DISCONTINUOUS CONTROL IN BANACH SPACES

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Abstract The application of state-discontinuous feedback laws to infinite-dimensional control systems, with particular reference to sliding motions, is discussed for linear systems with distributed control. Using differential inclusions a definition of generalized solutions for the discontinuous closed loop system is introduced. Sliding modes can both be defined as viable generalized solutions or by extending the equivalent control method to infinite dimensional systems. Regularity properties of the sliding manifold under which the two methods are equivalent are investigated. Then, a comparison between classical results obtained for finite dimensional spaces and properties of infinite dimensional sliding modes is made.

keywords: Variable Structure Systems; Infinite Dimensional Systems; Sliding Mode Control.

1. Introduction

Variable structure control methods and in particular sliding mode controls, are by now recognised as classical tools for the regulation of systems governed by ordinary differential equations in a finite dimensional setting. For an overview of the finite-dimensional theory see [21]. While being easy to design, they possess attractive properties of robustness and insensitivity with respect to disturbances and unmodeled dynamics. These characteristics are all the more important when dealing with infinite-dimensional systems. Recent research has been devoted to the extension of sliding mode control and therefore the use of discontinuous feedback laws, to the infinite-dimensional setting. The early works [14, 15, 17] were confined to some special classes of systems, but at present both theory and application of sliding mode control have been extended to a rather general setting [18, 16, 19]. In particular in [18] the key concept of

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equivalent control is extended to evolution equations governed by unbounded linear operators that generate C_0-semigroups.

The application of a state-discontinuous feedback law brings about the question of how to define what is the meaning of solution for the resulting closed loop. This issue becomes crucial for sliding mode control, since one seeks to constrain the evolution of the system to belong to the feedback discontinuity manifold. For ordinary differential equations the problem is solved by introducing Filippov solutions [4]. In the Banach space setting a generalised solution concept has been proposed in [10, 9] and a relationship between the equivalent control method and generalised solutions of infinite-dimensional systems with discontinuous right-hand side has been established, under some regularity assumptions. In Section 3 these results are extended to a more general setting by requiring less stringent hypotheses on the interaction between the evolution operator and the sliding surface. This allows for more flexibility in the construction of the sliding manifold and this is of primary importance for application purposes.

2. Generalized solutions for affine discontinuous control systems

The setting of the paper is the following: we consider controlled differential equations of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(x(t)) \\ x(0) = x_0, \end{cases} \tag{1}$$

where x is the state variable and u is the control variable.

ASSUMPTION 1 *The following conditions are assumed to hold:*

(i) $A : \mathcal{D}(A) \subset X \to X$ *is the infinitesimal generator of a* C_0*-semigroup* $K(t)$, $t \geq 0$, *on the reflexive Banach space* X*;*

(ii) U *is a Banach space and* $B : U \to X$ *is a continuous linear operator;*

(iii) $u : \mathcal{D}(u) \subset X \to U$ *is a densely defined function that satisfies the growth condition*

$$\|u(x)\| \leq M\|x\| + N, \qquad \forall x \in \mathcal{D}(u) \tag{2}$$

for some positive constants M *and* N.

Following [10, 9] we introduce the following multivalued function F

$$F(x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\, Bu(\overline{\mathcal{B}}(x,\varepsilon) \cap \mathcal{D}(u)), \quad x \in X, \tag{3}$$

where $\overline{\mathcal{B}}(x, r)$ is the closed ball of center x and radius r. We call *generalized solution* of (1) a mild solution of the differential inclusion

$$\begin{cases} \dot{x}(t) - Ax(t) \in F(x(t)) \\ x(0) = x_0, \end{cases} \tag{4}$$

A continuous function $x : [0, T] \to X$ is called a mild solution of (4) if there exists $g \in L^1(0, T; X)$ with $g(s) \in F(x(s))$ for almost all $s \in [0, T]$ such that

$$x(t) = K(t)x_0 + \int_0^t K(t-s)g(s)\, ds, \qquad t \in [0, T]$$

(see i.e. [22, 2] and references therein for a discussion about mild solutions and existence theorems).

THEOREM 2 *[10] If u satisfies (2), $F(x)$ is a non void, closed, convex and bounded subset of X for all $x \in X$. Moreover, F is strongly-weakly upper semi-continuous and locally bounded and therefore there always exist mild solutions of (4).*

In what follows, we will be particularly interested in the following class of solutions: if S is a subset of X and $x_0 \in S$ a mild solution of inclusion (4) that satisfies $x(t) \in S$ for all $t > 0$ is called *viable* on S. S is a *viable domain* for (4) if for any $x_0 \in S$ there exists a viable solution of the differential inclusion starting from x_0. A *generalized viable solution* of (1) is a viable solution of (4). The results by Cârjă and Vrabie in [2, 3] can be applied to our differential inclusion, so that we have necessary and sufficient conditions for the existence of viable generalized solutions.

3. Sliding modes on linear sliding manifolds

From now on the attention is restricted to a particular class of control functions u.

ASSUMPTION 3 *Let Y be a Banach space, $C : X \to Y$ a continuous linear operator, $C \neq 0$ and $\mathcal{D}(u) = X \setminus S$, $S = \ker C$.*

Thus S is a proper linear subspace of X, with void interior and $\mathcal{D}(u)$ is dense. A *sliding mode* is attained when, upon reaching the surface S, the state is henceforth constrained to remain (slide) on it. From the control view point the choice of C has to be done in such a way that, once the evolution is constrained on the sliding surface, the control goal is fulfilled. Let us suppose that S has been selected and the existence of the sliding mode has been proved (this can generally be done using Lyapunov-like techniques). Mimicking the finite dimensional case, an equivalent control can be defined as a feedback law that selects a constrained motion on S from those allowed by the system (1) with

$x_0 \in S$. As in the classical theory existence and uniqueness of such a control law is necessary for well-posedness (in some sense), we require it also in this context. Therefore it is assumed that

ASSUMPTION 4 *The operator $CB : U \to CB(U) \subset Y$ is continuously invertible and $X = S \oplus B(U)$*

(observe that, since C is not given by the problem, but is a control tool this just poses restrictions on the construction of S, not on the class of control systems under consideration). Then define

$$u_{\text{eq}}(x) := -(CB)^{-1}CAx, \qquad \forall x \in S, \tag{5}$$

and call $Q = B(CB)^{-1}C$, $P = I - Q$ the projections on $B(U)$ along S and vice-versa respectively. The projected equation obtained by substituting u_{eq} in (1) is

$$\begin{cases} \dot{x} = (A - QA)x \\ x(0) = x_0 \in S. \end{cases} \tag{6}$$

The above differential equation is well-posed in a "classical sense" (see [5] for a discussion about this issue) whenever the operator $A - QA$ generates a strongly continuous semigroup $\tilde{K}(t)$, $t \geq 0$ on S. The following result gives a condition under which the equivalent control method just described is meaningful and relate it to the generalized solutions of Section 2.

THEOREM 5 *Let $S \cap \mathcal{D}(A)$ be dense in S and suppose that Assumptions 1, 3 and 4 hold. Suppose moreover that QA is a perturbation of Miyadera-Voigt type, i.e. that there exist $t_0 > 0$ and $q < 1$ such that*

$$\int_0^{t_0} \|QAK(t)x\|\, dt \leq q\|x\|, \quad \forall x \in \mathcal{D}(A). \tag{7}$$

Then $\tilde{A} = A - QA$ generates a C_0-semigroup $\tilde{K}(t)$, $t \geq 0$ on S.
Moreover the trajectory on S obtained through the equivalent control method is a generalized solution of (1) viable on S if and only if $Bu_{\text{eq}}(x) \in F(x)$ for all $x \in S \cap \mathcal{D}(A)$.

PROOF. Condition (7) assures that $A - QA$ generates a C_0-semigroup $H(t)$, $t \geq 0$ on X by the perturbation theorem of Miyadera and Voigt (for a proof see for example [5], Section III.3.c). It is easy to prove that S is $H(t)$-invariant, so that the restriction $\tilde{K}(t)$ of $H(t)$ on S is a semigroup on S generated by $\tilde{A}$. The invariance of $H(\cdot)$ is equivalent to this property: there exists $\omega \in \mathbb{R}$ such that for any $\lambda > \omega$ one has $\mathcal{R}(\lambda; A - QA)S \subset S$ ([20] Theorem 5.1 p. 121). If $y = \mathcal{R}(\lambda; A - QA)x$ for some $x \in S$, then $\lambda y - Ay + QAy = x$. Applying C we get $\lambda Cy = 0$, therefore $y \in S$ if $\lambda \neq 0$ and thus $A - QA$ is a generator on S.

For the second part of the proof, we need the following results:

LEMMA 6 (COROLLARY 3.16 IN [5]) *Let A generate the C_0-semigroup $K(t)$, $t \geq 0$ on X and Q be a continuous linear operator such that QA satisfies condition (7) for some $t_0 > 0$ and $q \in [0, 1)$. Then the semigroup $H(t)$, $t \geq 0$ generated by $A - QA$ satisfies*

$$H(t)x = K(t)x + \int_0^t K(t-s)QAH(s)x\,ds \tag{8}$$

$$\int_0^{t_0} \|QAH(s)x\|\,ds \leq \frac{q}{1-q}\,\|x\|, \tag{9}$$

for all $x \in \mathcal{D}(A)$, and any $t \geq 0$.

LEMMA 7 (THEOREM 4.8.3 AND COROLLARY 4.8.1 IN [1]) *Suppose X is a reflexive Banach space and $f : [0, T] \to X$ is in $L^1(0, T; X)$. Then there exists a unique function $x : [0, T] \to X$ which is weakly continuous and such that for each $y \in \mathcal{D}(A^*)$ one has*

$$\langle x(t), y\rangle = \langle x_0, y\rangle + \int_0^t \langle x(s), A^*y\rangle\,ds + \int_0^t \langle f(s), y\rangle\,ds, \qquad 0 \leq t \leq T \tag{10}$$

and this function is given by

$$x(t) = K(t)x_0 + \int_0^t K(t-s)f(s)\,ds.$$

Let us go back to the proof of the theorem. By the density assumption there exist a sequence $\{x_n\}$ in $\mathcal{D}(A) \cap S$ such that $x_n \to x$. Setting $z_n(t) = \tilde{K}(t)x_n$, by (8) and (10) one has

$$\langle z_n(t), y\rangle = \langle x_n, y\rangle + \int_0^t \langle z_n(s), A^*y\rangle\,ds - \int_0^t \langle QAz_n(s), y\rangle\,ds,$$

for all $t \geq 0$ and $y \in \mathcal{D}(A^*)$. As $\tilde{K}(t)$, $t \geq 0$ is a C_0-semigroup it follows that $\|z_n(t) - z(t)\| \to 0$ uniformly on compact subsets of $[0, +\infty)$, therefore

$$\lim_{n\to+\infty} \int_0^t \langle QAz_n(s), y\rangle\,ds = -\langle z(t), y\rangle + \langle x, y\rangle + \int_0^t \langle z(s), A^*y\rangle\,ds. \tag{11}$$

For any $t \geq 0$ and any x the vector $\int_0^t \tilde{K}(s)x\,ds$ is in $\mathcal{D}(\tilde{A}) \subset \mathcal{D}(A)$ and $\tilde{K}(t)x - x = \tilde{A}\int_0^t \tilde{K}(s)x\,ds$, thus

$$\begin{aligned}\int_0^t \langle z(s), A^*y\rangle\,ds &= \langle(\tilde{A} + QA)\int_0^t z(s)\,ds, y\rangle \\ &= \langle z(t), y\rangle - \langle x, y\rangle + \langle QA\int_0^t z(s)\,ds, y\rangle.\end{aligned}$$

Combining the above results, by the density of $\mathcal{D}(A^*)$ it follows that

$$\int_0^t QAz_n(s)\,ds \rightharpoonup QA\int_0^t z(s)\,ds, \qquad \text{for all } t \geq 0. \tag{12}$$

Note that this just depends on the fact that $\tilde{A}$ is a generator and a perturbation of A. Condition (9) will now be exploited to show that the above convergence holds also in the abstract Sobolev space $W^{1,1}(0,T;X)$, thus proving the thesis. To simplify notations let $f_n(t) = \int_0^t QAz_n(s)\,ds$ and $f(t) = QA\int_0^t z(s)\,ds$ for $t \geq 0$. Obviously $f_n \in AC(0,T;X)$ for any n and $T > 0$ by the absolute continuity of the Bochner integral. Moreover by (9), for $T \leq t_0$ and any n, m

$$\|f_n' - f_m'\|_{L^1(0,T;X)} = \int_0^T \|QA\tilde{K}(s)(x_n - x_m)\|\,ds \leq \frac{q}{1-q}\|x_n - x_m\|.$$

Therefore $\{f_n'\}$ is a Cauchy sequence in $L^1(0,T;X)$ and since this space is complete, there exists $h \in L^1(0,T;X)$ such that $f_n' \to h$ in $L^1(0,T;X)$. Using the same arguments it is easy to see that $\{f_n\}$ is convergent in $L^1(0,T;X)$ and by (12) the limit has to be f. The only thing to prove now is that in fact f is absolutely continuous and $h = f'$ almost everywhere. This can be done by a standard argument involving derivatives in the distribution sense, applied to the abstract setting. In fact let $\mathcal{D}'(0,T;X)$ be the space of X-valued distributions on $(0,T)$, i.e. $\mathcal{D}'(0,T;X) = \mathcal{L}(\mathcal{D}(0,T),X)$. The derivative of a distribution in $\mathcal{D}'(0,T;X)$ is defined in the usual way and for $f \in L^1(0,T;X)$, $\varphi \in \mathcal{D}(0,T)$ it gives

$$\left(\frac{d}{dt}f\right)(\varphi) := -f(\varphi') := -\int_0^T f(s)\varphi'(s)\,ds.$$

Therefore for any $\varphi \in \mathcal{D}(0,T)$

$$\begin{aligned} -\int_0^T f(s)\varphi'(s)\,ds &= \lim_{n\to+\infty} -\int_0^T f_n(s)\varphi'(s)\,ds \\ &= \lim_{n\to+\infty} \int_0^T f_n'(s)\varphi(s)\,ds \\ &= \int_0^T h(s)\varphi(s)\,ds, \end{aligned}$$

that is $f' = h$ in $L^1(0,T;X)$.
From (11) it then follows that

$$\langle z(t), y\rangle = \langle x, y\rangle + \int_0^t \langle z(s), A^*y\rangle\,ds - \int_0^t \langle h(s), y\rangle\,ds,$$

where $h \in L^1(0,T;X)$

$$h(s) = \frac{d}{ds}QA\int_0^s z(r)\,dr, \qquad \text{a. e. } s \in [0,T]$$

and the thesis follows from Lemma 7. △

REMARK 8 *Observe that the condition on the equivalent control stated in the above result is also necessary in the finite dimensional setting in order that a sliding mode on S is feasible once the control law u has been chosen.*

Theorem 5 extends similar results in [10, 9], considerably enlarging the class of control systems for which the stated equivalence is valid. In [10] the operator A was assumed to generate a compact semigroup, while in [9] the requirement was the extendibility on S of the operator QA. Suppose that U is finite-dimensional, or for simplicity that the control is scalar. Then $Cx = \langle \gamma, x \rangle$ for some $\gamma \in X^*$ and QA admits an extension iff $\gamma \in \mathcal{D}(A^*)$, while this condition is not required for (7) to be verified. For example let $X = L^2(0,1)$ and A be the unbounded operator associated to the heat equation with Dirichlet boundary conditions, i.e. $Ax = x''$ with $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$. The input operator is $Bu = ub$ with $u \in I\!R$, $b \in X$ while C is chosen as above; to simplify matters suppose that $\gamma \in L^2(0,1)$ is such that $(\gamma, b) = 1$, where $(\cdot, \cdot)$ is the usual scalar product in X. Therefore $Qx = b(\gamma, x)$ and we have

$$\|QAK(t)x\|^2 = \int_0^1 b^2(\xi)(\gamma, AK(t)x)^2 d\xi = \|b\|^2(\gamma, AK(t)x)^2$$

so that

$$\int_0^h \|QAK(t)x\| = \|b\| \int_0^h |(\gamma, AK(t)x)| dt.$$

Recall now that A is the generator of an analytic semigroup, therefore fractional powers of A are well defined. Let $\gamma \in \mathcal{D}(A^{1/2})$. Then $(\gamma, AK(t)x) = (A^{1/2}\gamma, A^{1/2}K(t)x)$ and exploiting classical results about fractional powers of operators (see i.e. [20]) we get

$$\int_0^h |(\gamma, AK(t)x)| dt \leq \|A^{1/2}\gamma\| \int_0^h \|A^{1/2}K(t)x\| dt \leq C\|x\| \int_0^h t^{-1/2} dt$$

for small h. Since the right-hand side tends to zero for h tending to zero, there exist t_0 and q so that condition (7) is satisfied. Note that in this case a continuous extension of QA would require $\gamma \in \mathcal{D}(A) \subset \mathcal{D}(A^{1/2})$.

Observe also that in the finite dimensional case the equivalent control is defined everywhere on S and is continuous, therefore although the chosen feedback law is discontinuous, its effect in sliding motion is equivalent to the enforcement of a continuous control $u_{\text{eq}}(x)$. This is no more true for general Banach spaces. Now u_{eq} is only densely defined and A-bounded. The equation of motion on S is regulated by $\dot{x} = -Ax + QAx$ and QA is an unbounded perturbation. The control giving the constrained motion is not continuous, thus the application of a discontinuous control law in (1) results in an evolution obtained through an unbounded perturbation of the generator A. In the next example we show now how it is possible to interpret the equivalent control as a boundary feedback. Let $H = L^2(\Omega)$ with $\Omega \subset I\!R^N$ open, bounded with "smooth" boundary Γ. Consider the differential problem

$$\begin{cases} z_t(t,\xi) - A(\xi,\partial)z(t,\xi) = 0 & \text{in } (0,T] \times \Omega \\ z(0,\xi) = z_0(\xi), & \text{in } \Omega \\ z(t,\sigma) = (z(t,\cdot),\omega)g(\sigma) & \text{in } (0,T] \times \Gamma \end{cases}$$

where $A(\xi,\partial)$ is a second order elliptic differential operator, $z_0 \in H$ and $g \in L^2(\Gamma)$. Let γ be the trace operator of restriction on Γ and $\mathcal{D}(A) = H^2(\Omega) \cap \ker\gamma$, $(Ax)(\xi) = A(\xi,\partial)x(\xi)$ on $\mathcal{D}(A)$. Then $-A$ generates a C_0-semigroup on H. Let

$$D : L^2(\Gamma) \to \mathcal{D}(A^{1/4-\varepsilon}) = H^{1/2-2\varepsilon}$$

be the Dirichlet map $v = Dg$ iff $A(\xi,\partial)v = 0$, $\gamma v = g$.
The differential problem can be reformulated in semigroup form (see results in [6] for the parabolic case and [7] for hyperbolic systems)

$$\dot{z} = A_F z, \qquad z(0) = z_0$$

where

$$\begin{aligned} A_F z &= -A[I - Dg(z,\omega)] \\ \mathcal{D}(A_F) &= \{z \in H : z - Dg(z,\omega) \in \mathcal{D}(A)\}. \end{aligned}$$

Now as $A = A^*$

$$A_F^* = -Ax + (Ax, Dg)w, \quad \mathcal{D}(A_F^*) = \mathcal{D}(A^*).$$

Let us now consider a scalar sliding mode control for the abstract system

$$\dot{x} + Ax = bu, \qquad u \in I\!R,\ b \in H$$

with sliding surface $S = \{x \in H : (x,c) = 0\}$ with $(b,c) = 1$. From the application of the equivalent control method the evolution in sliding is governed by the projected differential equation

$$\dot{x} = -Ax + (Ax,c)b.$$

The relation with the Dirichlet boundary feedback problem above is now straightforward by considering as ω the function b corresponding to the input operator B and choosing c as the relevation Dg of an $L^2(\Gamma)$ function.

REMARK 9 *Note that in the above case, using the stated generation result of the feedback operator A_F, we can require weaker regularity on the function c in order that the equation of the sliding mode is well-posed, as opposed to the application of Theorem 5. Consider, however, that the above improvement of the result is strongly related to the analyticity properties of the semigroup that governs the evolution, while Theorem 5 is valid for a larger class of systems.*

References

[1] A. V. Balakrishnan. *Applied functional analysis*. Applications of Mathematics, No. 3, Springer-Verlag, 1976.

[2] O. Cârjă, I. I. Vrabie. Some new viability results for semilinear differential inclusions. *NoDEA Nonlinear Differential Equations Appl.* 4:401–424, 1997.

[3] O. Cârjă, I. I. Vrabie. Viability for semilinear differential inclusions via the weak sequential tangency condition. *J. Math. Anal. Appl.* 262:24–38, 2001.

[4] A. F. Filippov. *Differential equations with discontinuous righthand sides*. Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers Group, Dordrecht, 1988.

[5] K. J. Engel, R. Nagel. *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, 2000.

[6] I. Lasiecka, R. Triggiani. Feedback semigroups and cosine operators for boundary feedback parabolic and hyperbolic equations. *J. Differ. Equ.* 47:246–272, 1983.

[7] I. Lasiecka, R. Triggiani. Finite rank, relatively bounded perturbations of semigroups generators. I. Well-posedness and boundary feedback hyperbolic dynamics. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12:641–668, 1985.

[8] I. Lasiecka, R. Triggiani. Finite rank, relatively bounded perturbations of semigroups generators. II. Spectrum and Riesz basis assignment with applications to feedback systems. *Ann. Mat. Pura Appl. (4)*, 143:47–100,1986.

[9] L. Levaggi. Infinite dimensional systems' sliding motions. *Eur. J. Control* 8:508–516, 2002.

[10] L. Levaggi. Sliding modes in Banach spaces. *Differ. Integral Equ.* 15:167–189, 2002.

[11] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.

[12] J.-L. Lions. *Optimal control of systems governed by partial differential equations*. Die Grundlehren der mathematischen Wissenschaften, Band 170, Springer-Verlag, New York, 1971.

[13] J.-L. Lions, E. Magenes. *Non-homogeneous boundary value problems and applications. vol. i.* Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York, 1972.

[14] Yu. V. Orlov, V. I. Utkin. Use of sliding modes in distributed system control problems. *Automat. Remote Control* 43:1127–1135, 1982.

[15] Yu. V. Orlov, V. I. Utkin. Sliding mode control in indefinite-dimensional systems. *Automatica J. IFAC* 23:753–757, 1987.

[16] Y. Orlov, D. Dochain. Discontinuous feedback stabilization of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor. *IEEE Trans. Automat. Control* 47:1293–1304, 2002.

[17] Y. Orlov, V. I. Utkin. Unit sliding mode control in infinite-dimensional systems. *Appl. Math. Comput. Sci.* 8:7–20, 1998.

[18] Y. V. Orlov. Discontinuous unit feedback control of uncertain infinite-dimensional systems. *IEEE Trans. Automat. Control* 45:834–843, 2000.

[19] Y. Orlov, Y. Lou, P. D. Christofides. Robust stabilization of infinite-dimensional systems using sliding-mode output feedback control. *Internat. J. Control* 77:1115–1136, 2004.

[20] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983.

[21] V. I. Utkin. *Sliding modes in control and optimization*. Communications and Control Engineering Series, Springer-Verlag, Berlin, 1992.

[22] I. I. Vrabie. *Compactness methods for nonlinear evolutions*. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 32, Longman Scientific & Technical, Harlow, 1987.

[23] Tullio Zolezzi. Variable structure control of semilinear evolution equations, in *Partial differential equations and the calculus of variations - Vol. II*, Progr. Nonlinear Differential Equations Appl., Birkhäuser, Boston, MA, 1989, pp. 997–1018.