Solving different practical granular problems under the same system of equations

Discussion with the paper Kreinovich V. (Granular Computing 1(3):171–179, 2016)

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Abstract This paper contains discussion about the paper of Kreinovich (Granular Computing 1(3):171–179, 2016) in which the author suggests the thesis that in conditions of uncertainty “solving different practical problems we get different solutions of the same system of equations with the same granules”. Authors of the present paper are of the opinion that the above thesis is result of one-dimensional approach to interval analysis prevailing at present in scientific community and also used by Kreinovich. According to this approach the direct result of any arithmetic operation $* \in \{+, -, \times, /\}$ on intervals is also an interval. The main scientific contribution of the paper is showing that a granular problem analysis in an incomplete, low-dimensional space can lead to untrue conclusions because picture of the problem in this space is too poor and part of valuable, important information is lost. What is seen in the full-dimensional space of the problem space cannot be seen in an incomplete, low-dimensional space. The paper also shortly describes a multidimensional approach to interval arithmetic that is free of the above-described deficiencies.

Keywords Granular computing · Equations under uncertainty · Interval computations · RDM interval

arithmetic · One-dimensional interval arithmetic · Multidimensional interval arithmetic

1 Introduction

Granular computing (GC) comprises computing with different forms of uncertain, approximating data pieces, such as intervals, fuzzy numbers, vague numbers, grey numbers, intuitionistic sets, rough sets, etc (Chatterjee and Kar 2017; Livi and Sadeghian 2016). Lately, a novel interpretation of granules as patterns has been introduced. These patterns can be defined not only in geometric, but also in non-geometric input spaces (Livi and Sadeghian 2016). They find application, e.g. in document analysis, bio-molecules recognition, etc.

GC uses granules coming from experts, from computer analyses of Data Mining type and from physical sensors delivering various, uncertain perceptions that have to be aggregated to get a possibly simple and easily understandable information useful for people (D’Aniello et al. 2017). However, in the present paper we deal with a problem in which original, input granules are intervals and result granules are defined in higher dimensions of uncertainty. Authors of many papers on GC draw our attention to great difficulties accompanying solving GC-problems and to their high complicity level. For example, the paper by Wang et al. (2017) informs about various levels of a problem analysis on which granules of various precision have to be used to get coarse results (better comprehension, lower precision) or fine results (more difficult comprehension, higher precision). Authors of the paper Kovalerchuk and Kreinovich (2017) show not only how difficult can be solving of granular problems but also how differently the problem solution can be interpreted by people

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solving it. They give seven examples of different solution versions and show that the solution notion can have subjectively individual character resulting from interests of a person. However, it should be taken into account that also objective, mathematical interpretations and definitions of solutions (results) can be found (Popova 1998; Piegat and Landowski 2017b). Difficulty degree of a solved problem depends also on the uncertainty degree of information granules occurring in the problem. It is shown in the paper by Najariyan et al. (2017) in which fuzzy granules of Type-2 are used of third uncertainty level. However, it is difficult to find papers describing the importance of the space dimensionality in which original, input granules of a problem are defined and how this dimensionality influences dimensionality and precision of the achieved result. Papers on GC frequently do not differentiate between notions of the direct, main solution and indirect solutions derived from the direct solution that only represent certain features of it (Piegat and Landowski 2017b, 2018). It is a reason of the observed interpretative chaos and of misunderstandings among scientists.

The main motivation of this paper was correction of the message sent to GC-readers by the paper of Kreinovich (2016) speaking that in conditions of uncertainty “solving different practical problems we get different solutions of the same system of equations with the same granules” and explaining why this message is not true, if the problem is described in a full multidimensional space.

Scientific *contribution* of this paper consist of three parts C1–C3.

Contribution C1 The paper shows that granular problem analysis in incomplete, in low-dimensional space can lead to untrue conclusions because picture of the problem in this space is simplified. For example, projections on 2D-space of a sphere, of a half-sphere, and of a cylinder (of objects defined in 3D-space) can in this 2D-space be identical (a circle) and on their basis we are not able to say whether the real, projected objects are in the full 3D-space identical or not.

Contribution C2 The paper shows that application of interval arithmetic operating in 1D space of real-valued numbers (1-type arithmetical object arithmetic) is not recommended because arithmetical operations on intervals (1D-objects) give results existing in multidimensional space and not in 1D-space. These results can be called direct ones and we should differentiate between them and simplified indirect results that are derived from direct multidimensional results. Indirect results, e.g. span, cardinality (frequency) distribution, center of gravity of the direct multidimensional result are of lower dimensionality and they are used because they are easily understandable for people. Instead, the multidimensional direct results mostly cannot be imagined as 4D or as object of greater

dimensionality. Correct indirect results cannot be determined without correct multidimensional result. Hence, importance of direct multidimensional result for accuracy of problem solving is very great.

Contribution C3 The paper shows that the problem of passive understanding the world (passive WU-problem) and active changing the world (active WCH problem) which have the same mathematical models in terms of 1D interval arithmetic are described by different models in terms of multidimensional RDM arithmetic. This explains achievement of different solutions of both problems.

For many persons, interval arithmetic seems to be very simple, trivial and rather non-interesting. However, this view is a result of a rather shallow knowledge of this arithmetic. Its deep analyses based on multidimensional approach shows that interval computations are very complicated and hence they are fascinating, scientific tasks requiring careful investigations. Papers Kreinovich (2016) and Kovalerchuk and Kreinovich (2016) just try to show complicity level of GC problems and explain why in GC the specificity of each concrete, considered problem should be taken into account and why these problems should not to be solved schematically. Hence, both papers are very recommendable. Author of the paper Kreinovich (2016) has called his paper “pedagogical”, which means a paper of basic character for GC. However, authors of the present paper have detected in it certain inaccuracy resulting from application of one-dimensional interval arithmetic that can lead readers into error and would like to discuss and correct it. In the discussed paper Kreinovich (2016) tries to prove a thesis speaking that in conditions of uncertainty “solving different practical problems we get different solutions of the same system of equations with the same granules” which further on will shortly be called Kreinovich thesis. Kreinovich distinguishes, basing on classification presented in many papers of Shary, e.g. in Shary (1996) that in the practice two basic types of uncertain problems occur:

- problem of passive understanding the worlds (passive WU problem),
- problem of active changing the world (active WCH problem).

Kreinovich thesis is explained in Kreinovich (2016) mainly on two examples presented below.

Example 1 of passive WU. We have a reservoir. The start amount of water is equal to a but we know its value only approximately as $a \in A = [99, 101]$. From the reservoir an amount b of water was released. Its value is known also only approximately: $b \in B = [38, 42]$. The task is to determine the end-amount of water c left in the reservoir. To avoid possible misunderstandings concerning the notation, the problem was illustrated by Fig. 1. It is

Fig. 1 Illustration of Example 1 of the problem of passive world understanding

important because Kreinovich in Example 2 (Kreinovich 2016) has a little changed the notation.

As it was shown in Kreinovich (2016), problem 1 of passive WU can be solved heuristically or with the use of the naive computation method with formula (1), where C denotes set of all possible values of c of water left in the reservoir.

$$C = A - B = [99, 101] - [38, 42] = [57, 63] \quad (1)$$

The result set C given by (2) is in this case called *united solution set* because it unites (combines) all the values of c corresponding to all possible values of $a \in A$ and $b \in B$ (Shary 1996).

$$C = \{c : \exists a \in A \exists b \in B (c = a - b)\} \quad (2)$$

It should be noticed that in the above solution both information input sets A, B and the output set C were assumed to be sets of one and the same type: one-dimensional sets. It causes that the restoration principle cannot be kept. If we know the value of c we cannot restore values of a and b that created c . Each c value occurs in one-dimensional set only one time. But in a real system it can occur in many ways, from different doubles (a, b) . Hence, 1D interval form of the solution set C causes loss of information.

Example 2 problem of active changing the world. We know approximately that the original amount a of water in the reservoir is contained in interval $[99, 101]$ and we require that after releasing the amount b of water from the reservoir the remaining amount c will be contained between 38 and 42, i.e. that $c \in C = [38, 42]$. The task is to determine the amount of water b that should be released from the reservoir. The problem is illustrated in Fig. 2.

Let us notice that in Example 1 the original amount of water in the container had been $a \in [99, 101]$. After the release of amount $b \in [38, 42]$ the left amount was $c \in [57, 63]$. Hence, we could hope that if we would like to have the left water in amount $c \in [38, 42]$ we should release from the reservoir water in amount $b \in [57, 63]$. However, in Kreinovich (2016) it is shown that it is not true

Fig. 2 Illustration of the problem of the active changing the world from Kreinovich (2016)

(remark: in this paper notations b and c in Example 2 were changed in comparison with Example 1. In the present paper no such change was made and both examples have identical notations, the released water is always denoted by b and the left water by c). In Kreinovich (2016), the problem of active WCh first had been solved heuristically and then with use of Kaucher interval arithmetic. In the terminology used by Shary (1996), in Example 2 the aim was (citation)

to find the range B of all possible values b for which, for all $b \in B$, the value $c = a - b$ belongs to interval C . In other words we need to find the set

$$B = \{b : \forall a \in A \exists c \in C (c = a - b)\} \quad (3)$$

This is a particular case of tolerance solution. As we have mentioned to compute this set we can use Kaucher arithmetic. Since the variable a enters this formula with a universal quantifier $\forall$ instead of essential $\exists$ one, instead of the original interval $A = [99, 101]$, we need to consider an improper interval $A^* = [101, 99]$. For the resulting pair of intervals A^* and B , the above general rule of interval subtraction leads to

$$B = A^* - C = [101, 99] - [38, 42] = [59, 61] \quad (4)$$

(end of citation).

Summarizing of Examples 1 and 2

In the paper Kreinovich (2016) the author tried to show that “for different practical problems, we get different solutions to the same system of equations with the same granules”. And really, analysis of the results achieved in both examples with use of one-dimensional approach seems to prove this thesis considerably. Example 1 is described by (5) and Example 2 by (6).

$$[99, 101] - [38, 42] = [57, 63] \text{ passiveWU} \quad (5)$$

$$-[59, 61] = [38, 42] \text{ activeWCh} \quad (6)$$

In both examples input-information granules are the same $[99, 101]$ and $[38, 42]$ and achieved results are different $[57, 63]$ and $[59, 61]$. Kreinovich opinion that before solving a practical problem one should precisely analyse it and not to solve it schematically and rashly is right.

However, situation in which “for different practical problems, we get different solutions to the same system of equations with the same granules” seems from mathematical point of view rather strange and suspected. It seems that the above thesis has been achieved by Kreinovich as the result of using one-dimensional approach to problems described in Examples 1 and 2. This approach is used by both naive interval arithmetic, Kaucher arithmetic, and also all other interval arithmetic type known to the authors. The one-dimensional approach consists in the assumption that the direct result of arithmetical operations $\{+, -, \times, /\}$ on intervals is also an interval, i.e. the same mathematical object as input objects taking part in the operation. The above observation refers not only to interval arithmetic but also to fuzzy and probabilistic arithmetic. Why assuming interval as *direct result* of operations on intervals is an error? The answer on this question was described in two papers: Piegat and Landowski (2017b, 2018). Hence, in this paper complete argumentation will not be repeated but only one of more important arguments. A similar argument can be found in Dymova (2011). Let us assume that a system works according to the additive law $a + b = c$, where a and b are inputs and c is the output. This functioning law can be presented in four equivalent versions: (1) $a + b = c$, (2) $a = c - b$, (3) $b = c - a$, and (4) $a + b - c = 0$. If input values are not known precisely but only approximately as $a \in [\underline{a}, \bar{a}]$, $b \in [\underline{b}, \bar{b}]$ then we should ask what type of mathematical object will the result of $[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}]$ be? According to all known interval arithmetic types the sum is also an interval $[\underline{c}, \bar{c}]$ given by the following equation:

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{c}, \bar{c}] \quad (7)$$

If the interval $[\underline{c}, \bar{c}]$ really is the result of interval addition then it should satisfy all equivalent forms of addition operation (8–11) or with other words it should satisfy *the principle of the result universality* (Piegat and Landowski 2017b, 2018).

$$a + b = c \Rightarrow [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{c}, \bar{c}] \quad (8)$$

$$a = c - b \Rightarrow [\underline{a}, \bar{a}] = [\underline{c}, \bar{c}] - [\underline{b}, \bar{b}] \quad (9)$$

$$b = c - a \Rightarrow [\underline{b}, \bar{b}] = [\underline{c}, \bar{c}] - [\underline{a}, \bar{a}] \quad (10)$$

$$a + b - c = 0 \Rightarrow [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] - [\underline{c}, \bar{c}] = 0 \quad (11)$$

It is easy to check that particular forms (8–11) give the following results for $[\underline{c}, \bar{c}]$:

$$[\underline{c}, \bar{c}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \quad (12)$$

$$[\underline{c}, \bar{c}] = [\underline{a} + \bar{b}, \bar{a} + \underline{b}] \quad (13)$$

$$[\underline{c}, \bar{c}] = [\underline{a} + \bar{b}, \bar{a} + \underline{b}] \quad (14)$$

$$[\underline{c}, \bar{c}] = [\bar{a} + \underline{b}, \underline{a} + \bar{b}] \quad (15)$$

The example of interval addition delivers four different results, which shows that one-dimensional interval cannot be result of this operation because it does not satisfy the principle of the result universality. It also means that interval equation cannot be transformed into other form for calculating the result which can make solution determining impossible at all. This phenomenon characteristic for one-dimensional interval arithmetic types was called the UBM-phenomenon (unnatural behaviour phenomenon) in Mazandarani et al. (2017). If principle of solution universality cannot be kept then interval equations to be solved cannot be transformed into other appropriate forms and calculating their results can be impossible. The addition result cannot depend on form of the addition equation. The same can similarly be proved for other basic arithmetic operations $\{-, \times, /\}$. Why authors of all interval arithmetic types have assumed that the operation result also is an interval? The reason probably lies in *the closure axiom* (Bader and Nipkow 1998) and (Birkhoff 1967), which is formulated as follows: the set of intervals has a closure under an operation (e.g. of addition) if a performance of that operation on members of the set always produces a member of the same set, i.e. an interval. Just the closure axiom is the main cause of the one-dimensional approach to interval arithmetic, and also to fuzzy and probabilistic arithmetic. This axiom means transformation of thinking typical for conventional crisp numbers where result of addition $3 + 5 = 8$ is the same object as components of addition. Also, in the case of interval arithmetic and generally granular computing using the closure axiom leads to wrong results. Direct result of operations on intervals exists in space of higher level than 1. This level depends on number of intervals taking part in arithmetic operation. The next chapter will shortly present the multidimensional RDM interval arithmetic.

2 Short introduction into RDM interval arithmetic (RDM-IA)

RDM interval arithmetic was described in few publications, e.g. in Piegat and Landowski (2012, 2013, 2014) and Piegat and Tomaszewska (2013). It also has been applied in fuzzy arithmetic (RDM-FA) based on μ -cuts, Piegat and Plucinski (2015a, b, 2017), Piegat and Landowski (2015, 2017a) and Mazandarani et al. (2017). RDM-IA uses epistemic approach to interval calculations presented in Lodwick and Dubois (2015). The subject of arithmetic operations in RDM-IA is not the full set of values contained in interval $[x] = [\underline{x}, \bar{x}]$ but one single, true, precise value x^{pr} contained in the interval, which has really

occurred. Mathematical model of this value is given by the following equation:

$$\begin{aligned} X^{\text{gr}} : x^{\text{gr}} &= \underline{x} + \alpha_x^{\text{gr}}(\bar{x} - \underline{x}), \\ \alpha_x^{\text{gr}} &\in [0, 1], \text{card}X^{\text{gr}} = 1 \end{aligned} \quad (16)$$

X^{pr} denotes here the set of true x values. RDM variable α_x^{gr} informs about relative position of the true value x^{pr} in the interval $[x]$. Model (16) can be called “model of the precise value of variable x ” (PVV-model). However, though we have the PVV-model of the precise value x^{pr} , it does not mean that this value is precisely known. Formula (16) only informs that only one true value of variable x exists between $\underline{x}$ and $\bar{x}$. Formula (17) presents mathematical model of the set X^{poss} of all possible values x^{poss} that the precise value x^{pr} can take.

$$\begin{aligned} X^{\text{poss}} : x^{\text{poss}} &= \underline{x} + \alpha_x^{\text{poss}}(\bar{x} - \underline{x}), \\ \alpha_x^{\text{poss}} &\in [0, 1], \text{card}X^{\text{poss}} = c \text{ (continuum)} \end{aligned} \quad (17)$$

The main difference between RDM-IA and other types of IA consists in the fact that RDM-IA operations are realized not on sets (intervals) but on models of precise values of uncertain variables. Hence, the achieved result is also a model of a precise values and not an interval. This result is not of one-dimensional but of multidimensional character. Let us denote by x^{pr} and y^{pr} true values of variables about which we possess only approximate knowledge $x^{\text{pr}} \in [\underline{x}, \bar{x}]$ and $y^{\text{pr}} \in [\underline{y}, \bar{y}]$, and by $*$ one of arithmetic operations $*$ $\in \{+, -, \times, /\}$. Models of precise values of variables x and y are given by the following equations:

$$x^{\text{gr}} = \underline{x} + \alpha_x^{\text{gr}}(\bar{x} - \underline{x}), \quad \alpha_x^{\text{gr}} \in [0, 1], \quad \text{card}X^{\text{gr}} = 1 \quad (18)$$

$$y^{\text{gr}} = \underline{y} + \alpha_y^{\text{gr}}(\bar{y} - \underline{y}), \quad \alpha_y^{\text{gr}} \in [0, 1], \quad \text{card}Y^{\text{gr}} = 1 \quad (19)$$

Mathematical model of the precise value z^{pr} of any arithmetic operation $*$ is given by the following equation:

$$\begin{aligned} z^{\text{gr}}(\alpha_x^{\text{gr}}, \alpha_y^{\text{gr}}) &= x^{\text{gr}}(\alpha_x^{\text{gr}}) * y^{\text{gr}}(\alpha_y^{\text{gr}}), \\ \alpha_x^{\text{gr}}, \alpha_y^{\text{gr}} &\in [0, 1], \\ \text{card}X^{\text{gr}} &= \text{card}Y^{\text{gr}} = \text{card}Z^{\text{gr}} = 1 \end{aligned} \quad (20)$$

For example, in the case of addition the result is given by the following equation:

$$\begin{aligned} z^{\text{gr}} &= [\underline{x} + \alpha_x^{\text{gr}}(\bar{x} - \underline{x})] + [\underline{y} + \alpha_y^{\text{gr}}(\bar{y} - \underline{y})], \\ \alpha_x^{\text{gr}}, \alpha_y^{\text{gr}} &\in [0, 1], \\ \text{card}X^{\text{gr}} &= \text{card}Y^{\text{gr}} = \text{card}Z^{\text{gr}} = 1 \end{aligned} \quad (21)$$

The result set Z^{pr} contains only one element because in a real system only one state $(x^{\text{gr}}, y^{\text{gr}}, z^{\text{gr}})$ occurs in a considered, single instant. But in practical problems we also are interested in the set Z^{poss} containing all states (x, y, z) which can occur in a system when our knowledge about

inputs x, y is only approximate and of interval character. The set Z^{poss} contains an infinite number of elements and can be called *ontic set* (Lodwick and Dubois 2015) and its cardinality $\text{card}[Z^{\text{poss}}] = c$ (continuum). It is given by the following equation:

$$\begin{aligned} Z^{\text{poss}} : z^{\text{poss}} &= x^{\text{poss}}(\alpha_x^{\text{poss}}) * y^{\text{poss}}(\alpha_y^{\text{poss}}), \\ \alpha_x^{\text{poss}}, \alpha_y^{\text{poss}} &\in [0, 1], \\ \text{card}X^{\text{poss}} &= \text{card}Y^{\text{poss}} = \text{card}Z^{\text{poss}} = c \end{aligned} \quad (22)$$

Figure 3 shows the set Z^{poss} being addition result of two interval-valued variables x and y about which we know $x \in [1, 3]$ and $y \in [2, 5]$. In this case $x^{\text{poss}} = 1 + 2\alpha_x^{\text{poss}}$, $\alpha_x^{\text{poss}} \in [0, 1]$, $y^{\text{poss}} = 2 + 3\alpha_y^{\text{poss}}$, $\alpha_y^{\text{poss}} \in [0, 1]$, and $z^{\text{poss}} = (1 + 2\alpha_x^{\text{poss}}) + (2 + 3\alpha_y^{\text{poss}})$, $\alpha_x^{\text{poss}}, \alpha_y^{\text{poss}} \in [0, 1]$. For, e.g. $\alpha_x^{\text{poss}} = 0.1$ and $\alpha_y^{\text{poss}} = 0.3$ we achieve one of possible point addition results: $x^{\text{poss}} = 1.2$, $y^{\text{poss}} = 2.6$, $z^{\text{poss}} = 3.8$. The triple $(x^{\text{poss}}, y^{\text{poss}}, z^{\text{poss}}) = (1.2, 2.6, 3.8)$ can be interpreted as one of possible, conditional, addressed addition results (23). It also informs by which elements a and b this concrete result c was created and is of conditional character.

$$\text{IF } (x = 1.2) \text{ AND } (y = 2.6) \text{ THEN } (z = 3.8) \quad (23)$$

The set Z^{poss} is visualized in Fig. 3.

Figure 4 shows the set Z^{poss} in projection on 2D-space $X \times Y$ with marked numerical values of z^{poss} (isoclines $z = \text{const}$).

The set of possible results of arithmetic operations, especially of multiple operations is difficult to imagine. Hence, scientists use various simplified representations or indicators of it as, e.g. span, cardinality of distribution, center of gravity. It can be accepted. However, they frequently understand them as the direct, main result of an

Fig. 3 Visualization of the set Z^{poss} of possible point results of addition $x^{\text{poss}} + y^{\text{poss}} = z^{\text{poss}}$, $x^{\text{poss}} \in [1, 3]$, $y^{\text{poss}} \in [2, 5]$ existing in 3D-space $X \times Y \times Z$

Fig. 4 Projection of the set of all possible addition results z^{poss} from 3D space $X \times Y \times Z$ on 2D space $X \times Y$

operation. It can be observed in all present versions of interval arithmetic. However, the main, direct result of an arithmetic operation should be differentiated from its simplified representation and can upmost be understood as a secondary result derived from the direct result. Also particular arithmetic operations in a sequence of operations should not be realized with their simplified representations but with full multidimensional results. The mostly used indicator of the result set Z^{poss} is its span $s(Z^{\text{poss}})$. The span is expressed by the following equation:

$$s(Z^{\text{poss}}) = \left[\min_{\alpha_x, \alpha_y \in [0,1]} z^{\text{poss}}(\alpha_x, \alpha_y), \max_{\alpha_x, \alpha_y \in [0,1]} z^{\text{poss}}(\alpha_x, \alpha_y) \right] \quad (24)$$

In the case of interval addition the span is given by the known formula as follows:

$$s(Z^{\text{poss}}) = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] = [z_{\min}, z_{\max}] \quad (25)$$

Though the span $s(Z^{\text{poss}})$ is a valuable information, a much more valuable is cardinality distribution $\text{card}(z)$ of particular subsets of possible point results giving the same value of z . $\text{card}(z)$ can also be interpreted as solution frequency measure of particular subsets of the set Z^{poss} (Kovalerchuk and Kreinovich 2016). Figure 4 shows isoclines (lines of constant z values), e.g. $z = 4$, $z = 5$. Lengths of isoclines inform which z values of the result have greater and which have smaller frequency, the a priori chance to occur. As cardinality measure can length $L(z = \text{const})$ of segments corresponding to particular subsets $z = \text{const}$ be assumed. On the basis of Fig. 4 cardinality measures easily can be calculated. They are shown in Fig. 5.

Sometimes, in certain problems, cardinality distribution allows partition of the set of possible point results in sub-ranges of optimistic, pessimistic and neutral results.

Fig. 5 Not normalized distribution of cardinality (frequency) of particular subsets of interval addition results $z = x + y = \text{const}$ as a representative, indicator of the full set Z^{poss} of possible point results

Optimistic results can be interpreted as results in which the Nature, environment, circumstances favours us, is propitious to us. In the case of pessimistic results inversely. For e.g. in Kovalerchuk and Kreinovich (2016) a problem of travelling to an airport is presented. Favouring circumstances are such which allows us to arrive to the airport in short time and not favouring circumstances inversely can cause coming late. Small values of the travelling time have then sense of optimistic point solutions and large values of pessimistic solutions. Between optimistic and pessimistic solution range lies the neutral range. Third indicator of the set Z^{poss} is center of gravity of this set that can be calculated with the following formula:

$$z_{\text{COG}} = \frac{\int_{\min z}^{\max z} z \text{card}(z) dz}{\int_{\min z}^{\max z} \text{card}(z) dz} \quad (26)$$

In the case of the considered addition example $z_{\text{COG}} = 5.5$ (Fig. 6).

3 Solutions of examples of passive WU and active WCh from Kreinovich (2016) with use of RDM interval arithmetic

In Example 1 of passive WU the start amount of water in the reservoir had been equal to $a \in A = [99, 101]$. The released amount of water was $b \in [38, 42]$. The task is to calculate the amount of water $c = a - b$ left in the reservoir.

Fig. 6 Center of gravity of the set Z^{poss} of possible addition results of two intervals as simplified indicator of this set

Further on, mathematical model of the problem and its solution in for of the result set C_{AB}^{passiv} in terms of RDM-IA without use of superscripts poss, gr is presented.

$$\begin{aligned} a &\in A = [99, 101], \dim A = 1 \\ a &= 99 + 2\alpha_a, \alpha_a \in [0, 1] \\ b &\in B = [38, 42], \dim B = 1 \\ b &= 38 + 4\alpha_b, \alpha_b \in [0, 1] \\ c &= a - b = (99 + 2\alpha_a) - (38 + 4\alpha_b), \alpha_a, \alpha_b \in [0, 1] \\ C_{AB}^{\text{passiv}} &= \{(c, a, b) : c = a - b, \\ &(a, b) \in [99, 101] \times [38, 42]\} \\ \dim C_{AB}^{\text{passiv}} &= 3 \end{aligned} \quad (27)$$

The result set C_{AB}^{passiv} in simplified form, as projection on 2D-space $A \times B$ is presented in Fig. 7.

Determining the span $s(C_{AB}^{\text{passiv}})$ of the set of possible solutions C_{AB}^{passiv}

The span of the point solution set in respect of the result variable $c = a - b$ is given by the following equation:

$$s(C_{AB}^{\text{passiv}}) = \left[\min_{\alpha_a, \alpha_b \in [0, 1]} c, \max_{\alpha_a, \alpha_b \in [0, 1]} c \right] \quad (28)$$

The minimal and maximal value of $c = a - b = (99 + 2\alpha_a) - (38 + 4\alpha_b)$, $\alpha_a, \alpha_b \in [0, 1]$ can easily be determined because c monotonically increases with α_a and decreases with α_b . The span of the solution set is equal to $[57, 63]$.

Determining the cardinality (frequency) distribution $\text{card}(C_{AB}^{\text{passiv}})$ of solution subsets $c = \text{const}$ of the set C_{AB}^{passiv} of possible point solutions

Cardinality measure of particular subsets $c = a - b = \text{const}$ is length $L(c)$ of corresponding segments in Fig. 7. The smallest cardinality have values $c = 57$ and $c = 63$.

Fig. 7 Visualization of 3D set C_{AB}^{passiv} of conditional point results of the water c left in the reservoir in Example 1 of passive WU

The greatest cardinality $L(c) = \sqrt{8} = 2.828$ have values $c \in [59, 61]$. The cardinality distribution is shown in Fig. 8.

The cardinality distribution from Fig. 8 can be interpreted as a not-normalized, a priori distribution of probability density of particular c values. It shows that the smallest chance to occur have values $c = 57$ and 63 and the greatest chance the values lying between 59 and 61 . The distribution allows also for easy calculation of center of gravity $c_{\text{COG}}(C_{AB})$ of the result set C_{AB}^{passiv} that in this case is equal to $c_{\text{COG}} = 60$. It has sense of the expected value of the result set.

Solution of Example 2 of active WCh in terms of RDM interval arithmetic

The original amount of water in the reservoir is $a \in A = [99, 101]$. After releasing water in amount $b \in B = [\underline{b}, \bar{b}]$ its end-amount should with full certainty be equal to c that has to satisfy condition $c \in [38, 42]$. The calculation task is determining $B = [\underline{b}, \bar{b}]$. Our knowledge about the problem is as follows. The original amount of water:

$$\begin{aligned} a &\in [99, 101], \dim A = 1 \\ a &= 99 + 2\alpha_a, \alpha_a \in [0, 1]. \end{aligned}$$

where $\dim A$ is the dimension of the space in which the interval set A is defined (set of possible values of the start amount a of water in the reservoir).

We have no influence on this value. The amount of water to be released is at the beginning unknown, but we know that it will be contained in certain interval B :

$$B = [\underline{b}, \bar{b}], \dim B = 1.$$

where $\dim B$ is the dimension of the space in which the interval set B is defined (set of possible values of the released amount of water b).

Precise borders of this interval at the beginning are unknown but we certainly know that they are ≥ 99 and ≤ 101 (releasing all water from the reservoir). In terms of RDM IA the water b to be released is expressed by

Fig. 8 Not normalized cardinality (frequency) distribution of particular result c values of the set of possible results C_{AB}^{passiv}

$$b = \underline{b} + (\bar{b} - \underline{b})\alpha_b, \alpha_b \in [0, 1]$$

The amount $c = a - b$ of water left in the reservoir is given by the following equation:

$$c = a - b = (99 - 2\alpha_a) - [\underline{b} + (\bar{b} - \underline{b})\alpha_b], \underline{b} \leq \bar{b}, \alpha_a, \alpha_b \in [0, 1] \quad (29)$$

Set C_{AB}^{active} contains triples (c, a, b) which means that each of c values is addressed by a double (a, b) as follows:

$$C_{AB}^{\text{active}} = \{(c, a, b) : c = a - b, (a, b) \in [99, 101] \times [\underline{b}, \bar{b}]\}, \dim C_{AB}^{\text{active}} = 3 \quad (30)$$

The set C_{AB}^{active} of possible results c is shown in projection on 2D-space $A \times B$ in Fig. 9. As can be noticed, this set certainly is not one-dimensional as it is assumed in one-dimensional types of interval arithmetic.

Because the result set C_{AB}^{active} is three-dimensional then the requirement $c \in [38, 42]$ as one-dimensional condition refers not directly to the set C_{AB}^{active} but to its one-dimensional span $s(C_{AB}^{\text{active}})$:

$$s(C_{AB}^{\text{active}}) = [38, 42] = [\min c, \max c] \quad (31)$$

Because the function $c = f(\alpha_a, \alpha_b)$, formula (28) is monotonically increasing with α_a and decreasing with α_b , thus $\min c = c(\alpha_a = 0, \alpha_b = 1) = 99 - \bar{b}$ and $\max c = c(\alpha_a = 1, \alpha_b = 0) = 101 - \underline{b}$. Because we know that the required value $\min c = 38 = 99 - \bar{b}$, hence $\bar{b} = 61$, and the required value $\max c = 42 = 101 - \underline{b}$ hence $\underline{b} = 59$. Thus, the value of water b to be released, has to satisfy the condition:

$$b \in B = [59, 61] \quad (32)$$

The result (32) is identical with the result achieved in Kreinovich (2016). The present result (32) has been achieved without using improper intervals used by Kaucher arithmetic, Kaucher (1977) in which $\bar{b} \leq \underline{b}$. Improper intervals are not realizable physically and because of this

Fig. 9 Visualization of the result set C_{AB}^{active} of possible, addressed point results (c, a, b) in projection on the input space $A \times B$, $\underline{b} \leq \bar{b}$

reason they are not accepted by authors of the present paper and by other scientists such as Lodwick and Dubois (2015).

Figure 10 shows the visualization of the fully identified results set C_{AB}^{active} .

Figure 11 shows cardinality distribution of the result set C_{AB}^{active} .

Kreinovich (2016) suggested that “solving different practical problems we can get different solutions of the same system of equations with the same granules”. To support his thesis he presented two examples formulated in terms of 1D interval arithmetic.

Example 1 of passive WU in terms of one-dimensional interval arithmetic $[99, 101] - [38, 42] = [\underline{c}, \bar{c}]$ with solution $[\underline{c}, \bar{c}] = [57, 63]$.

Example 2 of active WCh in terms of one-dimensional interval arithmetic $[99, 101] - [\underline{c}, \bar{c}] = [38, 42]$ with solution $[\underline{c}, \bar{c}] = [59, 61]$.

Both examples contain the same information granules $[99, 101]$, $[38, 42]$ but solutions of them are different: $[57, 63]$ and $[59, 61]$. Thus, the examples that have been achieved with use of one-dimensional approach to interval analysis seem to prove Kreinovich thesis. However, application of multidimensional RDM interval arithmetic shows that mathematical formulations of both examples are different.

Example 1 of passive WU in terms of multidimensional RDM interval arithmetic

$$\begin{aligned} a \in A &= [99, 101], \dim A = 1, \\ a &= 99 + 2\alpha_a, \alpha_a \in [0, 1] \\ b \in B &= [38, 42], \dim B = 1, \\ b &= 38 + 4\alpha_b, \alpha_b \in [0, 1] \\ c &= a - b = (99 + 2\alpha_a) - (38 + 4\alpha_b), \\ \text{solutionset:} \end{aligned} \quad (33)$$

$$C_{AB}^{\text{passiv}} = \{(c, a, b) : c = a - b, (a, b) \in [99, 101] \times [38, 42]\}$$

$$\dim C_{AB}^{\text{passiv}} = 3$$

Fig. 10 Visualization of the fully identified results set C_{AB}^{active} of possible, addressed point results (c, a, b) in projection on the input space $A \times B$

Fig. 11 Not normalized cardinality distribution $\text{card}(c)$ of the water amount c left in the reservoir in Example 2 of active WCh with center of gravity $c_{\text{COG}} = 40$

* No requirement (constraint) is imposed on the result set C_{AB}^{passive} .

Span of the solution set:

$$s(C_{AB}^{\text{active}}) = [57, 63]$$

Example 2 of active WCh in terms of multidimensional RDM interval arithmetic

$$\begin{aligned} a &\in A = [99, 101], \dim A = 1, \\ a &= 99 + 2\alpha_a, \alpha_a \in [0, 1] \\ b &\in B = [\underline{b}, \bar{b}], \underline{b} \leq \bar{b} : \text{proper interval}, \dim B = 1, \\ b &= b + (b - b)\alpha_b, \alpha_b \in [0, 1] \\ c &= a - b = (99 + 2\alpha_a) - [b + (b - b)\alpha_b], \\ C_{AB}^{\text{active}} &= \{(c, a, b), c = a - b, \\ &\quad (a, b) \in [99, 101] \times [\underline{b}, \bar{b}]\}, \\ \dim C_{AB}^{\text{active}} &= 3 \end{aligned} \quad (34)$$

* Requirement (bound) imposed on span $s(C_{AB}^{\text{active}})$ in respect of c : $s(C_{AB}^{\text{active}}) = [\min c, \max c] = [38, 42]$.

Solution of the example:

$$B = [\underline{b}, \bar{b}] = [59, 61]$$

As mathematical formulations (33), (34) shown here exists important difference between the mathematical model of the passive WU and active WCh. These models are not identical. In the problem of passive WU interval B of released water is known from the beginning: $B = [38, 42]$. In the problem of active WCh we only know that value of b is bounded, $b \in [\underline{b}, \bar{b}]$ but the bound borders are not precisely known and have to be determined. In the problem of passive WU there is no requirement imposed on the multidimensional result set C_{AB}^{passive} . Instead, in the problem of active WCh the span of the result set C_{AB}^{active} is restricted by requirement $s(C_{AB}^{\text{active}}) = [38, 42]$. Because mathematical descriptions of both problems are different then the achieved solutions $[57, 63]$ and $[59, 61]$ also are different. It means that the multidimensional approach to IA delivers different results and conditions than one-dimensional

approach. Multidimensional RDM interval arithmetic allows for more precise description of problems than one-dimensional IA because it uses in calculations not only borders of intervals but also their insides. Therefore, it allows for deeper insights into problems than 1-D approach, which simplifies problem formulations, sometimes in a prohibited way and favours achieving imprecise results.

4 Conclusions

The paper, on the basis of discussion about the paper by Kreinovich (2016), has shown that a granular problem should not be solved in an incomplete, low-dimensional space, because it can lead to untrue conclusions such as “solving different practical problems we get different solution of the same system of equations with the same granules”. Reason of achieving such conclusion was application in Kreinovich (2016) of one-dimensional interval arithmetic according to which the main and direct result of arithmetic operations on intervals is also an interval, i.e. the same mathematical 1D-object. Next contribution of the paper is showing that equation systems describing practical problems of “passive understanding the world” and “active changing the world” are not the same but are different. It was possible by dimension increasing of the problems’ space and by applying multidimensional RDM interval arithmetic. Important contribution of the paper is also explaining the difference between direct and indirect results (solutions) of granular problems. Differentiation between these notions prevents misunderstandings between scientists.

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