On Edges Crossing Few Other Edges in Simple Topological Complete Graphs

Jan Kynčl and Pavel Valtr

Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

Abstract. We study the existence of edges having few crossings with the other edges in drawings of the complete graph (more precisely, in simple topological complete graphs). A *topological graph* $T = (V, E)$ is a graph drawn in the plane with vertices represented by distinct points and edges represented by Jordan curves connecting the corresponding pairs of points (vertices), passing through no other vertices, and having the property that any intersection point of two edges is either a common end-point or a point where the two edges properly cross. A topological graph is *simple*, if any two edges meet in at most one common point.

Let $h = h(n)$ be the smallest integer such that every simple topological complete graph on n vertices contains an edge crossing at most h other edges. We show that $\Omega(n^{3/2}) \leq h(n) \leq O(n^2/\log^{1/4} n)$. We also show that the analogous function on other surfaces (torus, Klein bottle) grows as cn^2.

1 Introduction

A *topological graph* $T = (V, E)$ is a graph drawn in the plane with vertices represented by distinct points and edges represented by Jordan curves connecting the corresponding pairs of points (vertices), passing through no other vertices, and having the property that any intersection point of two edges is either a common end-point or a point where the two edges properly cross. A topological graph is *simple*, if any two edges meet in at most one common point.

One of the traditional themes in the area of graph drawings is to realize a given abstract graph as a topological graph so that the number of edge crossings is minimized. Here we consider a variant of a "dual" problem. We study realizations of the complete graph where each edge crosses "many" other edges.

Consider a network model drawn as a topological graph where the edge crossings are used for the exchange of some commodities (or information) between the two crossing edges. In any such model, edges with few crossings can exchange only small amounts of the commodities with the other edges within a time unit. This leads to the question about the existence of drawings in which each edge crosses "many" other edges.

If we can choose the underlying abstract graph on n vertices, then we can realize it with each edge crossing $\Omega(n^2)$ other edges. E.g., take the vertices of a regular n-gon and connect each vertex by straight-line segments with the $\approx n/3$

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opposite vertices. Each edge in the obtained topological graph crosses at least $\approx n^2/9$ other edges. Moreover, each edge is realized by a straight-line segment, thus it is a so-called *rectilinear drawing* (sometimes also called a *geometric graph*). If the underlying graph is fixed then the situation is much more complicated. In this paper we restrict our attention to topological complete graphs, i.e., to realizations of (abstract) complete graphs. We are not aware of any result for other classes of graphs.

If any two edges are allowed to cross each other at most twice, then there are various realizations of the complete graph with each edge crossing $\Omega(n^2)$ other edges. E.g., take n points (vertices) on a short horizontal segment s and for any two vertices a, b, connect a and b by an arc constructed as follows. Let $U(a, b)$ be the unit circle going through a and b and having the center above s. Then the edge ab is drawn as the arc obtained from $U(a, b)$ by removing the part below the segment ab. Then any two edges with no common vertex cross once or twice. A different example of such a drawing is described in [14]. In this paper we show that the situation is different for simple topological (complete) graphs.

According to the so-called crossing lemma [1, 9], if T is a topological graph with n vertices and $e \geq (3 + \varepsilon)n$ edges then its crossing number is at least $\Omega(e^3/n^2)$, (i.e., it contains at least $\Omega(e^3/n^2)$ crossing pairs of edges). It follows that if T is a topological complete graph then its crossing number is $\Omega(n^4)$ (this has also a quite easy direct proof). If T is simple then there are at most $\binom{n-2}{2} = O(n^2)$ crossings on each edge. It follows that a simple topological complete graph on n vertices contains $\Omega(n^2)$ edges each of which crosses $\Omega(n^2)$ other edges.

We study the existence of edges with (much) fewer than cn^2 crossings. Let us remark that in any rectilinear drawing of K_n the edges on the boundary of the convex hull do not cross any other edge. On the other hand, Harborth and Thürmann [8] found a simple topological complete graph in which each edge crosses some other edges.

Let $h = h(n)$ be the smallest integer such that every simple topological complete graph on n vertices contains an edge crossing at most h other edges. Harborth and Thürmann [8] proved $h(n) > (\frac{3}{4} + o(1))n$. Other related questions were studied e.g. in [5, 6, 7, 16, 17]. It has been asked in the preliminary version of the book [2] whether $h(n) = O(n)$, and the final version of [2] contains a conjecture that $h(n) = o(n^2)$. In this paper we show that $h(n)$ grows much faster and we also give the first subquadratic upper bound on $h(n)$:

Theorem 1.

$$\Omega(n^{3/2}) \leq h(n) \leq O(n^2/\log^{1/4} n).$$

We describe two essentially different constructions giving the lower bound. We present both of them, since they may help in closing the gap between the bounds given in Theorem 1. We conjecture that the lower bound is closer to the asymptotic behavior of $h(n)$ than the upper bound, and maybe even $h(n) = \Theta(n^{3/2})$. We remark that our proof gives a reasonable constant involved in the $\Omega-$notation in the lower bound in Theorem 1. For simplicity of presentation, we do not compute the constants.

It is interesting that for other surfaces (torus, Klein bottle, real projective plane) it is possible to find simple topological complete graphs with each edge crossing $\Omega(n^2)$ other edges. This is discussed in the last section of the paper.

Brass, Moser, and Pach [2] describe a connection between the function $h(n)$ and the maximum number of disjoint edges in a topological graph. They have suggested the following greedy procedure: Select an edge intersecting the smallest number of other edges, delete these edges, and repeat the procedure. The lower bound in Theorem 1 indicates limits of this procedure in some cases. We remark that finding many disjoint edges and various similar questions on topological and geometric graphs have recently received a lot of attention, e.g. see [3, 4, 10, 11, 12, 13, 14, 15].

2 The Lower Bound

2.1 First Construction

Let $\mathcal{S}$ be the unit sphere in $\mathbf{R}^3$. Our topological complete graph giving the lower bound in Theorem 1 will be drawn on $\mathcal{S}$ by choosing an appropriate set P_n of n points on $\mathcal{S}$ and then connecting each pair of points of P_n by the shortest arc contained in $\mathcal{S}$. The points of P_n will be "well distributed" on $\mathcal{S}$ and *in general position*, meaning that no two points of P_n are antipodal and no three points of P_n lie on a common great circle of $\mathcal{S}$.

The crucial requirement on P_n is the following condition:

(C) If $d = d(P_n)$ denotes the minimum (Euclidean) distance of a pair of points of P_n then for any point $q \in \mathcal{S}$, the $1.1d$-neighborhood of q contains a point of P_n.

The set P_n is constructed as follows. First, we inductively construct n auxiliary points $a_1, \dots, a_n$. Choose a point $a_1 \in \mathcal{S}$ arbitrarily. Now, let $i \in \{1, \dots, n-1\}$ and suppose that $a_1, \dots, a_i$ have already been selected. Then we choose a_{i+1} as a point on $\mathcal{S}$ maximizing the quantity $\min\{||a_1 - a_{i+1}||, ||a_2 - a_{i+1}||, \dots, ||a_i - a_{i+1}||\}$. Clearly, we can slightly perturb the constructed set $\{a_1, \dots, a_n\}$ so that the perturbed set, P_n, is in general position and satisfies condition (C).

Observe that $(d =)\ d(P_n) = \Theta(1/\sqrt{n})$ follows from the following three facts by a simple counting argument: (i) the area of $\mathcal{S}$ is $\Theta(1)$, (ii) the $1.1d$-neighborhoods of the points of P_n cover $\mathcal{S}$, and (iii) the $0.49d$-neighborhoods of the points of P_n are pairwise disjoint.

Let $T = T_n$ be the simple topological complete graph on $\mathcal{S}$ such that $V(T) = P_n$ and that $E(T)$ consists of the shortest curves on $\mathcal{S}$ connecting the pairs of vertices. We have to show that every edge in T crosses $\Omega(n^{3/2})$ other edges.

We use the notions *equator, northern/southern hemisphere of* $\mathcal{S}$ in the obvious way. Clearly, for any two vertices a, b, the edge ab is a portion of the great circle containing a, b. Thus, it suffices to show that if a portion I of a great circle of $\mathcal{S}$ has length $|I| = d$ then it is intersected by at least $\Omega(n^{3/2})$ edges of T. We may suppose that I is a portion of the equator. We denote the end-points of I by s

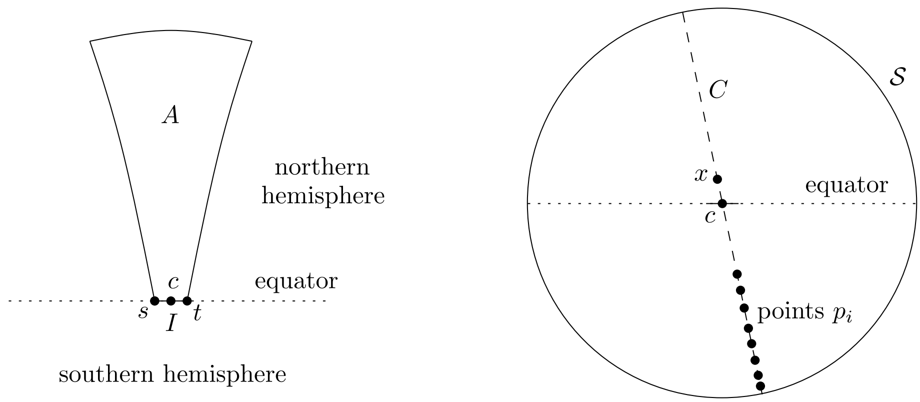

Fig. 1. The region A (left) and the points p_i (right)

and t. For a point $x \in \mathcal{S}$ not lying on the equator, *the spherical triangle stx* is the region on $\mathcal{S}$ bounded by I and by the two shortest arcs contained in $\mathcal{S}$ and joining x with the points s and t, respectively.

Let c be the mid-point of the arc I. We consider the region A on $\mathcal{S}$ of the points x on the northern hemisphere such that $||x - c|| < \frac{1}{100}$ and that the spherical triangle stx has the inner angles at s and t each at most 0.6π (see Figure 1). The region A is bounded by I and by three arcs of length $\Theta(1)$. Clearly, its area is $\Theta(1)$ and it contains $\Theta(n)$ points of P_n. It suffices to show that any point of $A \cap P_n$ is an end-point of $\Omega(\sqrt{n})$ edges intersecting I.

Let $x \in A \cap P_n$. Consider the great circle C going through the points x and c (see Figure 1). Since $d = \Theta(1/\sqrt{n})$, it is possible to select $\Theta(\sqrt{n})$ points $p_1, p_2, \ldots, p_t$ in the intersection of C with the southern hemisphere such that $\frac{1}{10} < ||c - p_i|| < \sqrt{2}$ (for each i) and $||p_i - p_j|| > 2.2d$ (for any $i \neq j$). In general, the points p_i do not lie in P_n. However, the $1.1d$-neighborhood of each p_i contains a point $p'_i \in P_n$. By the choice of the points p_i, the points p'_i are pairwise distinct. It is not difficult to verify that each of the $\Theta(\sqrt{n})$ edges xp'_i intersects the arc I. This completes the proof that any edge in $T = T_n$ crosses $\Omega(n^{3/2})$ other edges.

2.2 Second Construction

Our second construction giving the lower bound in Theorem 1 is only briefly outlined in this extended abstract. We start with any fixed simple topological complete graph T in which each edge has at least one crossing, e.g with the drawing on Fig. 2. Let $V(T) = \{v_1, v_2, \ldots, v_t\}$. Let $n \geq t$ and suppose for simplicity that $\sqrt{n/t}$ is an integer. We replace each vertex v_i by a set V_i of n/t vertices placed in a square lattice $\sqrt{n/t} \times \sqrt{n/t}$ of a very small diameter. Any two vertices in distinct sets $V_i, V_j, i \neq j$, will be connected by an edge contained in a small neighborhood of the edge v_iv_j of T. Let $i \in \{1, \ldots, t\}$ and suppose that the edges in T incident to v_i leave the vertex v_i in a counterclockwise order

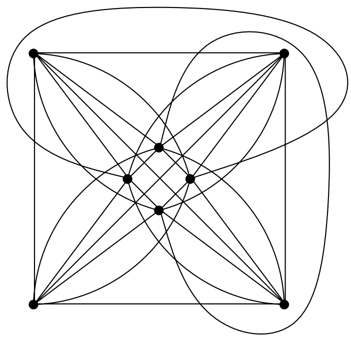

Fig. 2. A simple topological complete graph on 8 vertices in which each edge crosses another edge

$v_iv_{j_1}, v_iv_{j_2}, \ldots, v_iv_{j_{t-1}}$. In a small neighborhood of the convex hull of V_i we draw the edges leaving from the vertices of V_i so that any edge connecting a vertex of V_i with a vertex of V_{j_1} leaves the vertex of V_i along a vector parallel to some vector $(1, \varepsilon)$, where $\varepsilon > 0$ is very small (and different for different edges), and similarly any edge connecting any vertex of V_i with a vertex of V_{j_2}, V_{j_3}, or $V_{j_4} \cup \ldots \cup V_{j_{t-1}}$ (respectively) leaves the vertex of V_i along a vector parallel to some vector $(\varepsilon, 1)$, $(-1, \varepsilon)$, or $(\varepsilon, -1)$ (respectively). This ensures that after a very tiny perturbation of V_i and after connecting any two vertices of V_i by a straight-line segment, each such segment (edge) will be intersected by at least $\left(\sqrt{n/t} - 1\right) n/t = \Theta(n^{3/2})$ edges connecting vertices of V_i with the vertices of $V_{j_1} \cup V_{j_2} \cup V_{j_3} \cup V_{j_4}$. It is not too difficult to check that the whole construction can be done so that the resulting drawing is a simple topological (complete) graph. Moreover, any edge connecting vertices from distinct sets V_i, V_j has $(n/t)^2 = \Theta(n^2)$ crossings in a small neighborhood of the point where the edge v_iv_j crosses another edge of the graph T. Thus the obtained topological graph gives the lower bound in Theorem 1.

3 The Upper Bound

Topological graphs G, H are said to be *weakly isomorphic*, if there exists an incidence preserving one-to-one correspondence between $(V(G), E(G))$ and $(V(H), E(H))$ such that two edges of G intersect if and only if the corresponding two edges of H do. Let C_m denote a complete convex geometric graph with m vertices (note that all such graphs are weakly isomorphic to each other). A simple topological complete graph with m vertices is called *twisted* and denoted by T_m, if there exists a *canonical* ordering of its vertices $v_1, v_2, \ldots, v_m$ such that for every $i < j$ and $k < l$ two edges v_iv_j, v_kv_l cross if and only if $i < k < l < j$ or $k < i < j < l$ (see Figure 3). Figure 4 shows an equivalent drawing of T_m on the cylindric surface. If G, H are topological graphs, we say that G *contains* H, if G has a topological subgraph weakly isomorphic to H.

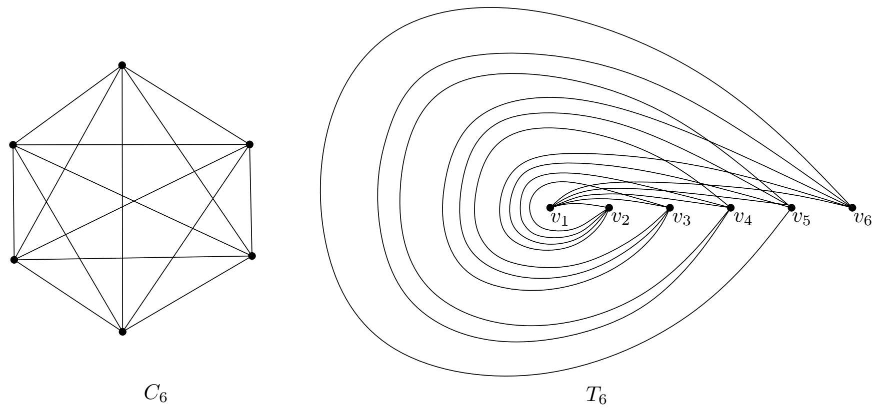

Fig. 3. The convex geometric graph C_6 and the twisted graph T_6

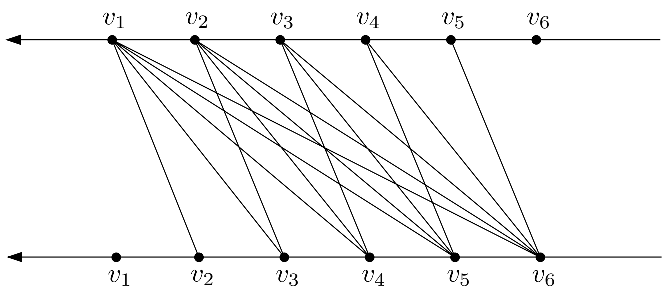

Fig. 4. Drawing of the twisted graph T_6 on the cylindric surface

In the proof of the upper bound, we will use the following asymmetric form of the result of Pach, Solymosi and Tóth [13]:

Theorem 2. [13] *There exists a* $c > 0$ *such that for all positive integers* n, m_1, m_2 *satisfying* $m_1 m_2 \le c \log^{1/4} n$ *every simple topological complete graph with* n *vertices contains* C_{m_1} *or* T_{m_2}.

We will use this theorem for $m_1 = c' \log^{1/4} n$ and m_2 constant.

Now we prove two lemmas, the first one related to the complete geometric graph C_m, the second one related to the twisted drawing T_5.

Lemma 1. *Let* G *be a simple topological complete graph with* n *vertices. If* G *contains* C_m, *then there exists an edge in* G *which crosses at most* $2n^2/m$ *other edges.*

Proof. Let H be a topological complete subgraph of G with m vertices weakly isomorphic to C_m. H has a face F that is bounded by a non-crossing Hamiltonian cycle C consisting of m edges. Without loss of generality, suppose that F is the outer face of H. Then all edges of H lie inside the region bounded by the cycle C. We denote this region by R.

Claim. Let c be a simple continuous curve which starts and ends inside F, does not go through any vertex of H and crosses each edge of H at most once. Then c crosses at most two edges of the cycle C.

Proof. For contradiction, suppose that c crosses more than two edges of C. Then the intersection of c with R consists of $k \geq 2$ disjoint arcs $c_1, c_2, \ldots, c_k$ (see Figure 5). In the region R, the arcs c_1, c_2 separate two portions of C, denoted by α, β, from each other (see Figure 5). Since $c \supseteq c_1 \cup c_2$ intersects each edge of C at most once, each of the arcs α, β contains a vertex of G. However, any edge e connecting a vertex on α with a vertex on β intersects both c_1 and c_2. Thus, it intersects c more than once — a contradiction.

Let c be an arbitrary edge of G and let k be the number of edges of C that are crossed by c. First, we delete from c a small neighborhood of its end-points, receiving a curve c' that is disjoint with all vertices of H and crosses the same edges as c does. If some of the end-points of c' lies inside the region R, we delete from c' the initial part between the end-point and the first point a, at which c' crosses C, including a small neighborhood of a. We receive a curve c'' that has both its end-points inside F and crosses at least $k-2$ edges from C. By the previous claim, c'' crosses at most 2 edges from C, thus $k \leq 4$.

G has less than $\frac{n^2}{2}$ edges, thus there are at most $2n^2$ crossings between the edges of G and the edges of C. By the pigeon-hole principle, among the m edges of C there is an edge, which crosses at most $2n^2/m$ edges of G.

Consider a simple topological complete graph H weakly isomorphic to the twisted graph T_m with the canonical ordering $v_1, v_2, \ldots, v_m$ of its vertices. The face incident with the vertices v_{m-1} and v_m only is called an *outer* face of H (it coincides with the outer face of the drawing of T_m at Figure 3), similarly the face incident with the vertices v_1 and v_2 only is called an *inner* face of H.

Lemma 2. *Let H be a simple topological complete graph weakly isomorphic to T_5. There does not exist a simple continuous curve c, which crosses each edge of H at most once, does not go through any vertex of H, begins and ends inside the outer face of H and intersects the inner face of H.*

Proof. Let $v_1, v_2, \ldots, v_5$ be the canonical ordering of the vertices of H. Consider a Hamiltonian cycle H_5, which is a subgraph of H with the edge set $E(H_5) =$

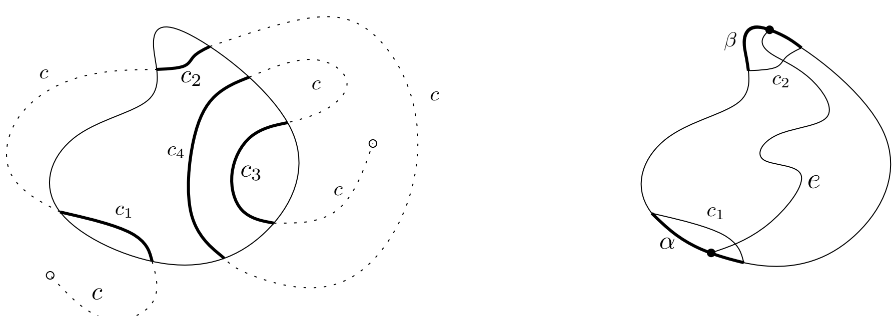

Fig. 5. The arcs c_i (left) and the arcs α, β (right)

$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$. Let F_1, F_2, F_3, F_4 be the four faces of H_5 such that F_1 is incident with the vertices v_1 and v_2 only and F_i borders with F_{i+1}, $i = 1, 2, 3$ (see Figure 6). Note that F_1 (F_4) is the inner (outer) face of H and that F_i does not border with F_j if $|i - j| \geq 2$.

For contradiction, suppose that there exists a simple continuous curve c starting and ending inside F_4 and passing through F_1, avoiding all vertices of H and crossing each edge of H at most once. Choose a point $p \in c \cap F_1$. By the previous observation, between the starting point and p, c has to pass through the faces F_2 and F_3, so it must cross at least three edges of H_5. Similarly, c crosses at least three edges of H_5 between the point p and its end-point. But H_5 has only five edges, thus at least one of them is crossed by c more than once, a contradiction.

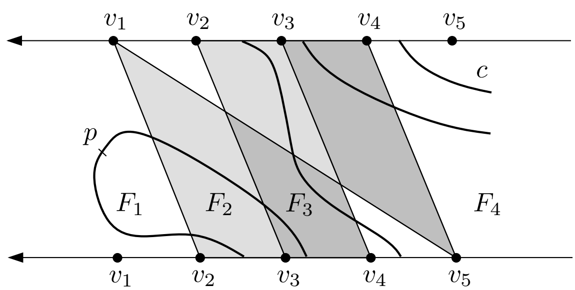

Fig. 6. The graph H_5 and a curve c with six crossings

The following theorem gives the upper bound in Theorem 1.

Theorem 3. *There exists a* $c > 0$ *such that in every simple topological complete graph with* n *vertices there exists an edge that crosses at most* $cn^2/\log^{1/4} n$ *other edges.*

Proof. Let G be a simple topological complete graph with n vertices. By Theorem 2, every induced subgraph of G with at least $n^{1/8}$ vertices contains T_{20} or $C_{\frac{c'}{2}\log^{1/4} n}$. If G contains $C_{\frac{c'}{2}\log^{1/4} n}$ then, by Lemma 1, G has an edge which crosses at most $\frac{4}{c'}n^2/\log^{1/4} n$ other edges. For the rest of the proof, suppose that G does not contain $C_{\frac{c'}{2}\log^{1/4} n}$, thus every induced subgraph of G with at least $n^{1/8}$ vertices contains T_{20}.

Let T_{20}^1 be a complete subgraph of G with 20 vertices weakly isomorphic to T_{20} and let $v_1^1, v_2^1, \ldots, v_{20}^1$ be a canonical ordering of its vertices. Consider a graph H^1 with the vertex set $V(H^1) = V(T_{20}^1)$ and the edge set $E(H^1) = \{v_1^1v_2^1, v_2^1v_3^1, \ldots, v_{19}^1v_{20}^1, v_1^1v_5^1, v_6^1v_{10}^1\}$ (see Figure 7). Denote the faces of H^1 as $F_1^1, F_2^1, \ldots, F_7^1$ such that F_1^1 is the inner face of T_{20}^1, F_7^1 contains the outer face of T_{20}^1 and F_i^1 borders with F_{i+1}^1, $i = 1, 2, \ldots, 6$ (as on the Figure 7).

Applying Lemma 2 on the twisted induced subgraph of T_{20}^1 with the vertices $v_1^1, v_2^1, \ldots, v_5^1$ we get that every edge of G, which crosses $v_1^1v_2^1$, has at least one

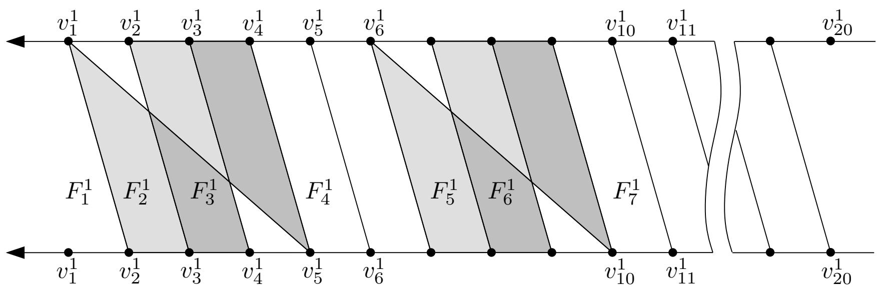

Fig. 7. The graph H^1 and its seven faces

end-point in the set $A^1 = F_1^1 \cup F_2^1 \cup F_3^1 \cup \{v_3^1, v_4^1, v_5^1\}$. Denote $a^1 = |A^1 \cap V(G)|$. If $a^1 < n^{1/8}$, then there are at most $n \cdot n^{1/8} = n^{9/8}$ edges with one end-point in A^1, thus at most $n^{9/8}$ edges cross the edge $v_1^1 v_2^1$. In the other case, the complete subgraph of G induced by the set $A^1 \cap V(G)$ has a subgraph T_{20}^2 weakly isomorphic to T_{20}. Consider a twisted subgraph H of T_{20}^1 induced by the vertices $v_{10}^1, v_9^1, v_8^1, v_7^1, v_6^1$ (in this canonical ordering). Every edge of T_{20}^2 has both its end-points inside the outer face of H, so it cannot intersect the inner face of H (by Lemma 2). This yields that all edges of T_{20}^2 lie in the set $B^1 = F_1^1 \cup F_2^1 \cup \ldots \cup F_6^1 \cup \{v_3^1, v_4^1, \ldots, v_{10}^1\}$. Denote $b_1 = |B^1 \cap V(G)|$. Note that $A^1 \subseteq B^1$, thus $a_1 \leq b_1$. It follows that at most one face of the graph T_{20}^2 does not lie in B^1. So we can choose a canonical ordering $v_1^2, v_2^2, \ldots, v_{20}^2$ of the vertices of T_{20}^2 such that the faces $F_1^2, F_2^2, \ldots, F_6^2$ of the graph H^2 (defined analogically as H^1 and its faces F_i^1) lie in B^1. We define sets A^2, B^2 and numbers a^2, b^2 analogically as A^1, B^1, a^1, b^1. B^2 is a proper subset of B^1, since all vertices of T_{20}^2 and faces $F_1^2, F_2^2, \ldots, F_6^2$ are contained in B^1, but, for example, vertex v_{11}^2 does not lie in B^2. It yields that $b_2 < b_1$. If $a_2 \geq n^{1/8}$, then there exists a twisted complete subgraph T_{20}^3 of G induced by some 20 vertices of the set $A^2 \cap V(G)$. Further we proceed by induction, similarly as above. In the i-th step, assuming that $a_{i-1} \geq n^{1/8}$, we find a twisted complete subgraph T_{20}^i of G with 20 vertices and define two integers a_i, b_i satisfying $0 \leq a_i \leq b_i < b_{i-i}$. After finitely many steps, we get a number a_i, which is less than $n^{1/8}$. It means that the edge $v_1^i v_2^i$ in the graph T_{20}^i is crossed by less than $n^{9/8} < n^2/\log^{1/4} n$ other edges of G.

4 Other Surfaces

Here we show that an analogue of the function $h(n)$ is quadratic for the torus and for the Klein bottle[1]:

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[1] The same result for the projective plane has been recently found by Attila Pór (personal communication). We describe Pór's construction at the end of this section. Since any drawing of a finite graph on the projective plane can be easily transformed to a drawing on the Klein bottle, Pór's construction can be used to obtain an alternative proof of Proposition 1 for the Klein bottle.

Proposition 1. *On the torus and on the Klein bottle, there exists a simple topological complete graph with each edge having at least cn^2 crossings.*

Proof. Consider a rectangle from which, after gluing its opposite sides, we get a torus. Place the vertices $v_1, v_2, \ldots v_n$ along its upper and lower side in this order. We draw the edges the following way: if $j - i \mod n \leq \lfloor \frac{n-1}{2} \rfloor$, or if $j - i \mod n = \frac{n}{2}$ and $i \leq \frac{n}{2}$, we represent the edge $v_i v_j$ as a segment starting at the upper vertex v_i, directing down and to the right, possibly leaving the rectangle on the right-hand side and entering on the left-hand side and ending at the lower vertex v_j. At Figure 8, you can see the representation of the edges incident to one vertex. It is clear that in this drawing each two edges intersect at most once and that every edge crosses at least cn^2 other edges.

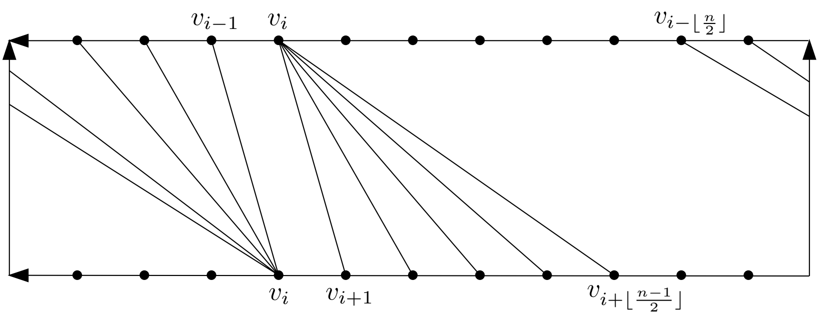

Fig. 8. Edges incident to the vertex v_i in the drawing of K_n on the torus

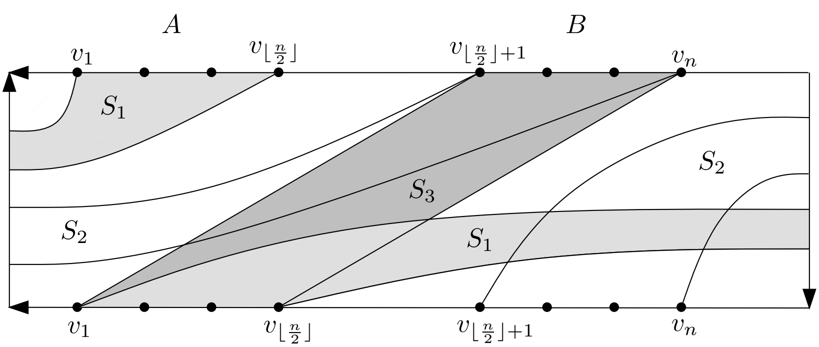

Fig. 9. Drawing of K_n on the Klein bottle

For the drawing on the Klein bottle, divide the vertices into two sets $A = \{v_1, v_2, \ldots, v_{\lfloor \frac{n}{2} \rfloor}\}$ and $B = \{v_{\lfloor \frac{n}{2} \rfloor+1}, \ldots, v_n\}$ and place all the edges into three strips S_1, S_2, S_3. S_1 contains all edges among the vertices of A, S_2 contains all edges among the vertices of B, and S_3 all edges between A and B (see Figure 9). Clearly, we can draw the edges such that no two of them intersect more than once. It is not difficult to verify that each edge crosses at least cn^2 other edges.

We now describe Attila Pór's construction of a simple topological complete graph on n vertices with each edge intersecting at least $\Omega(n^2)$ other edges. The projective plane can be obtained by adding a line at infinity to the real plane. We place the vertices of the constructed topological graph in the vertices of a regular n-gon P. Any two vertices are connected by the portion of the line through the two vertices outside of the polygon P. It is easy to see that any edge is intersected by $\Omega(n^2)$ other edges.

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