Groups with Exactly Ten Centralizers

S. M. Jafarian Amiri¹ · H. Madadi² · H. Rostami³

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Abstract

In this article, we determine all groups with exactly ten element centralizers. Also we obtain the maximum size of the pairwise non-commuting elements of such groups.

Keywords Finite group · Centralizer · CA -group

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1 Introduction

Throughout this paper all groups mentioned are assumed to be finite and we will use usual notation; for example, C_n^j and D_{2n} denote, respectively, the direct product of j copies of the cyclic group of order n and the dihedral group of order $2n$. For a finite group G , $\text{Cent}(G)$ denotes the set of all centralizers of single elements of G , i.e. $\text{Cent}(G) = \{C_G(g) : g \in G\}$, where $C_G(g)$ is the centralizer of the element g in G . For a positive integer n , a group G is called n -centralizer if $|\text{Cent}(G)| = n$ and a primitive n -centralizer group if $|\text{Cent}(G)| = |\text{Cent}(\frac{G}{Z(G)})| = n$ where $Z(G)$ denotes the center of G . Starting with Belcastro and Sherman [5], many authors have

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✉ S. M. Jafarian Amiri
sm_jafarian@znu.ac.ir; sm.jafariana110@gmail.com

H. Madadi
halime_madadi@m-iau.ac.ir

H. Rostami
h.rostami5991@gmail.com

¹ Department of Mathematics, Faculty of Sciences, University of Zanjan, Zanjan 4537138791, Iran

² Department of Mathematics, Miyaneh branch, Islamic Azad University, Miyaneh, Iran

³ Department of Education, Molla Sadra Teaching and Training Research Center, Zanjan 4513858899, Iran

studied the influence of $|Cent(G)|$ on the structure of a group G (see [1-4], [10-14] and [19-23]).

It is easy to see that G is non-abelian if and only if $|Cent(G)| \geq 4$. All n -centralizer groups have been characterized for $n \leq 9$ (see [1,2,5] and [9]). M. Zarrin [22] have generalized the previous results for infinite groups. He [19] showed that the derived length of a solvable group G is $\leq |Cent(G)|$. In [10] and [12] Jafarian Amiri et al. have determined all groups G with $|Cent(G)| \geq \frac{2}{3}|G|$. Recently, Jafarian Amiri and Rostami [13] have characterized groups having minimum number of the element centralizers among non-abelian groups of the same order.

The authors in [9] proved that the order of a 10-centralizer group is even. This result also was proved independently (in odd order case) by Foruzanfar and Mostaghim in [8], they prove that, there is no finite nonabelian group of odd order with $|Cent(G)| = 10$.

In this article, we determine the structure of finite groups with exactly ten centralizers. Our main result is:

Theorem 1.1 *Let G be a 10-centralizer group. Then $|\frac{G}{Z(G)}| \in \{16, 32, 56, 64\}$ and $\frac{G}{Z(G)}$ has one of the following structures:*

- (i) $C_2^4, C_4 \times C_4, (C_4 \times C_2) \rtimes C_2, D_{16}, C_2 \times D_8$.
- (ii) C_2^5 or C_2^6 .
- (iii) $C_2^3 \rtimes C_7$, the Frobenious group of order 56.

There are many 10-centralizer groups with the central factor of order 16. For example by GAP [17], there exist 10-centralizer groups

$$G_1 = \text{SmallGroup}(128, 6), G_2 = \text{SmallGroup}(128, 1400) \text{ and } G_3 = \text{SmallGroup}(128, 15)$$

such that $\frac{G_1}{Z(G_1)} \cong C_4 \times C_4$, $\frac{G_2}{Z(G_2)} \cong (C_2)^4$ and $\frac{G_3}{Z(G_3)} \cong (C_4 \times C_2) \rtimes C_2$.

In general, the converse of Theorem 1.1 is not true. For example by GAP [17], it can be checked that $\text{SmallGroup}(32, 50)$ has sixteen centralizers while its central factor group is isomorphic to C_2^4 and $\text{SmallGroup}(64, 14)$ is 12-centralizer while its central factor group is isomorphic to $(C_4 \times C_2) \rtimes C_2$. Also the central factor group of $\text{SmallGroup}(64, 132)$ is isomorphic to $C_2 \times D_8$ whereas this group is 12-centralizer, the central factor group of $\text{SmallGroup}(128, 2216)$ is isomorphic to C_2^5 whereas this group is 27-centralizer and $G = \text{SmallGroup}(128, 2326)$ is 64-centralizer group with $\frac{G}{Z(G)} \cong C_2^6$. But in Proposition 4.2 we will show that converse of Theorem 1.1 is true whenever $\frac{G}{Z(G)} \cong C_4 \times C_4$ or D_{16} . Also in Proposition 4.3, it is shown that if $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$, then $|Cent(G)| \in \{10, 14, 16\}$. However, we could not find a group G such that $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$ and $|Cent(G)| = 14$ or 16.

The following result is an immediate consequence of Theorem 1.1.

Corollary 1.2 *If G is a primitive 10-centralizer group, then $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$.*

2 Definitions, Notations and Preliminaries

There are interesting relations between centralizers and pairwise non-commuting elements in groups (see [1, Proposition 2.5 and Lemma 2.6]). Let G be a finite non-abelian group and let X be a subset of pairwise non-commuting elements of G such that $|X| \geq |Y|$ for any other set of pairwise non-commuting elements Y in G . Then the subset X is said to have the maximum size, and this size is denoted by $\omega(G)$. Also $\omega(G)$ is the maximum clique size in the non-commuting graph of a finite group G . The non-commuting graph of a group G is defined as a graph whose $G \setminus Z(G)$ is the set of vertices and two vertices are joined if and only if they do not commute. By a famous result of Neumann [15] answering a question of Erdős, the finiteness of $\omega(G)$ is equivalent to the finiteness of the factor group $\frac{G}{Z(G)}$ which follows that $|Cent(G)|$ is finite. In addition, if G has a finite number of centralizers, then it is easy to see that $\omega(G)$ is finite.

To prove the main theorem, we need the following results.

Remark 2.1 Let G be a group and $\{x_1, \dots, x_{\omega(G)}\}$ be a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{\omega(G)}$. Then we have the following.

- (i) $G = \bigcup_{i=1}^{\omega(G)} C_G(x_i)$ and $Z(G) = \bigcap_{i=1}^{\omega(G)} C_G(x_i)$ (see Theorem 5.1 of [18]).
- (ii) $\alpha_2 \leq \omega(G) - 1$ (see [18, Lemma 3.3]).
- (iii) If $\alpha_2 = \omega(G) - 1$, then $\alpha_i = \omega(G) - 1$ for each $i > 1$ and $C_G(x_i) \cap C_G(x_j) \leq C_G(x_1)$ for every $i \neq j$ (see [18, Lemma 3.3]). In addition, it is easy to see that $|\frac{G}{Z(G)}| = \sum_{i=2}^{\omega(G)} |\frac{C_G(x_i)}{Z(G)}|$.

Lemma 2.2 Let G be a finite group and $k \leq 6$ be a positive integer. Then, k is the number of non-abelian centralizers belonging to $Cent(G)$ if and only if $|Cent(G)| = \omega(G) + k$.

Proof See [11, Theorem 1.2]. □

Lemma 2.3 Let G be finite group and $x, y \in G$ such that $xy \neq yx$. If $C_G(x)$ is abelian of prime index p in G , then $C_G(x) \cap C_G(y) = Z(G)$ and $|\frac{C_G(x)}{Z(G)}| \leq p|G : C_G(y)|$.

Proof The proof is straightforward. □

Let G be a non-trivial group. A collection Π of non-trivial subgroups of G is said to be a *partition* of G if every non-trivial element of G belongs to a unique subgroup in Π . Recall that a group G is a *CA-group* ($G \in CA$) if $C_G(x)$ is abelian for every $x \in G \setminus Z(G)$. It is easy to see that if G is a CA-group, then $\{\frac{C_G(g)}{Z(G)} : g \in G \setminus Z(G)\}$ is a partition of $\frac{G}{Z(G)}$. By [1, Lemma 2.6], G is a CA-group if and only if $|Cent(G)| = \omega(G) + 1$.

3 Some Special Cases of 10-Centralizer Groups

In this section, we prove some results that we will need in the sequel. Suppose that G is a 10-centralizer group. By [1, Lemma 2.4], we have $5 \leq \omega(G) \leq 9$. In particular,

$\omega(G) = 9$ if and only if G is a CA -group. We first prove that if $\omega(G) \neq 9$, then $\frac{G}{Z(G)}$ is of order 16.

Lemma 3.1 *Suppose that G is a 10-centralizer group. If $\omega(G) \in \{5, 6, 7, 8\}$, then $|\frac{G}{Z(G)}| = 16$.*

Proof If $\omega(G) = 5$, then G is not a CA -group and so $|\frac{G}{Z(G)}|$ has at least four (not necessarily distinct) prime divisors by [4, Lemma 2.1]. In addition, we have $|\frac{G}{Z(G)}| \leq 16$ by [1, Remark 2.1(3)]. Therefore, $|\frac{G}{Z(G)}| = 16$, as wanted.

Suppose that $\omega(G) = 6$ and $\{x_1, \dots, x_6\}$ is a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_6$. By Lemma 2.2, $C_G(x_i)$ is abelian for each i and so G has three proper non-abelian centralizers, say $C_G(a)$, $C_G(b)$, $C_G(c)$. Also G is not a CA -group which implies that $|\frac{G}{Z(G)}|$ has at least four (not necessarily distinct) prime divisors. Therefore, $|\frac{G}{Z(G)}| \geq 16$. We have to prove that $|\frac{G}{Z(G)}| = 16$.

Suppose, for a contradiction, that $|\frac{G}{Z(G)}| > 16$. Since $\omega(G) = 6$, we have $\alpha_2 \leq 5$ by Remark 2.1(ii). If $\alpha_2 = 5$, then by Lemma 2.3, $|\frac{G}{Z(G)}| \in \{20, 25\}$ which follows that G is a CA -group, a contradiction. Hence $\alpha_2 = 4$. Similarly $\alpha_1 = 4$. Since $|\frac{G}{Z(G)}| > 16$ and $C_G(x_i)$ is abelian for $i \in \{1, 2\}$, we have $C_G(x_1) \cup C_G(x_2) \subseteq C_G(g)$ for some $g \in (C_G(x_1) \cap C_G(x_2)) \setminus Z(G)$ and so $G = C_G(g) \cup (\cup_{i=3}^6 C_G(x_i))$. Therefore, $\alpha_3 \leq 4$ by [18, Lemma 3.3] and hence $\alpha_1 = \dots = \alpha_6 = 4$. Consequently $C_G(x_i) \cap C_G(x_j) \neq Z(G)$ and $C_G(x_i) \cap C_G(x_j) \cap C_G(x_k) = Z(G)$ for every distinct i, j, k . Also by indices α_i 's and considering the covering of G by its centralizers, we conclude that each of the non-abelian centralizers $C_G(a)$, $C_G(b)$, $C_G(c)$ contains exactly two distinct centralizers $C_G(x_i)$ and $C_G(x_j)$ for some $i \neq j$, say $C_G(x_1) \cup C_G(x_2) \subseteq C_G(a)$, $C_G(x_3) \cup C_G(x_4) \subseteq C_G(b)$ and $C_G(x_5) \cup C_G(x_6) \subseteq C_G(c)$. On the other hand, we have $C_G(x_1) \cap C_G(x_3) \neq Z(G)$ which implies that $C_G(x_1) \cup C_G(x_3)$ are contained in one of $C_G(a)$, $C_G(b)$ or $C_G(c)$. Therefore, one of the non-abelian centralizers contains at least three centralizers $C_G(x_i)$'s, a contradiction. This completes the proof of this case.

The proof is similar when $\omega(G) = 7$ or 8 . □

In view of [1, Lemmas 3.3] and [9, 5.6], it is necessary to investigate all 10-centralizer groups G with $|\frac{G}{Z(G)}| = 24$ or 36 , separately.

Lemma 3.2 *If G is a 10-centralizer group, then $|\frac{G}{Z(G)}| \neq 24$ or 36 .*

Proof Let $\{x_1, \dots, x_{\omega(G)}\}$ is a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{\omega(G)}$. Put $\overline{G} = \frac{G}{Z(G)}$.

Suppose, for a contradiction, that $|\overline{G}| = 24$. It follows from Lemma 3.1 that $\omega(G) = 9$. Therefore, G is a CA -group and $\alpha_2 \leq 8$ by Remark 2.1(ii). Also $\Gamma = \{\frac{C_G(x_i)}{Z(G)} : 1 \leq i \leq 9\}$ is a partition of $\overline{G}$. If $\alpha_2 = 8$, then $\alpha_2 = \alpha_3 = \dots = \alpha_9 = 8$ by Remark 2.1(iii). Therefore, the number of the Sylow three-subgroups of $\overline{G}$ is eight, a contradiction. Since $C_G(x_1) \cap C_G(x_2) = Z(G)$, we have $\alpha_2 = 6$ and so $\alpha_1 = 4$ or 6 . It follows from [7, Theorem A] that either G has an abelian normal subgroup of prime index or $\overline{G}$ is a

Frobenius group with an abelian complement. By the order of $\overline{G}$ and indices α_i 's, the first case cannot be happened. Hence, $\overline{G}$ is Frobenius. If $\alpha_1 = 4$, then $\frac{C_G(x_1)}{Z(G)}$ is a normal cyclic subgroup of $\overline{G}$ of order six and so the center of $\overline{G}$ is nontrivial, a contradiction. Therefore, $\alpha_1 = \alpha_2 = 6$. By partition Γ and the order of $\overline{G}$, we see that $\frac{G}{Z(G)}$ has three Sylow two-subgroups. If the number of Sylow three-subgroups of $\frac{G}{Z(G)}$ is greater than one, then $\frac{G}{Z(G)} \cong S_4$ which implies that $11 = \omega(\overline{G}) \leq \omega(G)$, a contradiction. Thus, $\frac{G}{Z(G)}$ has a unique Sylow three-subgroup $\overline{Q}$ and so this is the Frobenius kernel of $\overline{G}$ since the orders of the kernel and complement of a Frobenius group is relatively prime. If $\frac{H}{Z(G)}$ is the Frobenius complement, then by [7, Theorem A], H is abelian which follows $|Cent(G)| \neq 10$ by [9, Proposition 3.1] and this is not possible.

Now suppose, for a contradiction, that $|\overline{G}| = 36$. Similarly, we have $\omega(G) = 9$ and so the result gets from [9, Corollary 3.2]. $\square$

We shall investigate the structure of 10-centralizer groups G such that $\frac{G}{Z(G)}$ is a two-group and this is one of the most important special cases (see [1, Lemma 3.2], [2, Corollary 3.3] and [9, Lemmas 4.2-4.6]). Suppose that G is a 10-centralizer group such that $|\frac{G}{Z(G)}| = 2^k$ for some positive integer k . If $\omega(G) \in \{5, 6, 7, 8\}$, then $|\frac{G}{Z(G)}| = 16$ by Lemma 3.1. If $\omega(G) = 9$, then $\alpha_2 \leq 8$ and G is a CA-group. It follows that $C_G(x_1) \cap C_G(x_2) = Z(G)$ and so $|\frac{G}{Z(G)}| \leq |G : C_G(x_1)||G : C_G(x_2)| \leq 64$. Therefore, we have the following two results.

Lemma 3.3 *If G is a 10-centralizer group such that $|\frac{G}{Z(G)}| = 32$, then $\frac{G}{Z(G)} \cong C_2^5$.*

Proof It follows from Lemma 3.1 that $\omega(G) = 9$. Consequently G is a CA-group. If $\{x_1, \dots, x_9\}$ is a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_9$, then $\Gamma = \{\frac{C_G(x_i)}{Z(G)} : 1 \leq i \leq 9\}$ is a partition of $\overline{G}$, $\alpha_1 = 4$ and $\alpha_2 = \dots = \alpha_9 = 8$. Therefore, the class sizes in G are $\{1, 4, 8\}$ and so $\exp(\frac{G}{Z(G)}) = 2$ by [7, Lemma 6.1]. This completes the proof. $\square$

Lemma 3.4 *If G is a 10-centralizer group such that $|\frac{G}{Z(G)}| = 64$, then $\frac{G}{Z(G)} \cong C_2^6$.*

Proof The proof is similar to Lemma 3.3. $\square$

4 Proofs of the Main Results

In what follows we determine all 10-centralizer groups G such that $\frac{G}{Z(G)}$ is not a two-group.

Proposition 4.1 *Let G be a 10-centralizer group. If $\frac{G}{Z(G)}$ is not a two-group, then $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$.*

Proof Suppose that $\{x_1, \dots, x_{\omega(G)}\}$ is a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{\omega(G)}$. It follows from Lemma 3.1 that $\omega(G) = 9$. Therefore, G is a CA-group and by Remark 2.1(ii), $\alpha_2 \leq 8$ which implies that $|\frac{G}{Z(G)}| \leq \alpha_1 \alpha_2 \leq 64$. By [20, Theorem A], G is solvable. Since $|G|$ is

even by [9, Proposition 5.8], we have either $\frac{G}{Z(G)}$ is Frobenius or G has an abelian normal subgroup of prime index by [7, Theorem A]. We claim that $|\frac{G}{Z(G)}| = 56$.

If G has an abelian normal subgroup of prime index p , then $|G'| = 8$ by [4, Theorem 2.3] and so $|\frac{G}{Z(G)}| = 8p$. Therefore, $p = 5$ or seven by Lemma 3.2. If $p = 5$, then $|\frac{G}{Z(G)}| = 40$ and $\alpha_2 = 8$. It follows that $\alpha_i = 8$ for each $i > 1$ and hence $\frac{G}{Z(G)}$ has at least eight Sylow five-subgroups which is a contradiction. Thus, $p = 7$ and the claim is proved in this case. So we may assume that G does not contain any normal abelian subgroup of prime index. Therefore, $\frac{G}{Z(G)}$ is Frobenius. Now, we want to show that $\alpha_2 = 8$.

It follows from [1, Proposition 2.5 (a)(2)] that $\alpha_2 \geq 4$. Clearly $\alpha_2 \neq 5$. If $\alpha_2 = 4$ or six, then $|\frac{G}{Z(G)}| \leq 36$ and so $|\frac{G}{Z(G)}| \in \{12, 18, 30\}$ by Lemma 3.2. If $|\frac{G}{Z(G)}| = 12$, then $\frac{G}{Z(G)} \cong A_4$ and so by [2, Corollary 3.6], we have $|Cent(G)| = 6$ or eight which is impossible. If $|\frac{G}{Z(G)}| = 30$, then G has an abelian normal subgroup of index two, a contradiction. If $|\frac{G}{Z(G)}| = 18$, then a Sylow three-subgroup of $\frac{G}{Z(G)}$ is the Frobenius kernel of order nine and so $|Cent(G)| \geq 11$ by [9, Proposition 3.1], a contradiction. Let $\alpha_2 = 7$. Then $|\frac{G}{Z(G)}| \leq 49$. By [4, Proposition 2.9], we have $|\frac{G}{Z(G)}| = 28$ or 42. Since there is no Frobenius group of order 28, $|\frac{G}{Z(G)}| = 42$. But by [9, Proposition 3.1], we have $|Cent(G)| = 9$ or greater than 10 which is impossible. Therefore, $\alpha_2 = 8$. It follows from Remark 2.1 (ii) that $\alpha_i = 8$ for $i > 1$ and $|\frac{G}{Z(G)}| = 56$. This completes the proof of the claim.

Since $\alpha_i = 8$ for $i \geq 2$ and $\alpha_1 = 7$, we conclude that $\frac{G}{Z(G)}$ is a Frobenius group. But among group of order 56, only $C_2^3 \rtimes C_7$ has trivial center. This completes the proof. $\square$

The group G is said to be *capable* if there exists a group H such that $G \cong \frac{H}{Z(H)}$. The *epicenter* of G , denoted by $Z^*(G)$, is the smallest normal subgroup of G such that $\frac{G}{Z^*(G)}$ is capable. In [6], it is proved that a group G is capable if and only if $Z^*(G) = 1$. Now, we are ready to prove the main result.

Proof of Theorem 1.1 Assume that $\{x_1, \dots, x_{\omega(G)}\}$ is a set of pairwise non-commuting elements of G such that $|G : C_G(x_i)| = \alpha_i$ with $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_{\omega(G)}$. Then $5 \leq \omega(G) \leq 9$ by [1, Lemma 2.4] and $\alpha_2 \leq \omega(G) - 1$ by Remark 2.1(ii).

We first assume that $\frac{G}{Z(G)}$ is a two-group. If $\omega(G) \neq 9$, then $|\frac{G}{Z(G)}| = 16$ by Lemma 3.1. But it is easily checked by GAP [17] that the only capable groups of order 16 are $C_4 \times C_4$, $(C_4 \times C_2) \rtimes C_2$, $C_2 \times D_8$, D_{16} , $(C_2)^4$, as desired. Now we may assume that $|\frac{G}{Z(G)}| > 16$. Then $\omega(G) = 9$ and so G is a CA-group which implies $|\frac{G}{Z(G)}| \leq \alpha_1 \alpha_2 \leq 64$. Thus, $|\frac{G}{Z(G)}| = 32$ or 64. We conclude that $\frac{G}{Z(G)} \cong C_2^5$ or C_2^6 by Lemmas 3.3 and 3.4, respectively.

Next, suppose that $\frac{G}{Z(G)}$ is not a two-group. Then $\frac{G}{Z(G)} \cong (C_2)^3 \rtimes C_7$ by Proposition 4.3. This completes the proof. $\square$

In the following result, we compute the number of centralizers for all groups with central factor isomorphic to $C_4 \times C_4$ or D_{16} . So in these cases convers of Theorem 1.1 is hold.

Proposition 4.2 *Let G be a non-abelian group such that $\frac{G}{Z(G)} \cong C_4 \times C_4$ or D_{16} . Then $|Cent(G)| = 10$.*

Proof Suppose that $\frac{G}{Z(G)} \cong C_4 \times C_4$ and $\{x_1, \dots, x_{\omega(G)}\}$ is a set of pairwise non-commuting element of G . Since $C_4 \times C_4$ has six cyclic subgroups of order four and $\frac{G}{Z(G)} = \bigcup_{i=1}^{\omega(G)} \frac{C_G(x_i)}{Z(G)}$, we have $\omega(G) \leq 6$. But $\omega(G) \geq 5$ by [9, Remark 2.4]. By [14, Theorem 1.3], $\omega(G) \neq 5$ and so $\omega(G) = 6$. It follows that $\frac{C_G(x_i)}{Z(G)} = \langle x_i Z(G) \rangle$ is of order four. Since $\frac{G}{Z(G)}$ has only three elements of order two, we may assume that $\frac{C_G(x_1)}{Z(G)} \cap \frac{C_G(x_2)}{Z(G)} = \langle x_1^2 Z(G) \rangle$, $\frac{C_G(x_3)}{Z(G)} \cap \frac{C_G(x_4)}{Z(G)} = \langle x_3^2 Z(G) \rangle$ and $\frac{C_G(x_5)}{Z(G)} \cap \frac{C_G(x_6)}{Z(G)} = \langle x_5^2 Z(G) \rangle$. Since $C_G(x_i) \subset C_G(x_i^2) \subset G$ for $i \in \{1, 3, 5\}$. Therefore,

$$Cent(G) = \{G, C_G(x_i) : 1 \leq i \leq 6\} \bigcup \{C_G(x_i^2) : i = 1, 3, 5\}$$

which implies that $|Cent(G)| = 10$.

Now, let $\frac{G}{Z(G)} \cong D_{16}$. It follow from [1, Proposition 2.2] that $|Cent(G)| = 10$ and the proof is complete. $\square$

Now, we compute $|Cent(G)|$ for any group G with $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$.

Proposition 4.3 *Let G be finite group. If $\frac{G}{Z(G)} \cong C_2^3 \rtimes C_7$, then $|Cent(G)| \in \{10, 14, 16\}$.*

Proof Since $\frac{G}{Z(G)}$ is a Frobenius group with the abelian Frobenius kernel and abelian Frobenius complement, G is a CA-group by [7, Theorem A] (see also [16]). Let $\frac{K}{Z(K)}$ be the Frobenius kernel of $\frac{G}{Z(G)}$. So by [16, Lemma 5.5], we consider two following cases:

Case 1. If $Z(G) \neq Z(K)$, then $|Cent(G)| = |K : Z(G)| + 2 = 10$.

Case 2. If $Z(G) = Z(K)$, then $|Cent(G)| = |K : Z(G)| + |Cent(K)|$. Since $\frac{K}{Z(K)} \cong C_2^3$, [1, Proposition 3.4] yields that $|Cent(K)| = 6$ or 8. Therefore, $|Cent(G)| = 14$ or 16 and we are done. $\square$

Proof of Corollary 1.2 Since $C_2^3 \rtimes C_7$ is a 10-centralizer group and the remainder groups listed in the Theorem 1.1 are not 10-centralizer, we have the result. $\square$

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