# Discrete Conformal Shape Representation and Reconstruction of 3D Mesh Objects

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**Abstract.** This paper studies shape representation of general 3D objects. In particular, it proposes a conformal representation for genus-zero mesh objects, by using the discrete conformal mapping technique. It also proposes a new method to reconstruct the original shape from its conformal representation. In order to simplify and robustify the computation, we made several improvements to the above two procedures. The modifications include planar graph drawing initialization, Moebius factorization and spring-embedding-based reconstruction, etc. Though being mostly incremental, these modifications do provide significant improvements on previous methods. Possible applications include 3D geometry compression and object classification/recognition, etc.

## 1 Introduction

3D object representation and recognition is one of the central topics in computer vision and pattern recognition research. A good *shape representation* scheme is at the heart of any practical shape recognition systems. This paper aims at developing a new 3D shape representation method for general mesh objects.

We intend to derive *complete* representation. By *complete* we mean, such representation must fully encode all the necessary information of the original shape. As the result, it should be possible to recover the original shape from the representation(up to some approximation). In mathematical sense, this is equivalent to finding a mathematical representation of the original geometric entity (i.e., the shape).

For closed genus-zero mesh object, the shape representation problem effectively reduces to a *surface parameterization* problem. Discrete conformal mapping (DCM) is a newly developed surface parameterization technique in computer graphics and geometric processing areas[6][9][5]. Though the underlying mathematical principles of conformal mapping were well known over a century ago, how to apply them to modern digital mesh surfaces is still unclear to most practitioners. Conformal mapping has many nice properties that make it especially suited to the application of surface parametrization. The most notable one is that it preserves angles, and therefore preserves local geometry. In addition, it depends only on surface geometry (the Riemannian metric), and therefore is very robust to changes of data triangulations, resolutions (level-of-detail) and noise.

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Our work in this paper basically follows [5] proposed by Gu and Yau et.al. It proposed a steepest-descend based iterative algorithm for global conformal parameterization of arbitrary genus objects, and presented many nice numerical results. It also showed the possibility for general 3D shape classification using conformal invariants[10].

However, when consider the application of 3D shape representation, their method have shortcomings. In particular, the mapping result may depend on human interactions, and the converge rate is rather slow. In order to better enjoy the nice properties of DCM while avoid most of the difficulties in its computation, we provide several necessary modifications in order to overcome most of these problems. Our new method is more efficient, and can automatically (no user interaction) produce complete shape representation. For demonstrating the completeness, we also proposed a shape reconstruction technique, which is used to reconstruct the original shape from its conformal representation. We have tested our method on a small set of mesh objects of different classes and complex geometries, and good results are obtained.

## 2 Discrete Conformal Mapping

Given two closed regular surfaces $M_1$ and $M_2$. According to the celebrated Riemann Mapping Theorem, for any two genus-zero surfaces there always exist conformal mappings between them. Therefore, a valid spherical parametrization of any genus-0 closed surfaces can always be found by such conformal mapping. However, the results are not unique (in fact, they are infinitely many). Nevertheless, all the feasible solutions actually form a low-dimensional subspace which is the Möbius group of 6-(real) parameters:

$$\mathbf{M}(z) = \frac{az+b}{cz+d}, ad-bc \neq 0, a,b,c,d \in \mathbf{C} \tag{1}$$

In practice, the conformal mapping is often approximated by a harmonic mapping, denoted by $f$. Namely, it must satisfy the following harmonic (Laplace) equation: $\triangle f = \mathbf{div}\ \mathbf{grad} f = 0$. For three-dimensional genus-0 surfaces, these two mappings are essentially the same. Therefore, the problem of finding a spherical conformal parametrization for genus-0 surface is reduced to a Laplace-on-Manifold problem, where the target manifold is the unit sphere $S^2$. Usually this is implemented by minimizing the following harmonic energy ([9][6]):

$$f = \arg\min_f E_H(f) = \frac{1}{2}\int_{M_2} \|\mathbf{grad} f\|^2 \tag{2}$$

## 3 Our New DCM Algorithm

Various methods for computing conformal mapping or discrete (triangulated) objects have been proposed. Our method basically follows paper[5]. To adapt this method for better fitting the 3D recognition purpose, we made several important improvements: (1) We introduce a new initialization method based on planar graph-drawing which effectively save many computations, and alleviate many fold-over problems. (2) We use the exponential map for solving Laplace-on-manifold diffusion problem, thus enlarges the valid area of neighborhood and improves the convergence. (3) We introduce an affine stratification algorithm for the Möbius-normalization. This algorithm is simple, effective and much faster than other existing algorithms. In the following part we will briefly describe these modifications. (For more details see [8].)

### 3.1 Initialization from Planar Graph Drawing

We start from a triangulated closed mesh object. We assume it is topologically valid, namely, a closed manifold surface, no isolate element exists. There are several softwares publicly available for such topological check. We assume the mesh has spherical topology, which can be simply verified by Euler's formula,say, test whether $V - E + F = 2$. The minimization algorithm of the harmonic energy is iterative. It therefore requires a good initialization which serves as the starting point. This initialization should be a spherical homeomorphic approximation of the final mapping. Paper [5] provided an initialization method using Spherical Tutte Embedding , where the Tutte Embedding itself is started from a Gauss map. However, though theoretically it has good convergence property, we find that it often fails to converge correctly for complex meshes.

Based on the fact that the connectivity(adjacent) graph of any genus-0 object is always a *planar graph*, where by definition a *planar graph* is graph that can be drawn on a plane in such a way that there are no edge crossings, we propose a simple method for the spherical initialization. Since our diffusion algorithm has a relatively large neighborhood, it does not require an accurate initialization, so long as the homeomorphism is guaranteed.

There exist a number of constrictive algorithms that are able to actually draw a graph on a plane without edge crossing. Such is obvious a homeomorphism of the original mesh. In fact, every planar graph can be drawn such that each edge is straight, so-called *straight-line planar embedding*. Moreover, very efficient linear time algorithm for straight line embedding are also available now.

Our initialization procedure is: first arbitrarily select one surface triangle as the boundary triangle, then apply a straight-line planar graph drawing on the whole graph, and followed by an inverse stereographic mapping to get our spherical initialization result. Figure-1 illustrates an example of such planar embedding of a wolf mesh. Although the result depends on specific choice of boundary triangle, this speciality, however, will soon be relaxed by the subsequent diffusion process.

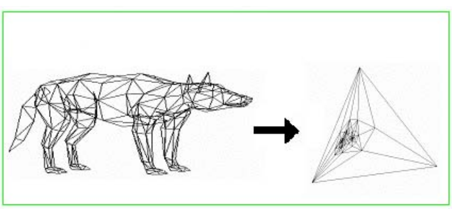

**Fig. 1.** A Planar graph drawing result of a wolf mesh

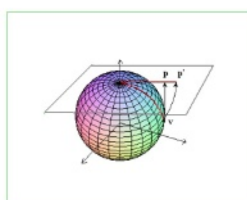

**Fig. 2.** Orthogonal map $\mathbf{p}$ and Exp map $\mathbf{p}'$ of the vertex $\mathbf{v}$

### 3.2 Harmonic Diffusion with the Exp-map

Having a homeomorphic spherical embedding as the initialization, the next step is to diffuse it to become a true conformal mapping. We accomplish this by solving a Laplace-diffusion equation on the unit sphere, namely, a Laplace-Beltrami Equation: $\triangle_{M_2} f = \mathbf{div}_{M_2}\mathbf{grad}_{M_2} f = 0$. Note that the Laplace operator has been adapted with respect to local geometry of the target manifold. Instead of directly solving this Laplace-Beltrami equation in Cartesian coordinates, which could be very involved, we adopt the use of tangent-plane-projection method. By this method the Laplace equation remains in its simple form, but acting on the tangent planes.

The purpose of tangent-plane-projection is to construct local coordinates systems on the manifold. Every tangent plane projection forms a local approximation, and can be regarded as a local representation at a local neighborhood (called a chart). Different mapping methods have different definitions of local neighborhood. Very often we would prefer a larger neighborhood definition, because by which we are able to use a less number of charts to approximate the whole manifold.

Orthogonal map is a simple method for tangent plane projection, and was adopted for DCM. However, acting on many manifolds it has a relatively smaller neighborhood compared with other methods. By orthogonal map, when two neighboring vertices are further apart than $\pi/4$ then they could not be included in a single chart.This will cause problem in computation.

We suggest the use of exponential map (exp-map, in short) to rectify this problem. The exp-map on manifold intuitively corresponds to expanding geodesic curve to tangent plane (See figure-2). It is easy to verify that for unit sphere the neighborhood of an exp-map is as large as $\pi$. In fact, this area can be further doubled if counting the direction of flow vectors. This means: by exp-map the valid neighborhood is as large as the whole sphere, which implies that all mesh vertices can find one-to-one maps on a single chart. This will ease the diffusion process, have better chance of convergence, and less depend on initial approximation. The computation of such exp-map on the unit sphere is also very simple thanks to the well-known Rodrigues formula.

### 3.3 Affine Factorization for Möbius Normalization

The solutions of conformal mapping from a surface to sphere are not unique. Simply applying another arbitrary automorphic conformal mapping to a solution will yield another valid solution. For the purpose of shape representation, we must ensure the uniqueness of the solution by using some *normalization* procedures.

Paper [5] suggested a nested-iteration algorithm for simultaneous diffusion and normalization. However, the required computations are extremely expensive especially for large scale meshes. Gotsman [9] use anchor point to normalize the solution, but the result depends on particular choice of the anchor point.

Though not unique, all the solutions actually have a relatively simple structure, say, all solutions follow a same Möbius transformation. They form a well-structured six dimensional *Möbius group*. Based on this important fact, we derive our new normalization algorithm, which significantly outperforms existing normalization algorithms.

Our method is based the concept *stratification*[2][3]. The main idea is to relax a fully Euclidean reconstruction to more general affine case, or projective case.

A Möbius Transformation has six degree-of-freedoms, which is a supergroup of the Euclidean group. We decide to decompose it into simple component transformations by analyzing its fixed points. It is easy to verify a Möbius group has at most two fixed points. If we keep the point at infinity invariant then we get an affine transformation, and the remaining one( i.e., the quotient) is proven to be a 3D rotation along the origin, which keeps the antipodal points remaining antipodal. This idea can be precisely clarified by the following operation.

Any Möbius Transformation can be represented by a 2x2 non-singular complex matrix $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Apply QR decomposition to this matrix, and after some algebras we get:

$$\mathbf{M} = \mathbf{Q} \cdot \mathbf{R} = \begin{bmatrix} q_1 & q_2 \\ -\overline{q}_2 & \overline{q}_1 \end{bmatrix} \cdot \begin{bmatrix} |r_1| & r_2 \\ 0 & |r_3| \end{bmatrix} \tag{3}$$

It is easy to verify that the orthonormal factor $\mathbf{Q}$ is a *quaternion* that precisely describes the 3D rotation along the origin, and the upper triangular factor $\mathbf{R}$ is indeed an affine transformation. Both factors has three degree-of-freedoms(not counting an arbitrary real scale factor), so each of them is a three-parameter subgroup.

Our strategy is to find a special affine factor $\mathbf{R}$ such that the following equation(which is obvious an invariant wrt. the 3D rotation) is satisfied:

$$\int_{S^2} \left[\phi^{-1} \circ \mathbf{R} \circ \phi\right] \circ f \, \mathbf{d}\sigma_{M_1} = 0 \tag{4}$$

where $\varphi$ is the stereographic mapping $\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right), x, y, z \in S^2$, $\mathbf{d}\sigma_{M_1}$ is the area-element on the original shape. Since the $f$ represents coordinates of vertices on target manifold, there are actually three equations. Now, our Möbius normalization procedure is reduced to very small-scale equations with the three *real* unknowns of the factor $\mathbf{R}$. Once solved, apply the corresponding spherical affine transformation $\mathbf{R}$ will give us a unique solution up to rotation.

## 4 Reconstruction from Spring Embedding

Now we obtain the spherical conformal parameterization of the input mesh (,for example, see figure-2). The next step will be to represent the original shape on this sphere parameterization. First we need to specify functions on the sphere, these functions themselves should be complete in the sense they can faithfully represent the original shape without information loss.

**Theorem:** *A closed surface $r(u, v)$ in $\mathbf{R}^3$ with parameter $(u, v)$ is determined by its first fundamental forms and its mean curvature $H(u, v)$ uniquely up to rigid motions.*

From the above theorem(the proof can be found elsewhere in a differential geometry text book), it is clear that the surface can be reconstructed from the first fundamental form and the mean curvature. Following [11], we also use the edge length and dihedral angle as the shape functions, because the first fundamental form is represented as the length of edges, the mean curvature is represented as dihedral angles of edges. Since these shape functions are complete, in turn it should be possible to uniquely (up to rigid motions) reconstruct the original surface from the two set of data.

Paper [11] suggests a reconstruction method, which is based on the solving of a set of simultaneous equations of local distances. In noise-free case this method works fairly good. However, when there is even small noise (for example, due to numerical precision), this algorithms may give rise to a very distorted shape. We here propose

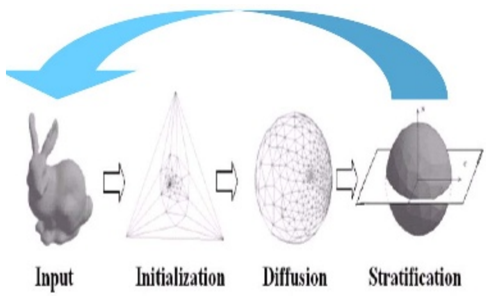


**Fig. 3.** Conformal shape representation and reconstruction. (The arrow above shows the direction of the reconstruction)

a global shape reconstruction method. This method is based on 3D graph drawing. In particular we apply a *Spring Embedding* algorithm for drawing a graph in 3D.

The spring embedding is a conceptually simple yet powerful technique for drawing 3D graph. For our application, the graph to be drawn is actually a planar graph (section 3-1). What we have now are both of its local distances (i.e., the edge lengthes) and second order distances. (The latter can be computed easily from the dihedral angles, see [4]). The total Springs Energy is thus written as:

$$W = \sum_{i=1}^{n-1} \sum_{j \in N_i} \frac{1}{2} k_{ij} \left( |v_i - v_j| - l_{ij} \right)^2 \tag{5}$$

where the $v_i, v_j$ are vertex coordinate vecteros, $l_{ij}$ represent both the edge lengthes and second order distances, $n$ is the total number of vertices, $N_i$ represents the up to second order nationhood. $k_{ij}$ is the spring constant, here we set it to the inverse of the corresponding distance.

We use a Gauss-Newton method to solve this minimization problem of eq 5. Experiments show that the convergence is very fast, the iteration finishes within 10 steps for a mesh object of about 2,000 edges. The recovered shapes are almost identical to the original ones.

## 5 Experiments and Results

We tested our algorithm on a set of mesh objects of difference classes. We first performed topological validation on them. For those that do not have valid spherical topologies we manually modified them. For example, some holes on the bottom of the Stanford bunny model had been filled.

Figure-4 shows examples of the DCM mapping results of different objects by our algorithm. The spheres showed in the right column are the result after Möbius normalization (affine stratification), so they are unit up to rotation. We have positively verified this by applying the algorithm again on a randomly rotated object.

We also tested the 3D reconstruction method based on spring embedding. Figure-5 show some results, the value of W gives the spring energy after converge. Figure-6 gives an example convergence curve for Bunny mesh.

We demonstrated the robustness of our methods with respect to different triangulations, resolution and different noise. We performed both subdivision-based refinement and edge-collapse-based simplification operations to the original meshes, and obtained same object with different triangulations and resolution. We further introduced isotropic Gaussian noise to the vertices coordinates (in the DCM stage) and to the edge distances(in the reconstruction stage), then apply our algorithms again on these distorted meshes. The new results are still very stable ( for space limit, we have not present the results here), which indicates the methods are robust.

**Fig. 4.** DCM mapping results by our method

## 6 Conclusions

We have proposed a conformal method for representing arbitrarily-complex genus-zero 3D mesh objects. We have also demonstrated the possibility of reconstructing the shape from its representation. This is only the first step toward describing more complex and more general 3D objects (for example, objects with higher-genus). It is expected that the

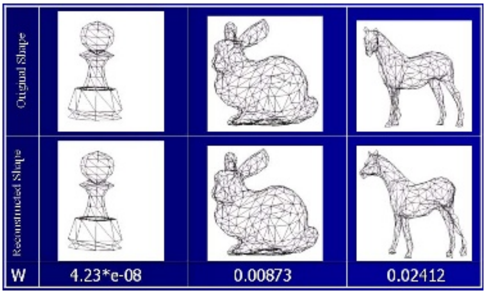


**Fig. 5.** Shape reconstruction from spring-embedding. (W is the final springs energy.)

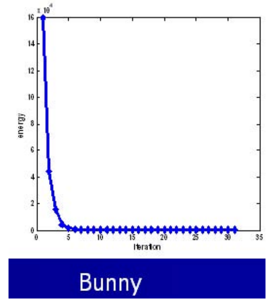


**Fig. 6.** A convergence curve. (energy .vs. iterations)

proposed method can find many practical applications, such as 3D geometry compression [1]and shape recognition[7][10]. For this purposes, more and much harder work still need to be done.

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