graph

Fig. 2. (a) Plane graph, (b) convex drawing, and (c) non-convex drawings

an outer convex polygon F_o^* has exactly three apices [6]. However, it is not known how many apices of F_o^* are necessary for a plane graph G to have a convex drawing.

In this paper, we show that the minimum number of apices of an outer convex polygon F_o^* necessary for G to have a convex drawing is, in effect, equal to the number of leaves in a triconnected component decomposition tree of a new graph constructed from G, and that a convex drawing of G having the minimum number of apices of F_o^* can be found in linear time.

The remainder of the paper is organized as follows. In Section 2 we give some definitions and two known lemmas. In Section 3 we present our results. Finally we conclude in Section 4.

2   Preliminaries

In this section, we give some definitions and two known lemmas.

We denote by $G = (V, E)$ an undirected connected simple graph with vertex set V and edge set E. An edge joining vertices u and v is denoted by (u, v). The

degree of a vertex v in G is the number of neighbors of v in G, and is denoted by $d(v)$.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph G divides the plane into connected regions, called *faces*. We denote by $F_o(G)$ the outer face of G. The boundary of $F_o(G)$ is also denoted by $F_o(G)$. A vertex on $F_o(G)$ is called an *outer vertex*, while a vertex not on $F_o(G)$ is called an *inner vertex*. An edge on $F_o(G)$ is called an *outer edge*, while an edge not on $F_o(G)$ is called an *inner edge*.

A polygonal drawing F_o^* of $F_o(G)$ is called an *outer convex polygon* if F_o^* is a convex polygon. A (geometric) vertex of a polygon F_o^* is called an *outer apex*. An outer convex polygon F_o^* is *extendable* if G has a convex drawing in which $F_o(G)$ is drawn as the convex polygon F_o^*. For example, the outer rectangle drawn by thick lines in Fig. 2(b) is extendable, while the outer triangle in Fig. 2(c) is not extendable. We denote by $G - F_o(G)$ the graph obtained from G by deleting all outer vertices.

We call a vertex v of a connected graph G a *cut vertex* if its removal from G results in a disconnected graph. A connected graph G is *biconnected* if G has no cut vertex. If G has a convex drawing, then G is biconnected.

The following necessary and sufficient condition for a plane graph G to have a convex drawing is known.

Lemma 1. [1, 9] Let G be a biconnected plane graph, and let F_o^* be an outer convex polygon of G. Assume that F_o^* is a k-gon, $k \geq 3$, and that $P_1, P_2, \cdots, P_k$ are the k paths in $F_o(G)$, each corresponding to a side of the polygon F_o^*. Then F_o^* is extendable if and only if the following Conditions (a)–(c) hold.

(a) For each inner vertex v with $d(v) \geq 3$, there exist three paths disjoint except v, each joining v and an outer vertex;

(b) The graph $G - F_o(G)$ has no connected component H such that all the outer vertices adjacent to vertices in H lie on a single path P_i, and no two outer vertices in each path P_i are joined by an inner edge (see Fig. 3); and

(c) Every cycle containing no outer edge has at least three vertices of degree ≥ 3.

We call a pair $\{u, v\}$ of vertices in a biconnected graph G a *separation pair* if its removal from G results in a disconnected graph, that is, $G - \{u, v\}$ is not connected. A biconnected graph G is *triconnected* if G has no separation pair. A plane biconnected graph G is *internally triconnected* if, for any separation pair $\{u, v\}$ of G, both u and v are outer vertices and each connected component of $G - \{u, v\}$ contains an outer vertex. In other words, G is internally triconnected if and only if it can be extended to a triconnected graph by adding a vertex in an outer face and joining it to all outer vertices. If a biconnected plane graph G is not internally triconnected, then G has a separation pair $\{u, v\}$ illustrated in Figs. 4(a)–(c) and a "split graph" H contains an inner vertex other than u and v.
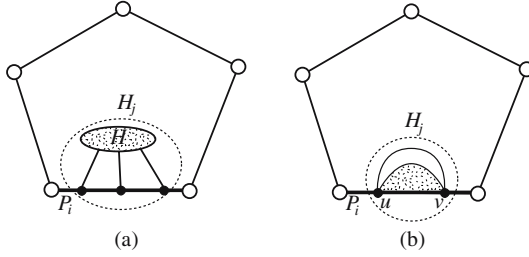
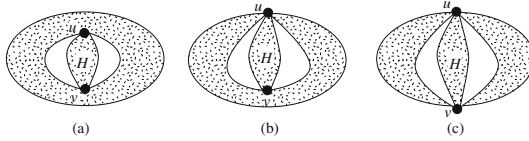**Fig. 3.** Examples violating Condition (b)

Fig. 4. Biconnected plane graphs which are not internally triconnected

Let $G = (V, E)$ be a biconnected graph, and let $\{u, v\}$ be a separation pair of G. Then, G has two subgraphs $G_1' = (V_1, E_1')$ and $G_2' = (V_2, E_2')$ such that

(a) $V = V_1 \bigcup V_2$, $V_1 \bigcap V_2 = \{u, v\}$; and
(b) $E = E_1' \bigcup E_2'$, $E_1' \bigcap E_2' = \emptyset$, $|E_1'| \geq 2$, $|E_2'| \geq 2$.

The graph G in Fig. 5(a) has six separation pairs $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$, $\{u_2, u_7\}$, $\{u_3, u_6\}$, and $\{u_4, u_5\}$.

For a separation pair $\{u, v\}$ of G, $G_1 = (V_1, E_1' + (u, v))$ and $G_2 = (V_2, E_2' + (u, v))$ are called the *split graphs* of G with respect to $\{u, v\}$. The new edges (u, v) added to G_1 and G_2 are called the *virtual edges*. Even if G has no multiple edges, G_1 and G_2 may have. Dividing a graph G into two split graphs G_1 and G_2 are called *splitting*. Reassembling the two split graphs G_1 and G_2 into G is called *merging*. Merging is the inverse of splitting. Suppose that a graph G is split, the split graphs are split, and so on, until no more splits are possible, as illustrated in Fig. 5(b) where virtual edges are drawn by dotted lines. The graphs constructed in this way are called the *split components* of G. The split components are of three types: triple bonds (i.e. a set of three multiple edges), triangles, and triconnected graphs. The *triconnected components* of G are obtained from the split components of G by merging triple bonds into a bond and triangles into a ring, as far as possible, where a *bond* is a set of multiple edges and a *ring* is a cycle. The graph in Fig. 5(a) is decomposed into seven triconnected components $H_1, H_2, \cdots, H_7$ as depicted in Fig. 5(c), where H_1, H_2 and H_6 are triconnected graphs, H_3, H_4 and H_7 are rings, and H_5 is a bond. The split components of G are not necessarily unique, but the triconnected components of G are unique [5].

Let T be a tree in which each node corresponds to a triconnected component H_i and T has an edge (H_i, H_j), $i \neq j$, if and only if H_i and H_j are triconnected components with respect to the same separation pair, as illustrated in Fig. 5(d).
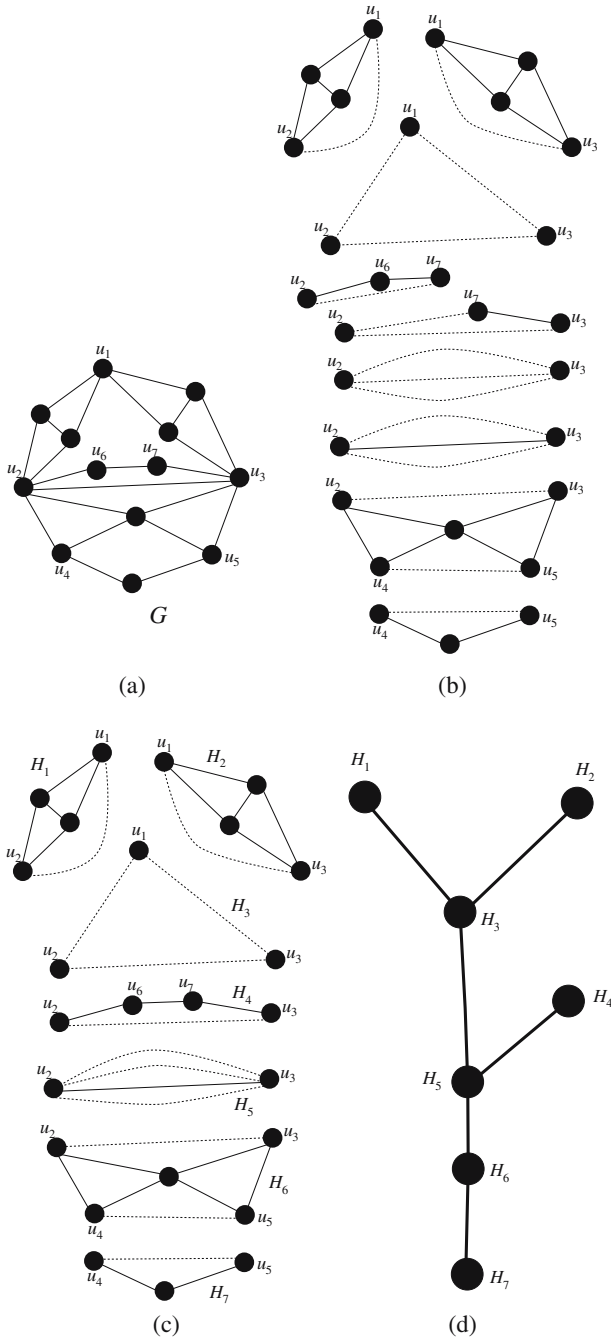
Fig. 5. (a) A biconnected graph G, (b) split components, (c) triconnected components, and (d) a triconnected component decomposition tree T of G

We call T a *triconnected component decomposition tree* of G [5]. Clearly every leaf of T does not correspond to a bond, but corresponds to a triconnected graph or a ring.

The following lemma is known.

Lemma 2. [5] A triconnected component decomposition tree T of a graph G can be found in linear time.

3 Convex Drawing of Minimum Outer Apices

One may obviously assume that a plane graph G is biconnected. One may further assume that every inner vertex of G has degree three or more; if an inner vertex v has degree two in G, then delete v from G, join the neighbors of v, and let G' be the resulting graph; clearly G has a convex drawing if and only if G' has no multiple edges and has a convex drawing. We can then newly formalize a necessary and sufficient condition for G to have a convex drawing, as follows.

Theorem 1. Let G be a plane biconnected graph in which every inner vertex has degree three or more, and let T be a triconnected component decomposition tree of G. Then the following (a)–(c) are equivalent with each other:

(a) G has a convex drawing;
(b) G is internally triconnected; and
(c) both of every separation pair are outer vertices, and every node of T corresponding to a bond has degree two in T.

Proof. We verify (a)⇔(b) and (b)⇔(c), as follows.

(a)⇒(b): Assume that G has a convex drawing. Then G has an extendable outer convex polygon F_o^*, and hence Conditions (a)–(c) in Lemma 1 hold for F_o^*. Suppose for a contradiction that G is not internally triconnected. Then G has a separation pair $\{u, v\}$ illustrated in Figs. 4(a)–(c). In a split graph H with respect to $\{u, v\}$, there is an inner vertex x other than u and v. Vertex x has degree three or more, and hence every path joining x and an outer vertex must pass through either u or v. Therefore there are no three paths disjoint except x, each joining x and an outer vertex. Thus Condition (a) does not hold, a contradiction.

(b)⇒(a): Assume that G is internally triconnected. Let F_o^* be an outer convex polygon in which every outer vertex of G is an apex of F_o^*. We show that F_o^* is extendable and hence G has a convex drawing. Since G is internally triconnected, clearly both Conditions (a) and (c) in Lemma 1 hold. Thus we shall show that Condition (b) holds for F_o^*. Each path P_i in F_o, corresponding to a side of F_o^*, consists of a single edge, and G is a simple graph. Therefore the two outer vertices in P_i are not joined by an inner edge. Since G is internally triconnected, one can easily know that $G - F_o(G)$ has no connected component H such that all the outer vertices adjacent to vertices in H lie on a single path P_i. Thus G satisfies Conditions (a)–(c), and hence by Lemma 1 F_o^* is extendable.
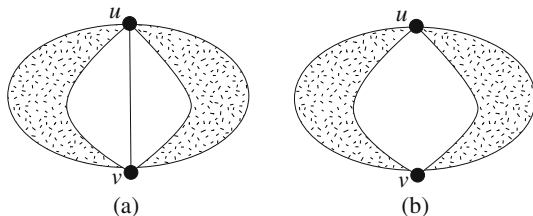
(a)   (b)

Fig. 6. Plane graphs having separation pairs corresponding to bonds

(b)$\Rightarrow$(c): Assume that G is internally triconnected. Then both of every separation pair of G are outer vertices, and G does not have a separation pair $\{u, v\}$ illustrated in Figs. 4(a)–(c). Therefore, for every separation pair $\{u, v\}$, $G - \{u, v\}$ has exactly two connected components, each containing an outer vertex, as illustrated in Figs. 6(a) and (b); G has an edge (u, v) in Fig. 6(a), while G has no edge (u, v) in Fig. 6(b). Hence one can easily observe that every node of T corresponding to a bond has degree two in T; the bond for Fig. 6(a) is a set of three multiple edges, two of which are virtual ones, and one of which is a real one; the bond for Fig. 6(b) is a pair of virtual multiple edges.

(c)$\Rightarrow$(b): Assume that both of every separation pair are outer vertices, and every node of T corresponding to a bond has degree two in T. We shall show that G does not have a separation pair $\{u, v\}$ illustrated in Figs. 4(a)–(c). Since both of every separation pair are outer vertices, G does not have a separation pair $\{u, v\}$ illustrated in Figs. 4(a) and (b). Assume for a contradiction that G has a separation pair $\{u, v\}$ illustrated in Fig. 4(c). Then a node of T corresponding to a bond containing virtual edges (u, v) has degree three or more, a contradiction. □

By Theorem 1 we may assume that G is internally triconnected and each inner vertex of G has degree three or more, as the graph in Fig. 1. Thus every vertex of degree two must be an outer vertex. Let $P = v_0, v_1, v_2, \cdots, v_{l+1}, l \geq 1$, be a path on $F_o(G)$ such that $d(v_0) \geq 3$, $d(v_1) = d(v_2) = \cdots = d(v_l) = 2$ and $d(v_{l+1}) \geq 3$. Such a path P is called an *outer chain* of G. (The graph in Fig. 1 has four outer chains drawn by thick lines.) We then have the following lemma.

Lemma 3. Let G be an internally triconnected plane graph, and let $P = v_0, v_1, \cdots, v_{l+1}$ be an outer chain of G. Then the following Propositions (a) and (b) hold:

(a) If G has an edge (v_0, v_{l+1}), then G has a convex drawing in which exactly one of the vertices $v_1, v_2, \cdots, v_l$ is an outer apex and G has no convex drawing in which none of them is an outer apex.

(b) If G has no edge (v_0, v_{l+1}), then G has a convex drawing in which none of the vertices $v_1, v_2, \cdots, v_l$ is an outer apex. (See Fig. 7.)

Proof. (a) Suppose that G has an edge (v_0, v_{l+1}). Since G is internally triconnected, by Theorem 1 G has a convex drawing. Let F_o^* be the outer convex
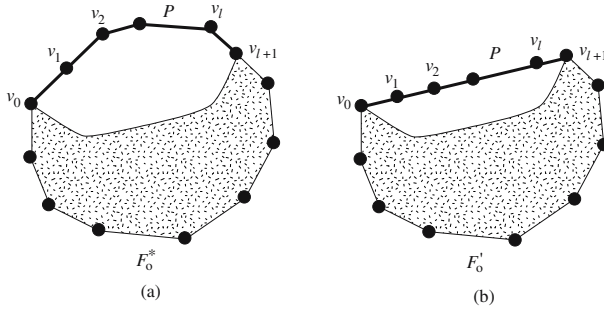
Fig. 7. Outer convex polygons F_o^* and F_o'

polygon of an arbitrary convex drawing of G. Clearly one or more of the vertices $v_1, v_2, \cdots, v_l$ are apices of F_o^*. One can easily modify the drawing to a convex drawing in which exactly one of them is an outer apex.

(b) Suppose that G has no edge (v_0, v_{l+1}). Let F_o^* be the outer convex polygon of an arbitrary convex drawing of G. We may assume that at least one of the vertices $v_1, v_2, \cdots, v_l$ is an apex of F_o^*, as illustrated in Fig. 7(a); otherwise, we have completed a proof for (b). We cut off the region enclosed by the path $P = v_0, v_1, \cdots, v_{l+1}$ and a straight line segment v_{l+1}, v_0 from the polygon F_o^*. Let F_o' be the resulting outer convex polygon, in which $v_0, v_1, \cdots, v_l$ lie on the line segment connecting v_0 and v_{l+1}, as illustrated in Fig. 7(b). We claim that F_o' is extendable and hence G has a convex drawing in which none of the vertices $v_1, v_2, \cdots, v_l$ is an apex of the outer convex polygon. Since G is internally tri-connected, both Conditions (a) and (c) in Lemma 1 hold for F_o'. Thus we shall show that Condition (b) holds for F_o'. Since Condition (b) holds for F_o^*, the definition of F_o' implies that Condition (b) holds for all paths of F_o', other than P, each corresponding to a side of the polygon F_o'. Since vertices $v_1, v_2, \cdots, v_l$ have degree two and there is no edge (v_0, v_{l+1}) in G, no two outer vertices in path P are joined by an inner edge. Since G is internally triconnected, $G - F_o(G)$ has no connected component H such that all the outer vertices adjacent to vertices in H lie on the single path P. Thus G satisfies Conditions (a)–(c), and hence F_o' is extendable.   □

By Lemma 3, in order to find a convex drawing of minimum outer apices, we may "contract" an outer chain $P = v_0, v_1, \cdots, v_{l+1}$ of G as follows: if G has an edge (v_0, v_{l+1}), then we replace P in G with an outer chain $P' = v_0, v_i, v_{l+1}$ for an arbitrary index i, $1 \le i \le l$; otherwise, we replace P in G with a single edge (v_0, v_{l+1}). Contract every outer chain of G as above, and let G' be the resulting graph. G' is called a *contracted graph* of G. Figure 8(b) illustrates a contracted graph of the graph G in Fig. 8(a). Since G is internally triconnected, G' is also internally triconnected and hence G' has a convex drawing. One can easily observe that the following lemma holds.

Lemma 4. *Let G be an internally triconnected plane graph, and let G' be a contracted graph of G. If an outer convex polygon F_o^* is extendable for G' (as*
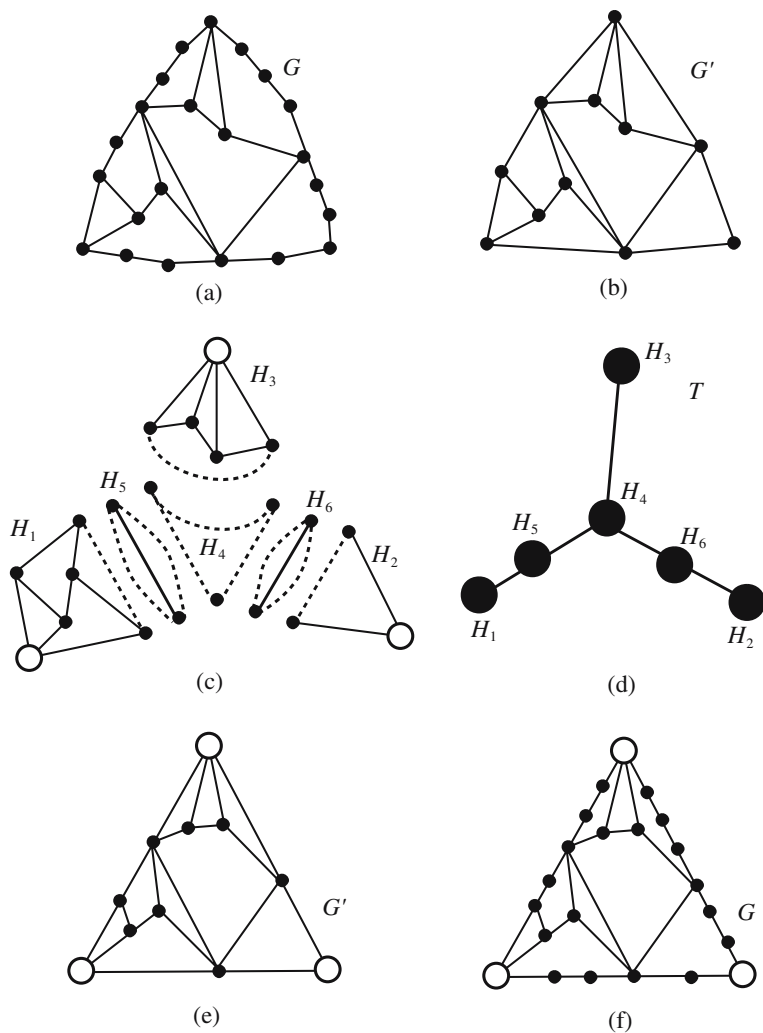
Fig. 8. (a) An internally triconnected graph G, (b) a contracted graph G' of G, (c) triconnected components of G', (d) a triconnected component tree T, (e) a convex drawing of G', and (f) a convex drawing of G

illustrated in Fig. 8(e)), then the same outer convex polygon F_o^* is extendable for G (as illustrated in Fig. 8(f)).

We now have the following lemma.

Lemma 5. Let G be an internally triconnected plane graph, let G' be a contracted graph of G, let T be a triconnected component decomposition tree of G', and let n_l be the number of leaves of T. Then the minimum number of apices of an extendable convex polygon of G' is equal to $\max\{3, n_l\}$.

Proof. If $n_l = 1$, that is, G' is either a cycle or is a triconnected graph, then G' has a convex drawing whose outer facial cycle is drawn as a triangle [6, 8]. Therefore, the minimum number of apices of an extendable convex polygon of G' is max$\{3, 1\} = 3$. We may thus assume that $n_l \geq 2$.

We first claim that G' has no extendable outer convex polygon such that the number of apices is less than max$\{3, n_l\}$. If $2 \leq n_l \leq 3$, then max$\{3, n_l\} = 3$ and hence G' has no extendable outer convex polygon such that the number of outer apices is less than max$\{3, n_l\}$. We may thus assume that $n_l \geq 4$. Suppose for a contradiction that G' has an extendable outer convex polygon F_o^* having less than max$\{3, n_l\} = n_l$ apices. Then G' has a triconnected component H_j, corresponding to a leaf of T, such that none of the outer vertices in H_j other than the separation pair is an apex of F_o^*. Hence all the outer vertices in H_j lie on a single side P_i of the polygon F_o^*. Since H_j corresponds to a leaf of T, H_j is either a triconnected graph or a ring. (H_1 and H_3 in Fig. 8(c) are triconnected graphs, while H_2 is a ring.) Consider first the case where H_j is a triconnected graph. Then $G' - F_o(G')$ has a connected component such that all the outer vertices adjacent to vertices in the component lie on the side P_i, as illustrated in Fig. 3(a). Consider next the case where H_j is a ring. Let H_p be the triconnected component corresponding to the node of T which is adjacent in T to the leaf corresponding to H_j. By the definitions of T and G', H_p must be a bond and contain an inner edge joining two outer vertices of G'. Therefore the two outer vertices on the side P_i are joined by the inner edge. Thus, in either case, Condition (b) does not hold for F_o^*. Hence F_o^* is not extendable, a contradiction. We have thus verified the claim.

We then choose a number $k = $ max$\{3, n_l\}$ of outer vertices as apices of an outer convex polygon F_o^* of G', as in the following two cases.

Case 1: $n_l \geq 3$.

In this case, as an apex, we choose an outer vertex v of G' from each triconnected component H_j corresponding to a leaf of T; v must be other than the vertices of the separation pair with respect to H_j. We set these n_l vertices chosen from all leaves of T as the apices of F_o^*. (In Fig. 8(c), there are three triconnected components H_1, H_2 and H_3 corresponding to leaves of T in Fig. 8(d), and for example the three vertices depicted as white circles are chosen from H_1, H_2 and H_3.)

Case 2: $n_l = 2$.

In this case, as apices, we first choose two outer vertices from the two leaves similarly as in Case 1 above. We then choose an arbitrary outer vertex other than the two vertices as an apex of F_o^*. We set these three vertices as the apices of F_o^*. (The triconnected component decomposition tree T of G' for the graph G in Fig. 1 has two leaves, and for example the three vertices depicted as white circles are chosen as outer apices.)

We finally claim that the k-gon F_o^* defined above is extendable. We shall show that Conditions (a)–(c) in Lemma 1 hold for F_o^*. Since G' is internally triconnected, both Conditions (a) and (c) hold. It thus suffices to show that Condition (b) holds for F_o^*.

Suppose for a contradiction that Condition (b) does not hold for F_o^*. Let P_1, $P_2, \cdots, P_k$ be the k paths in F_o^*, each corresponding to a side of the polygon F_o^*. Since Condition (b) does not hold for F_o^*, one of the following two cases occurs.
Case (i): *$G' - F_o(G')$ has a connected component H such that all the outer vertices adjacent to vertices in H lie on a single path P_i, as illustrated in Fig. 3(a).*
Case (ii): *two outer vertices in a path P_i are joined by an inner edge, as illustrated in Fig. 3(b).*

We first consider Case (i). Since G' is internally triconnected, H is contained in a triconnected component H_j of G' and H_j must be a leaf of T. Then none of the outer vertices in H_j other than the separation pair would be chosen as an outer apices, contrary to the definition of F_o^*.

We then consider Case (ii). Let (u, v) be the innermost one among all the inner edges satisfying Case (ii), and let H_j be the triconnected component containing edge (u, v). Then $\{u, v\}$ is a separation pair of G' and H_j is a leaf of T. None of the outer vertices in H_j other than u and v is an apex of F_o^*, contrary to the definition of F_o^*. □

Given a plane graph G together with an extendable outer convex polygon F_o^*, one can find in linear time a convex drawing of G which is an extension of F_o^* [1, 7]. Therefore, by Theorem 1 and Lemmas 2, 4 and 5, we have the following theorem.

Theorem 2. Let G be an internally triconnected graph, let T be a triconnected component decomposition tree of a contracted graph G' of G, and let n_l be the number of leaves of T. Then one can find a convex drawing of G having the minimum number of outer apices in linear time, and the minimum number is equal to $\max\{3, n_l\}$.

4   Conclusions

In this paper, we newly formalize a necessary and sufficient condition for a plane graph G to have a convex drawing, and show that a convex drawing having the minimum number of outer apices can be found in linear time.

In a convex grid drawing, all the vertices of G are put on grid points. It is known that if either G is triconnected or a triconnected component decomposition tree T of a contracted graph G' of G has at most three leaves then G has a convex grid drawing of size $(n - 1) \times (n - 1)$ where n is the number of vertices in G [2, 6]. The remaining problem is to obtain a good upper bound on the size of a convex grid drawing of G for which T has four or more leaves.

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