



Chaos suppression in fractional order systems using state-dependent noise

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Abstract

Noise play a creative role in the evolution of periodic and complex systems which are essential for continuous performance of the system. The interaction of noise generated within one component of a chaotic system with other component in a linear or nonlinear interaction is crucial for system performance and stability. These types of noise are inherent, natural and insidious. This study investigates the effect of state-dependent noise on the bifurcation of two chaotic systems. Circuit realization of the systems were implemented. Numerical simulations were carried out to investigate the influence of state dependent noise on the bifurcation structure of the Chen and Arneodo–Coullet fractional order chaotic systems. Results obtained showed that state dependent noise inhibit the period doubling cascade bifurcation structure of the two systems. These results pose serious challenges to system reliability of chaotic systems in control design, secure communication and power systems.

Keywords Noise · Fractional order · Chaos bifurcation · Chen system**PACS** 05.45.Pq · 05.45.Gg · 05.45.Ac

1 Introduction

Fractional order models are best suited to capture the memory effects in mathematical models [1]. The lack of appropriate methods of solution has made fractional order systems unattractive to scientists. However, in recent times, adequate mathematical representation and solutions has created a resurgence and renewed interest in the subject of fractional order. The order of fractional derivative is an index of memory [2] and as stable as their integer order counterpart [3]. Due to their relevance, fractional order models have been considered in finance [4], psychology [5] and electronic systems [6].

Noise is ubiquitous and inherent in nature. The role of noise in a system depends on how it was introduced and intended function of the system. Noise is desirable in some

systems while it is unwanted in others. The threshold of chaos can be increased or reduced through noise perturbations [7] and synchronization of chaotic oscillators with noise [8]. Chaos suppression due to noise has been reported in the Duffing oscillator [9], parametrically driven Lorenz system [10], Belousov-Zhabotinsky reaction [11] and logistic map [12]. Dennis et al. [13] suggests that noise cannot induce chaos. However, the introduction of noise has also been found to enhance synchronization of chaotic systems [14, 15] and induce chaos in systems [16–18]. Gao [19] showed that noise can induced chaos when the noise level is within a narrow range. It has been posited that real life neurons are robust to noise [20].

The possibility of generating noise from chaotic systems has been considered. The perceived similarity between chaos and random numbers make chaos a suitable

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Table 1 Statistical properties of state-dependent noise considered

	Chen system			Arneodo–Coullet system		
	U_1	U_2	$\xi(U_1, U_2)$	U_1	U_2	$\xi(U_1, U_2)$
Mean	0.8683	0.8808	−0.0042	1.8461	0.0809	0.0051
Median	1.6761	1.8132	0.3603	1.9403	0.3346	0.3774
STD	7.6955	8.3352	1.6158	0.8544	1.7650	0.9030
Skewness	−0.1833	−0.1776	0.0524	−0.3723	−0.4250	0.1719

candidate to realize truly random numbers. Murch and Bates [21] proposed a mechanism for generating coloured noise through by hierarchies of nonlinear recursive loops. This was extended to study the performance of random noise generated by bounded and unbounded chaos with a gaussian noise [22]. A chaotic digital phase-locked loop system has also been developed to generate pseudo-random noise [23]. The successful generation of large scale chaos random number generator was implemented using Duffing oscillators [24]. Random numbers generated from chaotic sequences have been employed in video encryption [25]. By using shifters and multiplexers, the phase space from the Lorenz and Lu systems have been utilized to generate random numbers on Field programmable gate arrays [26].

The role of different types of noise has been investigated on integer and discrete chaotic systems. There is the need to understand the dynamic influence of noise on the memory effect of fractional order chaotic systems. Since memory is an intrinsic property of a system, it is intuitive to understudy the interaction of noise generated within the system states and the system itself. Previous researches have considered the role of external noise on the dynamics of chaotic system. Although external noises play a significant role, the impact

of intrinsic noise cannot be ruled out. In our approach, we aim to generate noise from within a chaotic system and feed the same noise back into the system. This approach is different from the other approaches which uses external noise or generate noise from a chaotic system but do not consider its effect as feedback noise. This research is motivated by the need to understand noise generation by fractional order systems and the influence of the generated noise as a feedback into the system. The aim of this study is to investigate the impact of state-dependent noise in a class of fractional order chaotic systems. The potential of chaotic system in secure communication makes this study important. It will provide information to systems designer on the role of noise in the system, as well as, the impact of internally generated noise on system performance. In Sect. 2, the concept and definition of fractional order systems, state dependent noise, as well as, systems to be considered are introduced. In Sect. 3, we discuss the result of our analysis while our conclusions are presented.

2 Methodology

2.1 Fractional order derivatives

Fractional derivatives have been defined in different ways [27]. The Caputo’s definition of fractional order differ-integral equations are given as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - m)} \int_a^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha+1-m}} d\tau \tag{1}$$

where $m - 1 < \alpha \leq m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ is a fractional order of the differ-integral of the function $f(t)$ [28, 29].

Another common definition is Riemann–Liouville, which can be denoted as:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau \tag{2}$$

In Eq. 2, $\Gamma(\cdot)$ is the gamma function, where $n - 1 < \alpha < n$ and n is an integer. The Laplace transform of Eq. 2 can be written as

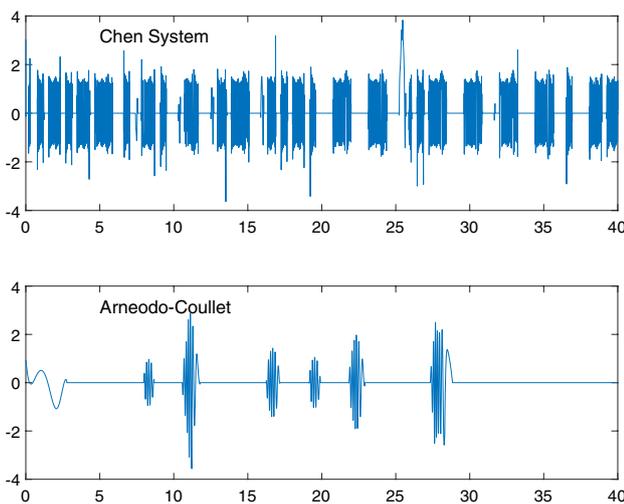


Fig. 1 State dependent noise for (top figure) the Chen system with system parameters $(a, b, c) = (40, 3, 28)$ using fractional order 0.9 (bottom figure) the Arneodo–Coullet system with parameters $(e, f) = (65.0)$ using fractional order 0.9

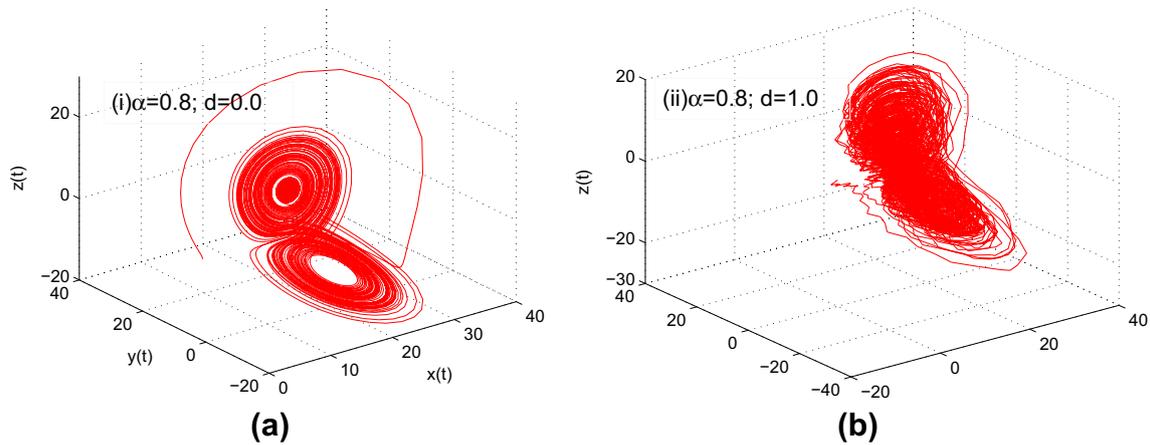


Fig. 2 3D Phase space representation of system 8 with fractional order $\alpha = 0.8$ and noise level, d **a** 0.0; and **b** 1.0

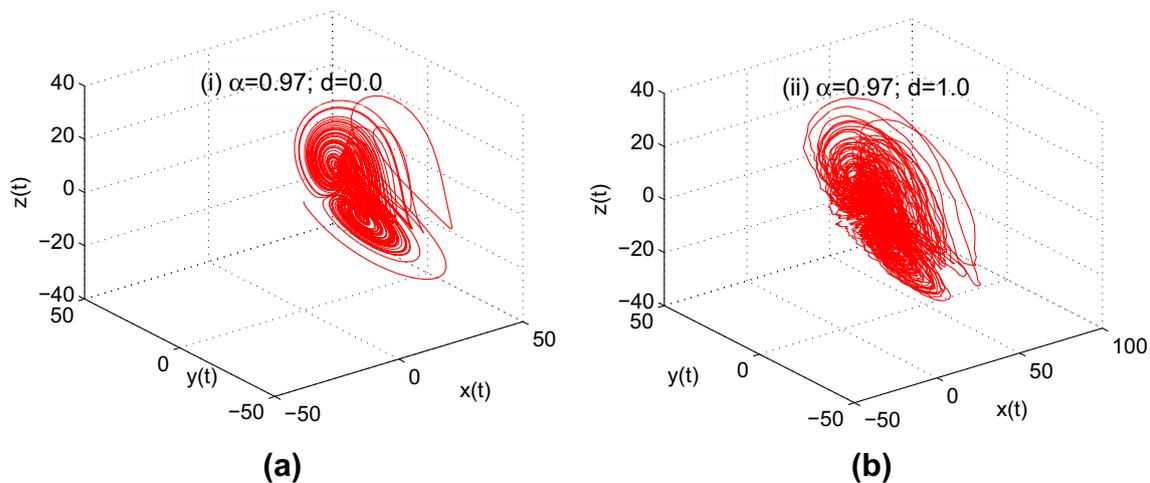


Fig. 3 3D Phase space representation of system 9 with fractional order $\alpha = 0.97$ and noise level, d **a** 0.0; and **b** 1.0

$$L \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha L \{ f(t) \} - \sum_{k=0}^{n-1} s^k \left\{ \frac{d^{\alpha-1-k} f(t)}{dt^{\alpha-1-k}} \right\}_{t=0} \quad (3)$$

If the initial conditions are zero for Eq. 3, it implies the Riemann–Liouville fractional derivative is

$$L \left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha L \{ f(t) \} \quad (4)$$

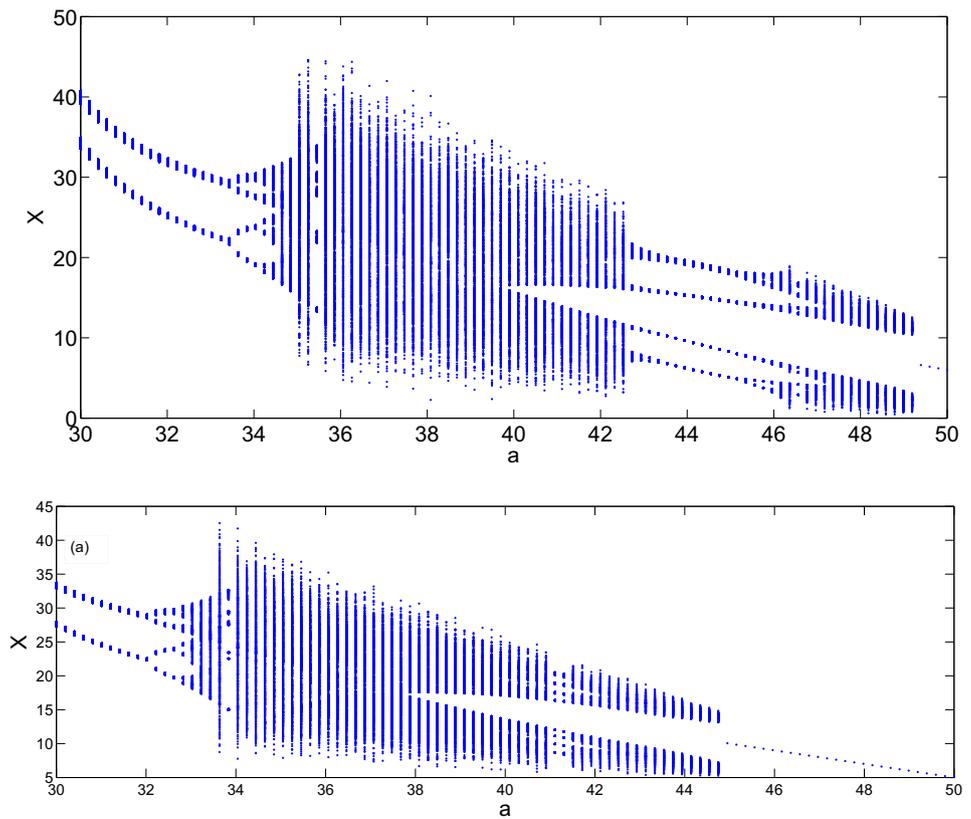
However, the transfer function $F(s)$ for the fractional integral operator of order α can therefore be expressed as $F(s) = \frac{1}{s^\alpha}$ in the frequency domain. With approximation approach, the errors of 2 dB have been deduce for fractional integral operator $\frac{1}{s^\alpha}$ [30]. Fractional order systems

are said to be commensurate if the orders are the same (ie $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$).

2.2 State dependent noise

We define a state-dependent noise as feedback noise arising from one or more components of the system. Consider a system defined as $f(x, y, z)$. If a fraction of the output y is made to interfere with the input x , a state-dependent noise has been introduced. This state-dependent noise can be a linear or nonlinear function of the state variables. Box and Muller [31] proposed that using Eq. 5, two random number streams (U_1 and U_2) can be used to generate standard random numbers. In this study, we consider

Fig. 4 Bifurcation diagram of the Chen system (Equation 8) with fractional order $\alpha = 0.97$ and noise level, d (top) 0.0; and (bottom) 1.0



two systems $f(x, y, z)$ where the state z is perturbed with feedback noise which is a nonlinear function of $x = U_1$ and $y = U_2$ states.

$$\xi(U_1, U_2) = d \left(\sqrt{-2 \log U_1} \cos 2\pi U_2 \right) \tag{5}$$

where d is the noise strength. If the states of the system are independently chaotic, the noise $\xi(x, y)$ can be regarded as a chaotic noise. The statistical properties of $\xi(U_1, U_2)$ are shown in Table 1 and the time series in Fig. 1.

2.3 Systems

The Chen and Ueta [32] and Arneodo et al. [33] chaotic systems were considered in this study. The integer order Chen system is defined as

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= (c - a)x - xz + cy \\ \dot{z} &= (xy - bz) \end{aligned} \tag{6}$$

This system has been shown to be chaotic when $a = 35$, $b = 3$, and $c = 28$. Chen system is related to but not topological equivalent to either the Lorenz [34] or Rössler [35] chaotic systems.

The Arneodo–Coulet chaotic system is defined as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= ex - fy - z - x^3 \end{aligned} \tag{7}$$

The fractional order form of the Chen and Arneodo–Coulet systems with continuous time, state-space noise (Sect. 2.2) are given in Eqs. 8 and 9 respectively.

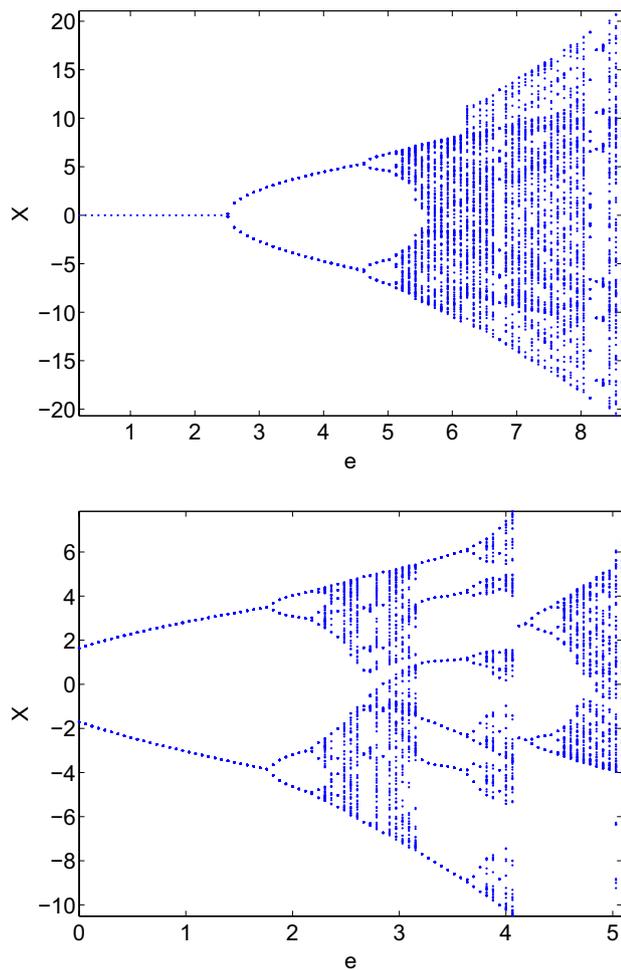


Fig. 5 Bifurcation diagram of the Arneodo–Coulet system (Equation 9) with fractional order $\alpha = 0.97$ and noise level, d (top) 0.0; and (bottom) 0.5

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y - x) \\ \frac{d^\alpha y}{dt^\alpha} &= (c - a)x - xz + cy \\ \frac{d^\alpha z}{dt^\alpha} &= (xy - bz) + \xi(x, y) \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= y \\ \frac{d^\alpha y}{dt^\alpha} &= z \\ \frac{d^\alpha z}{dt^\alpha} &= ex - fy - z - x^3 + \xi(x, y) \end{aligned} \tag{9}$$

The constants are given as $a = 35, b = 3, c = 28, e = 5.5$ and $f = 5.0$ for the two systems. The circuit realization and phase space for the two fractional order systems are shown in “Appendix A”.

3 Results and discussion

Numerical simulations of systems (8) and (9) using different fractional order, α and noise level were implemented. Using fractional orders of 0.8 (Fig. 2) and 0.97 (Fig. 3), the 3D phase space were found to experience distortion. With a fractional order of 0.97, the bifurcation diagram of Chen system was found to undergo significant changes (Fig. 4). Without the influence of state-dependent noise, the system is chaotic in the region $36 < a < 42.5$, however, the introduction of state-dependent noise with intensity $d = 1.0$, this regime was found in the range $34 < a < 44$. The periodic regime ($42.5 < a < 45$) and chaotic regime ($45 < a < 49$) were not observed when state dependent noise was introduced into the system. Similar distortions were found in the phase space (Fig. 3) and bifurcation (Fig. 5) diagram of the Arneodo–Coulet systems. In the bifurcation diagram of the Arneodo–Coulet system, the period doubling route to chaos was preserved but distorted. The introduction of state-dependent noise was found to suppress chaos in the two systems considered. For increased values of d , it is expected that the chaotic state of the system will be destroyed. The inhibition of the period doubling cascade by noise has also been reported in [19]. Intrinsic, although not state-dependent, noise have been found to induce synchronization, coherence and stochastic resonance in the CO₂ laser system [36]. The results presented supports several reports in literature which suggests that noise suppress chaos [9–12].

Serletis et al. [37] used extrinsic additive noise to show that stochastic noise changes the dynamics of low dimension chaotic systems. In this work, we have shown that intrinsic additive noise can also shift bifurcation points and produce noise-induced transitions. The impact of external stochastic noise on the phase space of Chen system has been reported [38]. The distortions reported were found to be distinctly different from what was obtained in this report. In this report, the phase space was found to be distorted while Bashkirtseva et al. [38] showed that small extrinsic noise perturbation

were found to create a stochastic attractor. It was reported in [39] that delayed self-controlling feedback noise caused stabilization of the system, however, this was not obvious in our analysis.

4 Conclusion

In this work, we have considered the effect of state dependent noise on the Chen and Arneodo–Coullet chaotic systems. The impact of external noise on the dynamics of chaotic systems have been investigated extensively. The role of internally generated noise fed back into the system has not been given much attention. The noise considered are dependent on feedback from other states of the system as against coloured noise. Hence, they can be regarded as intrinsic noise. The overall effect of state-dependent noise on the system studied is the suppression of chaos. The suppression of chaos by the intrinsic noise considered in this work was done by distortion of the route to chaos and phase space. In the two systems considered, the feedback noise from components of the system was found to break the period-doubling route to chaos. Thus, this approach will prove useful in the understanding of nonlinearity in systems susceptible to interference from other states of the system. The noise proposed in this study can be used as a

convenient way to control chaos in a system using the system variables. Further exploration can be made into the role of intrinsic feedback noise, as considered in this work, on the performance of chaotic systems in secure communication implementation. It will also be interesting to investigate stochastic and vibrational resonance in systems with intrinsic chaotic noise [40].

Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

Appendix A. Circuit diagrams and simulation

The circuit representation and resultant phase space are shown here. The analog implementation of the two proposed systems can likewise be generated with possibility of injecting Pseudo-random noise generator. The electronic circuits consists of chain fractance (combined capacitor and resistors), operational amplifiers (TL081CD), multipliers (A_3), resistors and capacitors with power supply unit. Figure 6 display the analog circuit realization of chaotic Chen and Arneodo–Coullet system as well as the phase space representation.

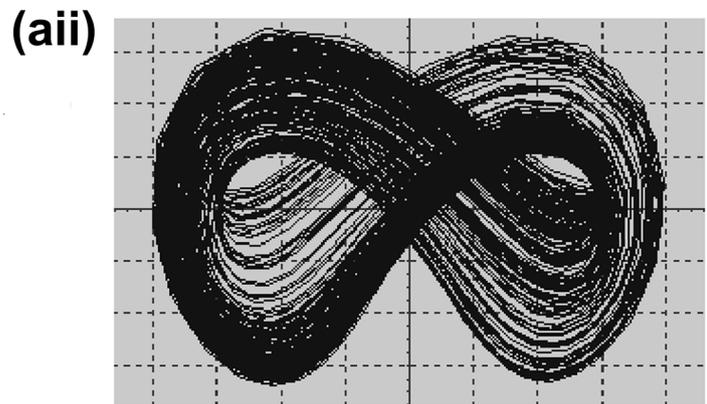
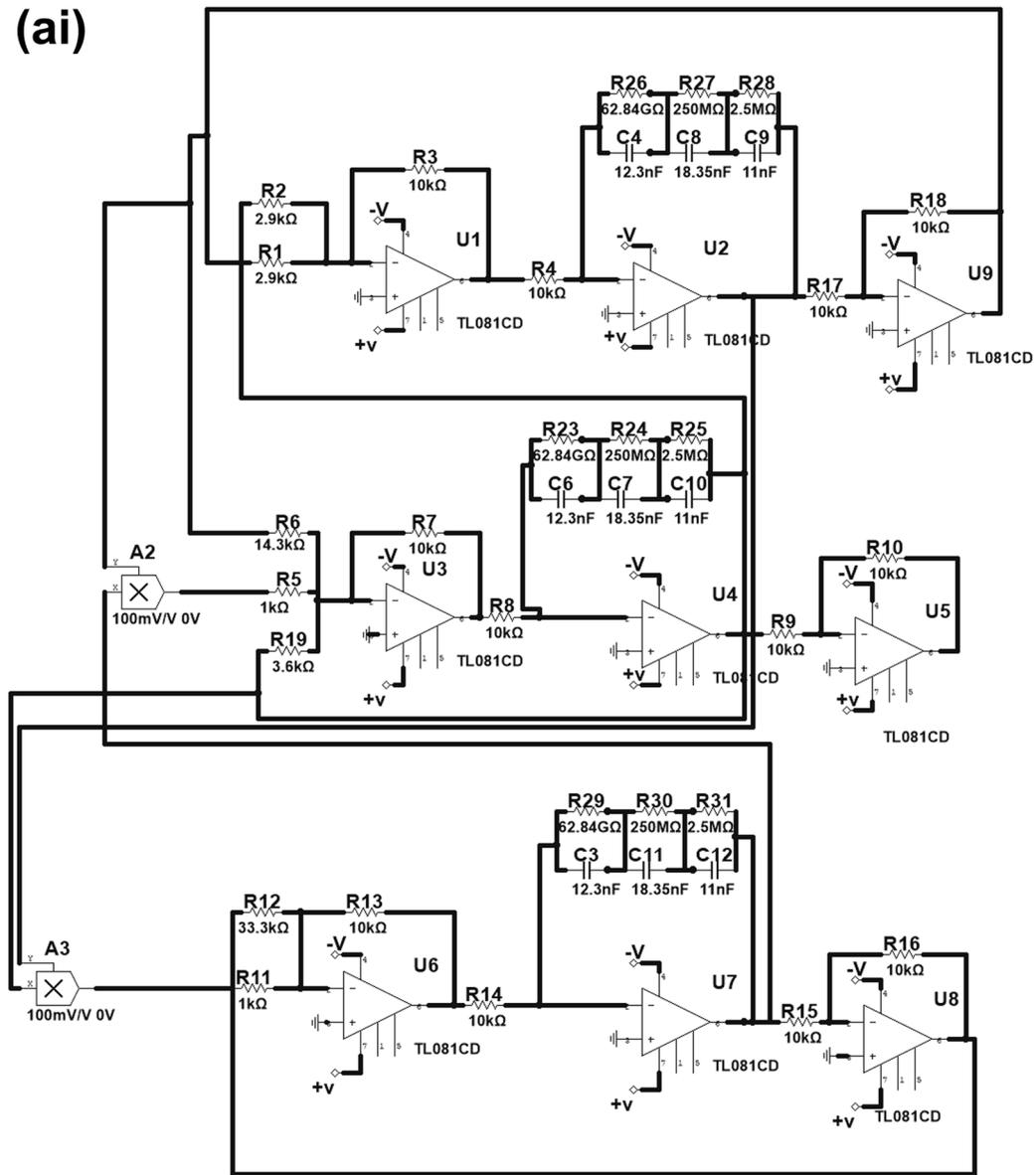


Fig. 6 Circuit diagram and phase space representation for the Chen (Eq. 8) and Arneodo–Coulet (Eq. 9) systems

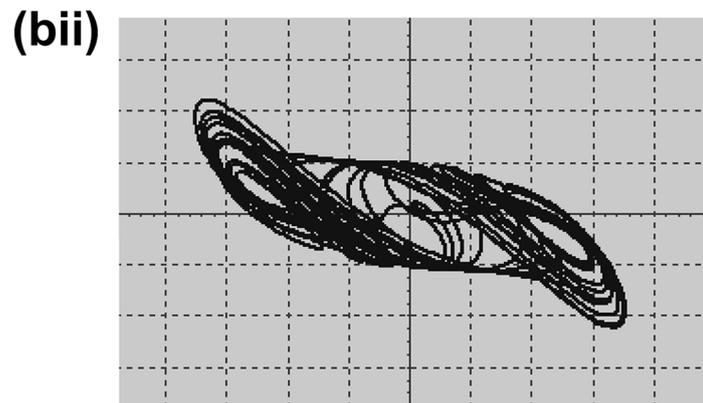
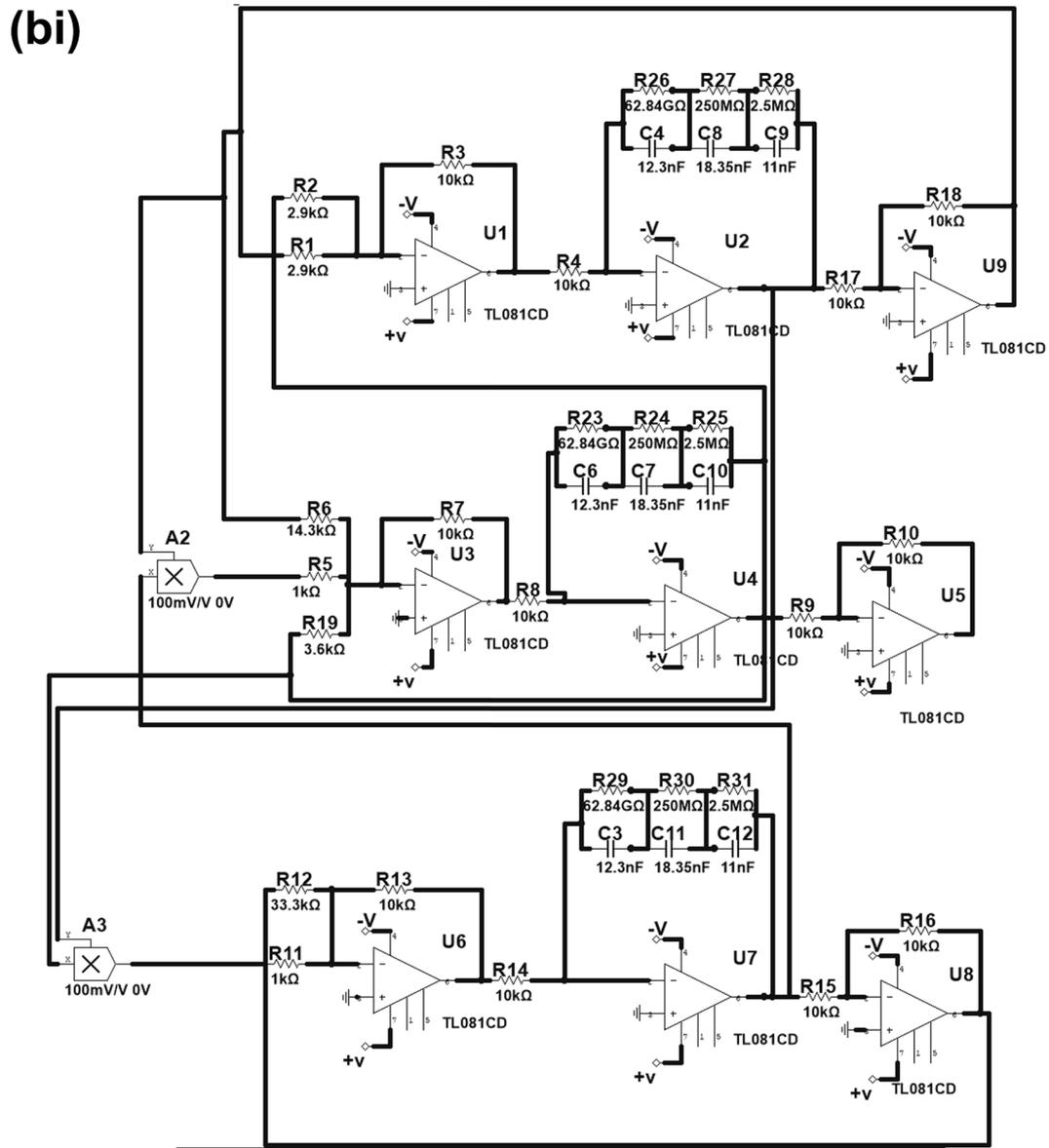


Fig. 6 (continued)

References

- Ogunjo ST, Ojo KS, Fuwape IA (2018) Chapter 15—multi-switching synchronization between chaotic fractional order systems of different dimensions. In: Azar AT, Radwan AG, Vaidyanathan S (eds) *Mathematical techniques of fractional order systems*. *Advances in nonlinear dynamics and chaos (ANDC)*. Elsevier, Amsterdam
- Du M, Zaihua W, Hu H (2013) Measuring memory with the order of fractional derivative. *Sci Rep* 3:3431
- Ahmed E, El-Sayed AMA, El-Saka HAA (2007) Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models. *J Math Anal Appl* 325(1):542–553
- Chen W-C (2008) Nonlinear dynamics and chaos in a fractional-order financial system. *Chaos Solitons Fractals* 36(5):1305–1314
- Ahmad WM, El-Khazali R (2007) Fractional-order dynamical models of love. *Chaos Solitons Fractals* 33(4):1367–1375
- Petrás I (2010) Fractional-order memristor-based Chua's circuit. *IEEE Trans Circuits Syst II Express Briefs* 57(12):975–979
- Fronzoni L, Giocondo M (1998) Controlling chaos with parametric perturbations. *Int J Bifurc Chaos* 8(08):1693–1698
- Astakhov VV, Anishchenko VS, Kapitaniak T, Shabunin AV (1997) Synchronization of chaotic oscillators by periodic parametric perturbations. *Phys D Nonlinear Phenom* 109(1–2):11–16
- Chacón R (1995) Suppression of chaos by selective resonant parametric perturbations. *Phys Rev E* 51(1):761
- Choe C-U, Höhne K, Benner H, Kivshar YS (2005) Chaos suppression in the parametrically driven lorenz system. *Phys Rev E* 72(3):036206
- Matsumoto K, Tsuda I (1983) Noise-induced order. *J Stat Phys* 31(1):87–106
- Rajasekar S (1995) Controlling of chaotic motion by chaos and noise signals in a logistic map and a Bonhoeffer–van der Pol oscillator. *Phys Rev E* 51(1):775
- Dennis B, Desharnais RA, Cushing JM, Henson SM, Constantinou RF (2003) Can noise induce chaos? *Oikos* 102(2):329–339
- Lin W, Chen G (2006) Using white noise to enhance synchronization of coupled chaotic systems. *Chaos Interdiscip J Nonlinear Sci* 16(1):013134
- Sanchez E, Matias MA, Perez-Munuzuri V (1997) Analysis of synchronization of chaotic systems by noise: an experimental study. *Phys Rev E* 56(4):4068
- Wei DQ, Luo XS (2009) Noise-induced chaos in single-machine infinite-bus power systems. *EPL (Europhys Lett)* 86(5):50008
- Zhou CS, Kurths J, Allaria E, Boccaletti S, Meucci R, Arecchi FT (2003) Noise-enhanced synchronization of homoclinic chaos in a co 2 laser. *Phys Rev E* 67(1):015205
- Qin YH, Li JC (2014) Random parameters induce chaos in power systems. *Nonlinear Dyn* 77(4):1609–1615
- Gao JB, Hwang SK, Liu JM (1999) When can noise induce chaos? *Phys Rev E* 82(6):1132
- Carroll TL (2005) Chaotic systems that are robust to added noise. *Chaos* 15:013901
- Murch AR, Bates RHT (1990) Colored noise generation through deterministic chaos. *IEEE Trans Circuits Syst* 37(5):608–613
- Demir K, Ergün S (2018) An analysis of deterministic chaos as an entropy source for random number generators. *Entropy* 20(12):957
- Bernstein GM, Lieberman MA (1990) Secure random number generation using chaotic circuits. *IEEE Trans Circuits Syst* 37(9):1157–1164
- Liao T-L, Wan P-Y, Yan J-J (2019) Design of synchronized large-scale chaos random number generators and its application to secure communication. *Appl Sci* 9(1):185
- Hui X, Tong X, Meng X (2016) An efficient chaos pseudo-random number generator applied to video encryption. *Optik* 127(20):9305–9319
- Rezk AA, Madian AH, Radwan AG, Soliman AM (2019) Reconfigurable chaotic pseudo random number generator based on fpga. *AEU Int J Electron Commun* 98:174–180
- Podlubny I, Petrás I, Vinagre BM, O'leary P, Dorčák L (2002) Analogue realizations of fractional-order controllers. *Nonlinear Dyn* 29(1–4):281–296
- Ogunjo ST, Ojo KS, Fuwape IA (2017) Comparison of three different synchronization schemes for fractional chaotic systems. In: Azar AT, Radwan AG, Vaidyanathan S (eds) *Fractional order control and synchronization of chaotic systems*. *Studies in computational intelligence*. Springer, Berlin, pp 471–495
- Dzieliński A, Sierociuk D, Sarwas G (2011) Some applications of fractional order calculus. *Bull Pol Acad Sci Tech Sci* 58(4):583–592
- Ahmad WM, Sprott JC (2003) Chaos in fractional-order autonomous nonlinear systems. *Chaos Solitons Fractals* 16(2):339–351
- Box GEP, Muller ME (1958) A note on the generation of random normal deviates. *Ann Math Stat* 29:610
- Chen G, Ueta T (1999) Yet another chaotic attractor. *Int J Bifurc Chaos* 09(07):1465–1466
- Arneodo A, Coulet P, Tresser C (1981) Possible new strange attractors with spiral structure. *Commun Math Phys* 79(4):573–579
- Lorenz EN (1963) Deterministic nonperiodic flow. *J Atmos Sci* 20(2):130–141
- Rössler OE (1976) An equation for continuous chaos. *Phys Lett A* 57(5):397–398
- Zhou CS, Kurths J, Allaria E, Boccaletti S, Meucci R, Arecchi FT (2003) Constructive effects of noise in homoclinic chaotic systems. *Phys Rev E* 67(6):066220
- Serletis A, Shahmoradi A, Serletis D (2007) Effect of noise on the bifurcation behavior of nonlinear dynamical systems. *Chaos Solitons Fractals* 33(3):914–921
- Bashkirtseva I, Chen G, Ryashko L (2012) Analysis of noise-induced transitions from regular to chaotic oscillations in the Chen system. *Chaos Interdiscip J Nonlinear Sci* 22(3):033104
- Pyragas K (1992) Continuous control of chaos by self-controlling feedback. *Phys Lett A* 170(6):421–428
- Adelakun AO, Ogunjo ST, Fuwape IA (2019) Chaos suppression in fractional order systems using state-dependent noise arXiv preprint arXiv, [arXiv:1902.05526](https://arxiv.org/abs/1902.05526)

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