Interval-valued intuitionistic multiplicative aggregation in group decision making

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Abstract Interval-valued intuitionistic multiplicative information uses an unsymmetrical scale (Saaty's 1–9) instead of the ordinary symmetrical scale to reflect our intuition which is more objective in some special situations. Recently, studies on aggregation techniques with interval-valued intuitionistic multiplicative information for group decision making (GDM) have received increasing attention, and several techniques have been proposed. However, none of them have considered the use of pseudo-multiplication or Sugeno measure, and solved the GDM problems with interrelationships among input information. Since a pseudo-multiplication operator is given by weakening the axiomatic condition which is a key of fuzzy measure, and Sugeno measure is a representative non-additive measure, they are suitable for comprehensive evaluation problems which exist interactions. In this paper, we investigate interval-valued intuitionistic multiplicative aggregation operators and study the GDM problems where the preference information is given by the correlated decision makers. Based on the pseudo-multiplication, we first develop two aggregation operators (the interval-valued intuitionistic multiplicative weighted and ordered weighted averaging operators) and obtain several specific results when some specific forms are assigned. Then, for the

situations where the input arguments have correlations or connections, we give the aggregation operator based on Choquet integral and Sugeno measure, and also study their properties. Based on the proposed aggregation operators, we present a method for GDM with interval-valued intuitionistic multiplicative preference relations, where some correlations or connections are reflected among the decision makers. Finally, we illustrate the given method with a practical example concerning the design of a satellite communication system.

Keywords Interval-valued intuitionistic multiplicative set · Group decision making · Aggregation operator

1 Introduction

Nowadays, decision making exists everywhere in people's daily life, and everyone faces it every day. We even cannot live without it in such a high complex society. Due to the changing and complicated socioeconomic environment, group decision making (GDM) has become a prevalent topic in the last decades (Conde and Pérez 2010; Fang et al. 2012; Jiang et al. 2013, 2015a, b, 2016; Li et al. 2017; Liang et al. 2017; Liu 2009; Orlovsky 1978; Pinar et al. 2017; Qin 2017; Ren et al. 2017; Sanchez et al. 2017; Xia et al. 2013; Xu 2007, 2010; Xu and Chen 2008; Yu 2015a, b; Yu and Shi 2015). In GDM, it is not necessary that a single decision maker should consider all important aspects of practical decision-making problems. As one of the useful tools in the process of GDM, the preference relation can express the decision makers' preferences over the given alternatives (objects or criteria), and it has quickly caught much attention and derived many desirable

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results. The preference relations can be roughly classified into two widely used forms:

1. *Fuzzy preference relations* (Orlovsky 1978). A fuzzy preference relation R on the set $X = \{x_1, x_2, \dots, x_n\}$ is represented by a complementary matrix $R = (r_{ij})_{n \times n}$ with $0 \leq r_{ij} \leq 1$, $r_{ij} + r_{ji} = 1$, $r_{ii} = 0.5$, for all $i, j = 1, 2, \dots, n$, where r_{ij} denotes the preference degree of the alternative x_i over x_j .
2. *Multiplicative preference relations* (Saaty 1980). A multiplicative preference relation P on the set $X = \{x_1, x_2, \dots, x_n\}$ is defined as a reciprocal matrix $P = (p_{ij})_{n \times n}$ under the conditions: $1/9 \leq p_{ij} \leq 9$, $p_{ij} \cdot p_{ji} = 1$, $p_{ii} = 1$, for all $i, j = 1, 2, \dots, n$, where p_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i to x_j .

The main differences between these two kinds of preference relations are the measured scales describing information, the former is based on the 0–1 scale which is a symmetrical distribution around 0.5 (Orlovsky 1978; Xia et al. 2013), while the latter is based on Saaty's 1–9 scale which is an asymmetric distribution around 1 (Saaty 1977) (see Table 1 for more details). Since this paper focuses on the preference information measured by Saaty's 1–9 scale, we will not pay much attention on the preference information measured by the 0–1 scale.

Sometimes, due to the time pressure and lack of information or knowledge, it is difficult for the decision makers to provide their preferences clearly. To describe the imprecision, the interval-valued fuzzy preference relations (Xu and Chen 2008) and the interval-valued multiplicative preference relations (Conde and Pérez 2010; Liu 2009; Wang and Elhag 2007) are provided to allow the decision makers to use the interval-valued numbers (Zadeh 1975) to express their preferences. It is obvious to find that the preference relations mentioned above are only characterized by a membership function describing the strength that one alternative is prior

to another, but never consider the degree that one alternative is not prior to another. To deal with such cases, the intuitionistic fuzzy preference relation (Xu 2007) and the intuitionistic multiplicative preference relation (Xia et al. 2013) are defined to depict the degree that one alternative is prior to another (positive information) and the degree that one alternative is not prior to another (negative information) simultaneously. Especially, the intuitionistic multiplicative preference relation (Xia et al. 2013) can be utilized to deal with the unbalanced distribution commonly found in real-life. Generally speaking, the gap between the grades expressing good information should be larger than the one between the grades reflecting bad information. One example is the law of diminishing marginal utility in economics (Jiang et al. 2013; Xia et al. 2013). It is stated that as the amount of one factor of production increases as all other factors of production hold the same, the marginal return (extra output gained by adding an extra unit) decreases. Another example is the rain attenuation prediction for satellite communication. According to the prediction model in ITU-R (International Telecommunication Union-Radio Communication Sector) (ITU-R RPN.618-4) (1996), with improving a certain frequency, it increases more rain attenuation in the higher operating frequencies than in the lower frequencies, and causes more cost to compensate the signal outage.

However, all the positive and negative information in both the intuitionistic fuzzy and intuitionistic multiplicative preference relations need to be answered with exact values by the decision makers. Then in order to adapt the more complex characteristics of practical GDM problems, based on the interval-valued intuitionistic fuzzy sets (Atanassov and Gargov 1989), Xu and Chen extended the intuitionistic fuzzy preference relation and presented the interval-valued intuitionistic fuzzy preference relation (Xu 2010); Ren et al. proposed a simplified interval-valued intuitionistic fuzzy number which was characterized by two intuitionistic fuzzy numbers (Ren et al. 2016), in which both the positive and negative information are allowed to be given by interval-valued numbers. Recently, as an extension of the intuitionistic multiplicative set (Xia et al. 2013), Jiang et al. introduced another type of set called the interval-valued intuitionistic multiplicative set (IVIMS) (Jiang et al. 2014) and studied its operations and distance measures. Similar to the interval-valued intuitionistic fuzzy set, the IVIMS was developed in view of the cases where the positive or negative information cannot be easily given by an exact value but a probable degree range. In addition, all components of the IVIMS are measured by Saaty's 1–9 scale, not the symmetric 0–1 scale. As an extension of the intuitionistic multiplicative preference relation, Yu et al. developed a new type of preference relation which is called interval-valued intuitionistic multiplicative preference fuzzy relation (Yu et al. 2013) (here we call it as the

Table 1 Saaty's 1–9 scale

1–9 scale	Meaning
1/9	Extremely not preferred
1/7	Very strongly not preferred
1/5	Strongly not preferred
1/3	Moderately not preferred
1	Equally preferred
3	Moderately preferred
5	Strongly preferred
7	Very strongly preferred
9	Extremely preferred
Other values between 1/9 and 9	Intermediate values used to present compromise

interval-valued intuitionistic multiplicative preference relation (IVIMPR), with regard to the homogeneity of the terms) and gave some aggregation operators based on algebraic operational laws to aggregate intuitionistic multiplicative preference information. Although many methods for decision making with interval-valued intuitionistic multiplicative information have been developed (Jiang et al. 2014; Yu 2015a, b; Yu et al. 2013), especially the aggregation techniques, there is no study on the basis of pseudo-multiplication, and no studies have considered Sugeno measure combined with Choquet integral and the situations where there exist the interrelationships among input information. Since Sugeno measure, as one type of the fuzzy measures, is a representative non-additive measure, which is useful for comprehensive evaluation problems which exist interactions in GDM. Based on the pseudo-multiplication operator, which is given by weakening the axiomatic condition, it should be effective for aggregating the interval-valued intuitionistic multiplicative information in GDM.

In this paper, based on the pseudo-multiplication, we give some different aggregation techniques which are more general to aggregate the intuitionistic multiplicative information. Besides, in order to deal with the correlated interval-valued intuitionistic multiplicative information, we apply the Choquet integral (Beliakov et al. 2007; Choquet 1953; Grabisch 1996; Yager 2004) and the Sugeno measure (also called λ -fuzzy measure) (Beliakov et al. 2007; Sugeno 1974) to propose some aggregation operators to solve the decision-making problems. Then we propose an approach to deal with the situations where all correlated decision makers provide their preference information over alternatives by using IVIMPRs.

In order to do that, the remainder of this paper is organized as follows: Sect. 2 briefly reviews some basic concepts. In Sect. 3, we mainly give some concepts related to IVIMPRs. In order to aggregate the preference information in IVIMPRs, Sect. 4 develops some interval-valued intuitionistic multiplicative aggregation operators, including the ones whose input information is non-correlated and correlated with each other. Section 5 presents an approach for GDM with IVIMPRs, and a numerical example concerning the design of a satellite communication system is given to illustrate the proposed approach by comparing it with the existing method. Section 6 ends the paper with some concluding remarks.

2 Preliminaries

In this section, we review some basic concepts. Let $X = \{x_1, x_2, \dots, x_n\}$, then an IVIMS $\tilde{D}$ on X is defined as follows:

Definition 2.1 (Jiang et al. 2014) Let X be fixed, an interval-valued intuitionistic multiplicative set (IVIMS) is defined as

$$\tilde{D} = \{\langle x, \tilde{\rho}_D(x), \tilde{\sigma}_D(x) \rangle \mid x \in X\}, \quad (1)$$

where $\tilde{\rho}_D(x) \subseteq [1/9, 9]$ and $\tilde{\sigma}_D(x) \subseteq [1/9, 9]$ are intervals satisfying $0 < \sup \tilde{\rho}_D(x) \cdot \sup \tilde{\sigma}_D(x) \leq 1$, for every $x \in X$.

Especially, if $\inf \tilde{\rho}_D(x) = \sup \tilde{\rho}_D(x)$ and $\inf \tilde{\sigma}_D(x) = \sup \tilde{\sigma}_D(x)$, then the IVIMS $\tilde{D}$ reduces to an intuitionistic multiplicative set. Here, the pair $(\tilde{\rho}_D(x), \tilde{\sigma}_D(x))$ denotes an interval-valued intuitionistic multiplicative number (IVIMN). For convenience, $(\tilde{\rho}_D(x), \tilde{\sigma}_D(x))$ can be simplified as $(\tilde{\rho}_D, \tilde{\sigma}_D) = ([\rho_D^-, \rho_D^+], [\sigma_D^-, \sigma_D^+])$ without ambiguity, where $0 < \rho_D^+ \sigma_D^+ \leq 1$. Let $\tilde{\tau}_D(x)$ be the range of uncertainty of x to X , where $\tilde{\tau}_D(x) \subseteq [1, 81]$, and $\tilde{\tau}_D(x)$ can be simplified as $\tilde{\tau}_D = [\tau_D^-, \tau_D^+]$ with the conditions $\tau_D^- = 1/\rho_D^+ \sigma_D^+$ and $\tau_D^+ = 1/\rho_D^- \sigma_D^-$.

Definition 2.2 (Yu et al. 2013) Let $\tilde{a}$, $\tilde{a}_1$, and $\tilde{a}_2$ be three IVIMNs, $s(\tilde{a}) = \sqrt{\rho_a^+ \rho_a^- / \sigma_a^- \sigma_a^+}$ be the score function of $\tilde{a}$, and $r(\tilde{a}) = \sqrt{\rho_a^- \rho_a^+ \sigma_a^- \sigma_a^+}$ be the accuracy function of $\tilde{a}$, then

1. If $s(\tilde{a}_1) > s(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;
2. If $s(\tilde{a}_1) = s(\tilde{a}_2)$, then
 - (i) If $r(\tilde{a}_1) > r(\tilde{a}_2)$, then $\tilde{a}_1 > \tilde{a}_2$;
 - (ii) If $r(\tilde{a}_1) = r(\tilde{a}_2)$, then $\tilde{a}_1 = \tilde{a}_2$.

Pseudo-multiplications are a special kind of associative binary relations satisfying a series of properties. It is given by weakening the axiomatic condition (Benvenuti and Mesiar 2004; Sugeno 1974; Sugeno and Murofushi 1987). Based on the pseudo-multiplication $\odot$ which is defined as: $x \odot y = g^{-1}(g(x)g(y))$, where g is a strictly decreasing function such that $g(t) : (0, \infty) \rightarrow (0, \infty)$ (Pap and Ralević 1998), some interval-valued intuitionistic multiplicative operations about IVIMNs were also given (Jiang et al. 2014):

Definition 2.3 (Jiang et al. 2014) Let $\tilde{a} = (\tilde{\rho}_a, \tilde{\sigma}_a) = ([\rho_a^-, \rho_a^+], [\sigma_a^-, \sigma_a^+])$ and $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ ($i = 1, 2$) be three IVIMNs, and $\lambda > 0$, then we have

1. $\tilde{a}_1 + \tilde{a}_2 = \left(\left[h^{-1} \left(h(\rho_{a_1}^-) h(\rho_{a_2}^-) \right), h^{-1} \left(h(\rho_{a_1}^+) h(\rho_{a_2}^+) \right) \right], \left[g^{-1} \left(g(\sigma_{a_1}^-) g(\sigma_{a_2}^-) \right), g^{-1} \left(g(\sigma_{a_1}^+) g(\sigma_{a_2}^+) \right) \right] \right)$;
2. $\tilde{a}_1 \cdot \tilde{a}_2 = \left(\left[g^{-1} \left(g(\rho_{a_1}^-) g(\rho_{a_2}^-) \right), g^{-1} \left(g(\rho_{a_1}^+) g(\rho_{a_2}^+) \right) \right], \left[h^{-1} \left(h(\sigma_{a_1}^-) h(\sigma_{a_2}^-) \right), h^{-1} \left(h(\sigma_{a_1}^+) h(\sigma_{a_2}^+) \right) \right] \right)$;
3. $\lambda \tilde{a} = \left(\left[h^{-1} \left((h(\rho_a^-))^\lambda \right), h^{-1} \left((h(\rho_a^+))^\lambda \right) \right], \left[g^{-1} \left((g(\sigma_a^-))^\lambda \right), g^{-1} \left((g(\sigma_a^+))^\lambda \right) \right] \right)$;

$$4. \quad \tilde{a}^\lambda = \left(\left[g^{-1} \left((g(\rho_a^-))^\lambda \right), g^{-1} \left((g(\rho_a^+))^\lambda \right) \right], \right. \\ \left. \left[h^{-1} \left((h(\sigma_a^-))^\lambda \right), h^{-1} \left((h(\sigma_a^+))^\lambda \right) \right] \right),$$

where g is a strictly decreasing function such that $g(t) : (0, \infty) \rightarrow (0, \infty)$, and $h(t) = g(1/t)$.

Furthermore, the following relations and operations are also proven to be valid in Jiang et al. (2014):

Theorem 2.1 (Jiang et al. 2014) Let $\tilde{a} = (\tilde{\rho}_a, \tilde{\sigma}_a) = ([\rho_a^-, \rho_a^+], [\sigma_a^-, \sigma_a^+])$ and $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ ($i = 1, 2$) be three IVIMNs, and $\lambda, \lambda_1, \lambda_2 > 0$, then we have

1. $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$;
2. $\tilde{a}_1 \cdot \tilde{a}_2 = \tilde{a}_2 \cdot \tilde{a}_1$;
3. $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda\tilde{a}_1 + \lambda\tilde{a}_2$;
4. $(\tilde{a}_1 \cdot \tilde{a}_2)^\lambda = \tilde{a}_1^\lambda \cdot \tilde{a}_2^\lambda$;
5. $\lambda_1\tilde{a} + \lambda_2\tilde{a} = (\lambda_1 + \lambda_2)\tilde{a}$;
6. $\tilde{a}^{\lambda_1} \cdot \tilde{a}^{\lambda_2} = \tilde{a}^{\lambda_1 + \lambda_2}$.

Let $\tilde{a}^C = (\tilde{\sigma}_a, \tilde{\rho}_a)$ denote the complement of $\tilde{a}$, then

7. $(\tilde{a}^C)^\lambda = (\lambda\tilde{a})^C$;
8. $\lambda(\tilde{a}^C) = (\tilde{a}^\lambda)^C$;
9. $\tilde{a}_1^C + \tilde{a}_2^C = (\tilde{a}_1 \cdot \tilde{a}_2)^C$;
10. $\tilde{a}_1^C \cdot \tilde{a}_2^C = (\tilde{a}_1 + \tilde{a}_2)^C$.

3 Interval-valued intuitionistic multiplicative preference relations

Preference relation is one of the convenient techniques to express the decision makers' preferences over alternatives, by which the ranking of alternatives can be obtained. For a set of alternatives $X = \{x_1, x_2, \dots, x_n\}$, the decision makers compare each pair of alternatives and construct judgment matrixes, respectively. If every component in the judgment matrixes is an IVIMN, then the concept of the IVIMPR can be given. For better and further studies, it is necessary to give a precise definition based on Yu et al. (2013) as follows:

Definition 3.1 Let $X = \{x_1, x_2, \dots, x_n\}$ be n alternatives, then the interval-valued intuitionistic multiplicative preference relation is defined as $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, where $\tilde{a}_{ij} = (\tilde{\rho}_{ij}, \tilde{\sigma}_{ij})$ is an IVIMN, for all $i, j = 1, 2, \dots, n$, $\tilde{\rho}_{ij}$ indicates the preference degree of the alternative x_i over x_j , while $\tilde{\sigma}_{ij}$ indicates the preference degree of the alternative x_j over x_i , and both of them should satisfy the following condition:

$$\tilde{\rho}_{ij} = \tilde{\sigma}_{ji}, \quad \tilde{\sigma}_{ij} = \tilde{\rho}_{ji}, \\ 0 < \sup \tilde{\rho}_{ij} \cdot \sup \tilde{\sigma}_{ij} \leq 1, \quad \tilde{\rho}_{ij}, \tilde{\sigma}_{ij} \subseteq [1/9, 9] \quad (2)$$

We can easily get $\tilde{\rho}_{ii} = \tilde{\sigma}_{ii} = ([1, 1], [1, 1])$ for any IVIMPR. Moreover, if $\tau_{ij}^- = 1/\rho_{ij}^+ \sigma_{ij}^+$ and $\tau_{ij}^+ = 1/\rho_{ij}^- \sigma_{ij}^-$, then $\tilde{\tau}_{ij} = [\tau_{ij}^-, \tau_{ij}^+]$ denotes the uncertainty degree of x_i over x_j , where $\tilde{\tau}_{ij} \subseteq [1, 81]$.

Especially, if $\rho_{ij}^+ = \rho_{ij}^-$, $\sigma_{ij}^- = \sigma_{ij}^+$, for all i, j , then the IVIMPR $\tilde{A}$ reduces to an intuitionistic multiplicative preference relation, besides, if $\tau_{ij}^- = \tau_{ij}^+ = 1$, then the IVIMPR $\tilde{A}$ reduces to a traditional multiplicative preference relation.

It is noted that: (a) Compared with the intuitionistic multiplicative preference relations (Xia et al. 2013), IVIMPRs allow the decision makers to give their preferences over alternatives by using probable degree ranges, not exact values necessarily which are sometimes difficult to be given due to the time pressure and lack of information or knowledge. (b) Compared with the interval-valued multiplicative preference relations (Conde and Pérez 2010; Liu 2009; Wang and Elhag 2007), the IVIMPRs contain the decision makers' preferences including the positive and negative information, without ignoring the degree that one alternative is not prior to another. (c) Compared with the interval-valued fuzzy and intuitionistic fuzzy preference relations (Xu 2007; Xu and Chen 2008), the IVIMPRs permit the decision makers to express their preferences over alternatives by an asymmetric measured scale (Saaty's 1–9 scale) which can be used to model the unbalanced distribution (Xia et al. 2013) commonly found in real-life, but not necessarily symmetrical scale.

In what follows, we give some corresponding concepts of the IVIMPR:

Definition 3.2 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an IVIMPR, where $\tilde{a}_{ij} = (\tilde{\rho}_{ij}, \tilde{\sigma}_{ij})$, then $\tilde{A}^T = (\tilde{a}_{ji}^T)_{n \times n} = (\tilde{a}_{ji})_{n \times n}$ is called the transposition of $\tilde{A}$, where $\tilde{a}_{ji} = (\tilde{\rho}_{ji}, \tilde{\sigma}_{ji})$, for all $i, j = 1, 2, \dots, n$.

Definition 3.3 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an IVIMPR, where $\tilde{a}_{ij} = (\tilde{\rho}_{ij}, \tilde{\sigma}_{ij})$, then $S(\tilde{A}) = (s_{ij})_{n \times n}$ is called the score matrix of $\tilde{A}$, where s_{ij} is derived by the score function, such that

$$s_{ij} = \sqrt{\rho_{ij}^+ \rho_{ij}^- / \sigma_{ij}^- \sigma_{ij}^+}, \quad i, j = 1, 2, \dots, n. \quad (3)$$

Definition 3.4 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an IVIMPR, where $\tilde{a}_{ij} = (\tilde{\rho}_{ij}, \tilde{\sigma}_{ij})$, then $R(\tilde{A}) = (r_{ij})_{n \times n}$ is called the accuracy matrix of $\tilde{A}$, where r_{ij} is derived by the accuracy function, such that

$$r_{ij} = \sqrt{\rho_{ij}^+ \rho_{ij}^- \sigma_{ij}^+ \sigma_{ij}^-}, \quad (i, j = 1, 2, \dots, n). \quad (4)$$

Score and accuracy functions are useful tools in the process of matrix calculation. Usually, they are used for solving the decision-making problems, especially GDM.

Theorem 3.1 Let $\tilde{A}^T = (\tilde{a}_{ji}^T)_{n \times n}$ be the transposition of the IVIMPR $\tilde{A}$, then we have $\tilde{A}^T = (\tilde{a}_{ij}^C)_{n \times n}$, where $\tilde{a}_{ij}^C = (\tilde{\sigma}_{ij}, \tilde{\rho}_{ij})$, for all $i, j = 1, 2, \dots, n$.

Proof From Eq. (2), we have $\tilde{\rho}_{ij} = \tilde{\sigma}_{ji}$ and $\tilde{\sigma}_{ij} = \tilde{\rho}_{ji}$ then $\tilde{A}^T = (\tilde{a}_{ji}^T)_{n \times n} = (\tilde{\rho}_{ji}, \tilde{\sigma}_{ji})_{n \times n} = (\tilde{\sigma}_{ij}, \tilde{\rho}_{ij})_{n \times n} = (\tilde{a}_{ij}^C)_{n \times n}$.

Obviously, the transposition of the symmetric IVIMPR equals to itself. Next, we continue to study the score and accuracy matrices.

Theorem 3.2 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an IVIMPR, then $S^T(\tilde{A}) = S(\tilde{A}^T)$, $S(\tilde{A}) \circ S(\tilde{A}^T) = Z_{n \times n}$ (5)

for all $i, j = 1, 2, \dots, n$, where “ $\circ$ ” denotes the Hadamard product and Z indicates a matrix whose elements are all equal to 1.

Proof Due to that $\tilde{A}^T = (\tilde{a}_{ji}^T)_{n \times n}$ is the transposition of $\tilde{A}$, $S(\tilde{A}) = (s_{ij})_{n \times n}$ is the score matrix of $\tilde{A}$, $S^T(\tilde{A})$ is the transposition of $S(\tilde{A})$, and $S(\tilde{A}^T)$ is the score matrix if $\tilde{A}^T$, we have

$$\tilde{A} = \left(\left[\rho_{ij}^-, \rho_{ij}^+ \right], \left[\sigma_{ij}^-, \sigma_{ij}^+ \right] \right)_{n \times n},$$

$$\tilde{A}^T = \left(\left[\rho_{ji}^-, \rho_{ji}^+ \right], \left[\sigma_{ji}^-, \sigma_{ji}^+ \right] \right)_{n \times n},$$

$$S^T(\tilde{A}) = \left(\sqrt{\rho_{ij}^+ \rho_{ij}^- / \sigma_{ij}^- \sigma_{ij}^+} \right)_{n \times n}^T = \left(\sqrt{\rho_{ji}^+ \rho_{ji}^- / \sigma_{ji}^- \sigma_{ji}^+} \right)_{n \times n} = S(\tilde{A}^T).$$

Then based on Theorem 3.1, $\tilde{A}^T = (\tilde{a}_{ij}^C)_{n \times n}$, and thus, we get

$$S(\tilde{A}) \circ S(\tilde{A}^T) = \left(\sqrt{\rho_{ij}^+ \rho_{ij}^- / \sigma_{ij}^- \sigma_{ij}^+} \right)_{n \times n} \circ \left(\sqrt{\rho_{ji}^+ \rho_{ji}^- / \sigma_{ji}^- \sigma_{ji}^+} \right)_{n \times n}$$

$$= \left(\sqrt{\rho_{ij}^+ \rho_{ij}^- / \sigma_{ij}^- \sigma_{ij}^+} \cdot \sqrt{\rho_{ij}^+ \rho_{ij}^- / \sigma_{ij}^- \sigma_{ij}^+} \right)_{n \times n} = Z_{n \times n}.$$

Theorem 3.3 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ be an IVIMPR, then $R^T(\tilde{A}) = R(\tilde{A})$, $R(\tilde{A}) = R(\tilde{A}^T)$ (6)

which implies that R is symmetric, for all $i, j = 1, 2, \dots, n$.

Proof Since $R(\tilde{A}) = (r_{ij})_{n \times n}$ is the accuracy matrix of $\tilde{A}$, $R^T(\tilde{A})$ is the transposition of $R(\tilde{A})$ and $R(\tilde{A}^T)$ is the score matrix if $\tilde{A}^T$, then

$$R(\tilde{A}) = \left(\sqrt{\rho_{ij}^+ \rho_{ij}^- \sigma_{ij}^+ \sigma_{ij}^-} \right)_{n \times n}$$

and from Eq. (2), we have

$$R^T(\tilde{A}) = \left(\sqrt{\rho_{ji}^+ \rho_{ji}^- \sigma_{ji}^+ \sigma_{ji}^-} \right)_{n \times n} = \left(\sqrt{\sigma_{ij}^- \sigma_{ij}^+ \rho_{ij}^- \rho_{ij}^+} \right)_{n \times n} = R(\tilde{A})$$

and then according to Theorem 3.1, we have

$$R(\tilde{A}^T) = \left(\sqrt{\sigma_{ij}^- \sigma_{ij}^+ \rho_{ij}^- \rho_{ij}^+} \right)_{n \times n} = R(\tilde{A}).$$

Next, we use an example to illustrate the IVIMPR:

Example 1 Let $\tilde{A} = (\tilde{a}_{ij})_{n \times n} (i, j = 1, 2, \dots, n)$ be an IVIMPR such that

$$\tilde{A} = \begin{pmatrix} ([1, 1], [1, 1]) & ([3, 5], [1/6, 1/5]) & ([2, 3], [1/8, 1/6]) \\ ([1/6, 1/5], [3, 5]) & ([1, 1], [1, 1]) & ([4, 6], [1/8, 1/7]) \\ ([1/8, 1/6], [2, 3]) & ([1/8, 1/7], [4, 6]) & ([1, 1], [1, 1]) \end{pmatrix}_{3 \times 3}.$$

Then based on Definition 3.2, the transposition of $\tilde{A}$ is $\tilde{A}^T = (\tilde{a}_{ji}^T)_{n \times n}$:

$$\tilde{A}^T = \begin{pmatrix} ([1, 1], [1, 1]) & ([1/6, 1/5], [3, 5]) & ([1/8, 1/6], [2, 3]) \\ ([3, 5], [1/6, 1/5]) & ([1, 1], [1, 1]) & ([1/8, 1/7], [4, 6]) \\ ([2, 3], [1/8, 1/6]) & ([4, 6], [1/8, 1/7]) & ([1, 1], [1, 1]) \end{pmatrix}_{3 \times 3}$$

and based on Definition 3.3, the score matrixes of $\tilde{A}$ and $\tilde{A}^T$ are as follows:

$$S(\tilde{A}) = \begin{pmatrix} 1 & 0.0471 & 0.0589 \\ 21.2132 & 1 & 0.0273 \\ 16.9706 & 36.6606 & 1 \end{pmatrix}_{3 \times 3},$$

$$S(\tilde{A}^T) = \begin{pmatrix} 1 & 21.2132 & 16.9706 \\ 0.0471 & 1 & 36.6606 \\ 0.0589 & 0.0273 & 1 \end{pmatrix}_{3 \times 3}.$$

We can easily find $S^T(\tilde{A}) = S(\tilde{A}^T)$, and

$$S(\tilde{A}) \circ S(\tilde{A}^T) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{3 \times 3}.$$

Based on Definition 3.4, we get the accuracy matrixes of $\tilde{A}$ and $\tilde{A}^T$:

$$R(\tilde{A}) = \begin{pmatrix} 1 & 0.7071 & 0.3536 \\ 0.7071 & 1 & 0.6547 \\ 0.3536 & 0.6547 & 1 \end{pmatrix}_{3 \times 3}$$

$$= R(\tilde{A}^T) = R^T(\tilde{A}).$$

4 Interval-valued intuitionistic multiplicative aggregation operators

To aggregate the interval-valued intuitionistic multiplicative information, in what follows, we develop some interval-valued intuitionistic multiplicative aggregation operators including the ones whose input information is non-correlated or correlated.

4.1 Some aggregation operators with interval-valued intuitionistic multiplicative operations

In this section, we mainly apply the operational laws reviewed in Sect. 2 (Definition 2.3 and Theorem 2.1) to aggregate the interval-valued intuitionistic multiplicative information:

Definition 4.1 (Yu et al. 2013) Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{a}_i$,

where ω_i indicates the importance degree of $\tilde{a}_i$ satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, if

$$\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{i=1}^n \omega_i \tilde{a}_i \quad (7)$$

then the function IVIMWA is called the interval-valued intuitionistic multiplicative weighted averaging (IVIMWA) operator. Here, we call the IMIFWA operator in (Yu et al. 2013) as the IVIMWA operator with regard to the homogeneity of the terms.

Especially, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the IVIMWA operator reduces to the interval-valued intuitionistic multiplicative averaging (IVIMA) operator:

$$\text{IVIMA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \frac{1}{n} \sum_{i=1}^n \tilde{a}_i. \quad (8)$$

Theorem 4.1 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of them, where ω_i indicates the importance degree of $\tilde{a}_i$ satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, then the aggregated value by using the IVIMWA operator is also an IVIMN, and

$$\begin{aligned} \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^-) \right)^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^+) \right)^{\omega_i} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^-) \right)^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^+) \right)^{\omega_i} \right) \right] \right). \end{aligned} \quad (9)$$

Proof By using the mathematical induction on n , for $n = 2$, we have

$$\begin{aligned} \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2) &= \sum_{i=1}^2 \omega_i \tilde{a}_i = \omega_1 \tilde{a}_1 + \omega_2 \tilde{a}_2 \\ &= \left(\left[h^{-1} \left(h \left(h^{-1} \left(\left(h(\rho_{a_1}^-) \right)^{\omega_1} \right) \cdot h \left(h^{-1} \left(\left(h(\rho_{a_2}^-) \right)^{\omega_2} \right) \right) \right), h^{-1} \left(h \left(h^{-1} \left(\left(h(\rho_{a_1}^+) \right)^{\omega_1} \right) \cdot h \left(h^{-1} \left(\left(h(\rho_{a_2}^+) \right)^{\omega_2} \right) \right) \right) \right], \right. \right. \\ &\quad \left. \left[g^{-1} \left(g \left(g^{-1} \left(\left(g(\sigma_{a_1}^-) \right)^{\omega_1} \right) \cdot g \left(g^{-1} \left(\left(g(\sigma_{a_2}^-) \right)^{\omega_2} \right) \right) \right), h^{-1} \left(h \left(h^{-1} \left(\left(g(\sigma_{a_1}^+) \right)^{\omega_1} \right) \cdot h \left(h^{-1} \left(\left(g(\sigma_{a_2}^+) \right)^{\omega_2} \right) \right) \right) \right] \right) \right] \\ &= \left(\left[h^{-1} \left(\left(h(\rho_{a_1}^-) \right)^{\omega_1} \cdot \left(h(\rho_{a_2}^-) \right)^{\omega_2} \right), h^{-1} \left(\left(h(\rho_{a_1}^+) \right)^{\omega_1} \cdot \left(h(\rho_{a_2}^+) \right)^{\omega_2} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\left(g(\sigma_{a_1}^-) \right)^{\omega_1} \cdot \left(g(\sigma_{a_2}^-) \right)^{\omega_2} \right), g^{-1} \left(\left(g(\sigma_{a_1}^+) \right)^{\omega_1} \cdot \left(g(\sigma_{a_2}^+) \right)^{\omega_2} \right) \right] \right). \end{aligned}$$

Suppose that Eq. (9) holds for $n = k$, that is

$$\begin{aligned} \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k) &= \sum_{i=1}^k \omega_i \tilde{a}_i = \omega_1 \tilde{a}_1 + \omega_2 \tilde{a}_2 + \dots + \omega_k \tilde{a}_k \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^+))^{\omega_i} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] \right), \end{aligned}$$

then we have

$$\begin{aligned} \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_k, \tilde{a}_{k+1}) &= \sum_{i=1}^k \omega_i \tilde{a}_i + \omega_{k+1} \tilde{a}_{k+1} \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^+))^{\omega_i} \right) \right], \left[g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] \right) \\ &\quad + \left(\left[h^{-1} \left((h(\rho_{a_{k+1}}^-))^{\omega_{k+1}} \right), h^{-1} \left((h(\rho_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right], \left[g^{-1} \left((g(\sigma_{a_{k+1}}^-))^{\omega_{k+1}} \right), g^{-1} \left((g(\sigma_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right] \right) \\ &= \left(\left[h^{-1} \left(h \left(h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^-))^{\omega_i} \right) \right) h \left(h^{-1} \left((h(\rho_{a_{k+1}}^-))^{\omega_{k+1}} \right) \right) \right), h^{-1} \left(h \left(h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^+))^{\omega_i} \right) \right) h \left(h^{-1} \left((h(\rho_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right) \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(g \left(g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^-))^{\omega_i} \right) \right) g \left(g^{-1} \left((g(\sigma_{a_{k+1}}^-))^{\omega_{k+1}} \right) \right) \right), g^{-1} \left(g \left(g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^+))^{\omega_i} \right) \right) g \left(g^{-1} \left((g(\sigma_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right) \right) \right] \right) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^-))^{\omega_i} \cdot (h(\rho_{a_{k+1}}^-))^{\omega_{k+1}} \right), h^{-1} \left(\prod_{i=1}^k (h(\rho_{a_i}^+))^{\omega_i} \cdot (h(\rho_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^-))^{\omega_i} \cdot (g(\sigma_{a_{k+1}}^-))^{\omega_{k+1}} \right), g^{-1} \left(\prod_{i=1}^k (g(\sigma_{a_i}^+))^{\omega_i} \cdot (g(\sigma_{a_{k+1}}^+))^{\omega_{k+1}} \right) \right] \right) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^{k+1} (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^{k+1} (h(\rho_{a_i}^+))^{\omega_i} \right) \right], \left[g^{-1} \left(\prod_{i=1}^{k+1} (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^{k+1} (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] \right), \end{aligned}$$

i.e., Eq. (9) holds for $n = k + 1$. Thus, Eq. (9) holds for all n .

In addition, we have known that $h(t) = g(1/t)$, and g is a strictly decreasing function such that $g(t) : (0, +\infty) \rightarrow (0, +\infty)$; therefore, $h(t)$ is a strictly increasing function implying that

$$\begin{aligned} 0 &< h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right), \quad h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right), \\ g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right), \quad g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) &< +\infty. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{1}{9} &= \frac{1}{9} \sum_{i=1}^n \omega_i = h^{-1} \left(\prod_{i=1}^n (h(1/9))^{\omega_i} \right) \leq h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right) \\ &\leq h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \leq h^{-1} \left(\prod_{i=1}^n (h(9))^{\omega_i} \right) = 9 \sum_{i=1}^n \omega_i = 9 \\ \frac{1}{9} &= \frac{1}{9} \sum_{i=1}^n \omega_i = g^{-1} \left(\prod_{i=1}^n (g(1/9))^{\omega_i} \right) \leq g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right) \\ &\leq g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \leq g^{-1} \left(\prod_{i=1}^n (g(9))^{\omega_i} \right) = 9 \sum_{i=1}^n \omega_i = 9. \end{aligned}$$

Then, we get

$$\begin{aligned} \left[h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \right] &\subseteq [1/9, 9] \\ \text{and} \quad \left[g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] &\subseteq [1/9, 9] \end{aligned}$$

and

$$\begin{aligned} &h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \cdot g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \\ &\leq h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \cdot g^{-1} \left(\prod_{i=1}^n (g(1/\rho_{a_i}^+))^{\omega_i} \right) \\ &= h^{-1} \left(\sum_{i=1}^n \omega_i h(\rho_{a_i}^+) \right) \times \left(1 / h^{-1} \left(\sum_{i=1}^n \omega_i h(\rho_{a_i}^+) \right) \right) = 1. \end{aligned}$$

Based on the pseudo-multiplication, we can prove the following desirable properties of the IVIMWA operator.

Property 1 If all IVIMNs $\tilde{a}_i (i = 1, 2, \dots, n)$ are equal, i.e., $\tilde{a}_i = \tilde{a} = (\tilde{\rho}_a, \tilde{\sigma}_a)$ for all i , then

$$\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a}. \quad (10)$$

Proof Let $\tilde{a}_i = \tilde{a} = (\tilde{\rho}_a, \tilde{\sigma}_a)$, then

$$\begin{aligned} \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \text{IVIMWA}(\tilde{a}, \tilde{a}, \dots, \tilde{a}) = \sum_{i=1}^n \omega_i \tilde{a} \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n (h(\rho_a^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n (h(\rho_a^+))^{\omega_i} \right) \right], \right. \\ &\quad \left. \times \left[g^{-1} \left(\prod_{i=1}^n (g(\sigma_a^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n (g(\sigma_a^+))^{\omega_i} \right) \right] \right) \\ &= ([h^{-1}(h(\rho_a^-)), h^{-1}(h(\rho_a^+))], [g^{-1}(g(\sigma_a^-)), g^{-1}(g(\sigma_a^+))]) \\ &= ([\rho_a^-, \rho_a^+], [\sigma_a^-, \sigma_a^+]) = \tilde{a}. \end{aligned}$$

Property 1 can be called the idempotence law of the IVIMWA operator.

Property 2 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ and $\tilde{b}_i = (\tilde{\rho}_{b_i}, \tilde{\sigma}_{b_i})$ ($i = 1, 2, \dots, n$) be two collections of IVIMNs, if $\rho_{a_i}^- \leq \rho_{b_i}^-$, $\rho_{a_i}^+ \leq \rho_{b_i}^+$, $\sigma_{a_i}^- \geq \sigma_{b_i}^-$ and $\sigma_{a_i}^+ \geq \sigma_{b_i}^+$, for all i , then

$$\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq \text{IVIMWA}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) \quad (11)$$

Proof Due to $h(t) = g(1/t)$, and $g(t) : (0, +\infty) \rightarrow (0, +\infty)$ is a strictly decreasing function, $h(t)$ is a strictly increasing function. Since $\rho_{a_i}^- \leq \rho_{b_i}^-$, $\rho_{a_i}^+ \leq \rho_{b_i}^+$, $\sigma_{a_i}^- \geq \sigma_{b_i}^-$ and $\sigma_{a_i}^+ \geq \sigma_{b_i}^+$, then

$$h^{-1}\left(\sum_{i=1}^n \omega_i h(\rho_{a_i}^-)\right) \leq h^{-1}\left(\sum_{i=1}^n \omega_i h(\rho_{b_i}^-)\right)$$

$$h^{-1}\left(\sum_{i=1}^n \omega_i h(\rho_{a_i}^+)\right) \leq h^{-1}\left(\sum_{i=1}^n \omega_i h(\rho_{b_i}^+)\right)$$

$$g^{-1}\left(\sum_{i=1}^n \omega_i g(\sigma_{a_i}^-)\right) \geq g^{-1}\left(\sum_{i=1}^n \omega_i g(\sigma_{b_i}^-)\right)$$

$$g^{-1}\left(\sum_{i=1}^n \omega_i g(\sigma_{a_i}^+)\right) \geq g^{-1}\left(\sum_{i=1}^n \omega_i g(\sigma_{b_i}^+)\right)$$

$$\text{IVIMWA}(\tilde{a}_1 + \tilde{b}, \tilde{a}_2 + \tilde{b}, \dots, \tilde{a}_n + \tilde{b})$$

$$\begin{aligned} &= \left(\left[h^{-1}\left(\prod_{i=1}^n (h(h^{-1}(h(\rho_{a_i}^-) \cdot h(\rho_b^-)))\right)^{\omega_i} \right), h^{-1}\left(\prod_{i=1}^n (h(h^{-1}(h(\rho_{a_i}^+) \cdot h(\rho_b^+)))\right)^{\omega_i} \right) \right], \\ &\left[g^{-1}\left(\prod_{i=1}^n (g(g^{-1}(g(\sigma_{a_i}^-) \cdot g(\sigma_b^-)))\right)^{\omega_i} \right), g^{-1}\left(\prod_{i=1}^n (g(g^{-1}(g(\sigma_{a_i}^+) \cdot g(\sigma_b^+)))\right)^{\omega_i} \right) \right] \\ &= \left(\left[h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^-) \cdot h(\rho_b^-))^{\omega_i} \right), h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^+) \cdot h(\rho_b^+))^{\omega_i} \right) \right], \left[g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^-) \cdot g(\sigma_b^-))^{\omega_i} \right), g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^+) \cdot g(\sigma_b^+))^{\omega_i} \right) \right] \right) \end{aligned}$$

$$\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \tilde{b}$$

$$\begin{aligned} &= \left(\left[h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \right], \left[g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] \right) + ([\rho_b^-, \rho_b^+], [\sigma_b^-, \sigma_b^+]) \\ &= \left(\left[h^{-1}\left(h\left(h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right) \cdot h(\rho_b^-)\right)\right), h^{-1}\left(h\left(h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \cdot h(\rho_b^+)\right)\right) \right], \right. \\ &\left. \left[g^{-1}\left(g\left(g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right) \cdot g(\sigma_b^-)\right)\right), g^{-1}\left(g\left(g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \cdot g(\sigma_b^+)\right)\right) \right] \right) \\ &= \left(\left[h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \cdot h(\rho_b^-)\right), h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \cdot h(\rho_b^+)\right) \right], \left[g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \cdot g(\sigma_b^-)\right), g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \cdot g(\sigma_b^+)\right) \right] \right) \\ &= \left(\left[h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^-) \cdot h(\rho_b^-))^{\omega_i} \right), h^{-1}\left(\prod_{i=1}^n (h(\rho_{a_i}^+) \cdot h(\rho_b^+))^{\omega_i} \right) \right], \left[g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^-) \cdot g(\sigma_b^-))^{\omega_i} \right), g^{-1}\left(\prod_{i=1}^n (g(\sigma_{a_i}^+) \cdot g(\sigma_b^+))^{\omega_i} \right) \right] \right) \end{aligned}$$

Thus, based on Definition 2.2, we have

$$s(\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)) \leq s(\text{IVIMWA}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)).$$

Property 2 tells us that if $S(\tilde{a}_i) < S(\tilde{b}_i)$, then $\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq \text{IVIMWA}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n)$. Namely, the IVIMWA operator is a monotonically increasing function with respect to the input argument variable.

Property 3 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ ($i = 1, 2, \dots, n$) be a collection of IVIMNs, if $\tilde{b} = (\tilde{\rho}_b, \tilde{\sigma}_b)$ is an IVIMN, then,

$$\begin{aligned} &\text{IVIMWA}(\tilde{a}_1 + \tilde{b}, \tilde{a}_2 + \tilde{b}, \dots, \tilde{a}_n + \tilde{b}) \\ &= \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \tilde{b}. \end{aligned} \quad (12)$$

Proof Based on Definition 2.3, we have

$$\begin{aligned} \tilde{a}_i + \tilde{b} &= \left(\left[h^{-1}\left(h(\rho_{a_i}^-) \cdot h(\rho_b^-)\right), h^{-1}\left(h(\rho_{a_i}^+) \cdot h(\rho_b^+)\right) \right], \right. \\ &\left. \left[g^{-1}\left(g(\sigma_{a_i}^-) \cdot g(\sigma_b^-)\right), g^{-1}\left(g(\sigma_{a_i}^+) \cdot g(\sigma_b^+)\right) \right] \right) \end{aligned}$$

Therefore,

Then we can get $\text{IVIMWA}(\tilde{a}_1 + \tilde{b}, \tilde{a}_2 + \tilde{b}, \dots, \tilde{a}_n + \tilde{b}) = \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \tilde{b}$.

Property 4 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, if $r > 0$, then

$$\text{IVIMWA}(r\tilde{a}_1, r\tilde{a}_2, \dots, r\tilde{a}_n) = r\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n). \quad (13)$$

Proof According to Definition 4, we have

$$r\tilde{a} = \left[\left[h^{-1} \left((h(\rho_a^-))^r \right), h^{-1} \left((h(\rho_a^+))^r \right) \right], \right. \\ \left. \left[g^{-1} \left((g(\sigma_a^-))^r \right), g^{-1} \left((g(\sigma_a^+))^r \right) \right] \right].$$

Then,

$$\begin{aligned} & \text{IVIMWA}(r\tilde{a}_1, r\tilde{a}_2, \dots, r\tilde{a}_n) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h \left(h^{-1} \left(h(\rho_{a_i}^-) \right)^{r\omega_i} \right) \right) \right), \right. \right. \\ & \quad \left. \left. h^{-1} \left(\prod_{i=1}^n \left(h \left(h^{-1} \left(h(\rho_{a_i}^+) \right)^{r\omega_i} \right) \right) \right) \right], \right. \\ & \quad \left[g^{-1} \left(\prod_{i=1}^n \left(g \left(g^{-1} \left(g(\sigma_{a_i}^-) \right)^{r\omega_i} \right) \right) \right), \right. \\ & \quad \left. \left. g^{-1} \left(\prod_{i=1}^n \left(g \left(g^{-1} \left(g(\sigma_{a_i}^+) \right)^{r\omega_i} \right) \right) \right) \right] \right) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^-) \right)^{r\omega_i} \right), h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^+) \right)^{r\omega_i} \right) \right], \right. \\ & \quad \left. \left[g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^-) \right)^{r\omega_i} \right), g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^+) \right)^{r\omega_i} \right) \right] \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} & r\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \\ &= \left(\left[h^{-1} \left(\left(h \left(h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^-) \right)^{\omega_i} \right) \right) \right)^r \right), \right. \right. \\ & \quad \left. \left. h^{-1} \left(\left(h \left(h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^+) \right)^{\omega_i} \right) \right) \right)^r \right) \right], \right. \\ & \quad \times \left[g^{-1} \left(\left(g \left(g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^-) \right)^{\omega_i} \right) \right) \right)^r \right), \right. \\ & \quad \left. \left. g^{-1} \left(\left(g \left(g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^+) \right)^{\omega_i} \right) \right) \right)^r \right) \right] \right). \end{aligned}$$

Then we can obtain $\text{IVIMWA}(r\tilde{a}_1, r\tilde{a}_2, \dots, r\tilde{a}_n) = r\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$.

Properties 3 and 4 imply that the IVIMWA operator satisfies the associativity and distributivity laws. According to Properties 3 and 4, we can get Property 5 easily:

Property 5 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, if $r > 0$ and $\tilde{b} = (\tilde{\rho}_b, \tilde{\sigma}_b)$ is an IVIMN, then

$$\begin{aligned} & \text{IVIMWA}(r\tilde{a}_1 + \tilde{b}, r\tilde{a}_2 + \tilde{b}, \dots, r\tilde{a}_n + \tilde{b}) \\ &= r\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \tilde{b}. \end{aligned} \quad (14)$$

Property 6 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ and $\tilde{b}_i = (\tilde{\rho}_{b_i}, \tilde{\sigma}_{b_i}) (i = 1, 2, \dots, n)$ be two collections of IVIMNs, then

$$\begin{aligned} & \text{IVIMWA}(\tilde{a}_1 + \tilde{b}_1, \tilde{a}_2 + \tilde{b}_2, \dots, \tilde{a}_n + \tilde{b}_n) \\ &= \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \text{IVIMWA}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n). \end{aligned} \quad (15)$$

Proof Based on Definition 2.3, we have

$$\begin{aligned} \tilde{a}_i + \tilde{b}_i &= \left(\left[h^{-1} \left(h(\rho_{a_i}^-) h(\rho_{b_i}^-) \right), h^{-1} \left(h(\rho_{a_i}^+) h(\rho_{b_i}^+) \right) \right], \right. \\ & \quad \left. \left[g^{-1} \left(g(\sigma_{a_i}^-) g(\sigma_{b_i}^-) \right), g^{-1} \left(g(\sigma_{a_i}^+) g(\sigma_{b_i}^+) \right) \right] \right). \end{aligned}$$

Then

$$\begin{aligned} & \text{IVIMWA}(\tilde{a}_1 + \tilde{b}_1, \tilde{a}_2 + \tilde{b}_2, \dots, \tilde{a}_n + \tilde{b}_n) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h \left(h^{-1} \left(h(\rho_{a_i}^-) \cdot h(\rho_{b_i}^-) \right) \right) \right)^{\omega_i} \right), \right. \right. \\ & \quad \left. \left. h^{-1} \left(\prod_{i=1}^n \left(h \left(h^{-1} \left(h(\rho_{a_i}^+) \cdot h(\rho_{b_i}^+) \right) \right) \right)^{\omega_i} \right) \right], \right. \\ & \quad \left[g^{-1} \left(\prod_{i=1}^n \left(g \left(g^{-1} \left(g(\sigma_{a_i}^-) \cdot g(\sigma_{b_i}^-) \right) \right) \right)^{\omega_i} \right), \right. \\ & \quad \left. \left. g^{-1} \left(\prod_{i=1}^n \left(g \left(g^{-1} \left(g(\sigma_{a_i}^+) \cdot g(\sigma_{b_i}^+) \right) \right) \right)^{\omega_i} \right) \right] \right) \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^-) \cdot h(\rho_{b_i}^-) \right)^{\omega_i} \right), \right. \right. \\ & \quad \left. \left. h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_i}^+) \cdot h(\rho_{b_i}^+) \right)^{\omega_i} \right) \right], \right. \\ & \quad \left[g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^-) \cdot g(\sigma_{b_i}^-) \right)^{\omega_i} \right), \right. \\ & \quad \left. \left. g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_i}^+) \cdot g(\sigma_{b_i}^+) \right)^{\omega_i} \right) \right] \right) \end{aligned}$$

$$\begin{aligned}
& \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) + \text{IVIMWA}(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_n) \\
&= \left(\left[h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \right], \left[g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \right] \right) \\
&+ \left(\left[h^{-1} \left(\prod_{i=1}^n (h(\rho_{b_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n (h(\rho_{b_i}^+))^{\omega_i} \right) \right], \left[g^{-1} \left(\prod_{i=1}^n (g(\sigma_{b_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n (g(\sigma_{b_i}^+))^{\omega_i} \right) \right] \right) \\
&= \left(\left[h^{-1} \left(h \left(h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-))^{\omega_i} \right) \right) h \left(h^{-1} \left(\prod_{i=1}^n (h(\rho_{b_i}^-))^{\omega_i} \right) \right) \right), h^{-1} \left(h \left(h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+))^{\omega_i} \right) \right) h \left(h^{-1} \left(\prod_{i=1}^n (h(\rho_{b_i}^+))^{\omega_i} \right) \right) \right) \right], \right. \\
&\left. \left[g^{-1} \left(g \left(g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-))^{\omega_i} \right) \right) g \left(g^{-1} \left(\prod_{i=1}^n (g(\sigma_{b_i}^-))^{\omega_i} \right) \right) \right), g^{-1} \left(g \left(g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+))^{\omega_i} \right) \right) g \left(g^{-1} \left(\prod_{i=1}^n (g(\sigma_{b_i}^+))^{\omega_i} \right) \right) \right) \right] \right) \\
&= \left(\left[h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^-) \cdot h(\rho_{b_i}^-))^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n (h(\rho_{a_i}^+) \cdot h(\rho_{b_i}^+))^{\omega_i} \right) \right], \left[g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^-) \cdot g(\sigma_{b_i}^-))^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n (g(\sigma_{a_i}^+) \cdot g(\sigma_{b_i}^+))^{\omega_i} \right) \right] \right).
\end{aligned}$$

Good properties of IVIMNs have laid the foundation for our further investigation.

If the function g is assigned different forms, some specific IVIMWA operators can be obtained as follows:

Case 1 If $g(t) = (1+t)/t$, then $h(t) = t+1$, the IVIMWA operator reduces to

$$\begin{aligned}
\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\prod_{i=1}^n (1 + \rho_{a_i}^-)^{\omega_i} - 1, \prod_{i=1}^n (1 + \rho_{a_i}^+)^{\omega_i} - 1 \right], \right. \\
&\left. \left[\frac{\prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}{\prod_{i=1}^n (1 + \sigma_{a_i}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}, \frac{\prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}}{\prod_{i=1}^n (1 + \sigma_{a_i}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}} \right] \right). \quad (16)
\end{aligned}$$

Case 1 is the same as the IMIFWA operator in (Yu et al. 2013).

Case 2 If $g(t) = (2+t)/t$, then $h(t) = 2t+1$, the IVIMWA operator reduces to

$$\begin{aligned}
\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\frac{\prod_{i=1}^n (1 + 2\rho_{a_i}^-)^{\omega_i} - 1}{2}, \frac{\prod_{i=1}^n (1 + 2\rho_{a_i}^+)^{\omega_i} - 1}{2} \right], \right. \\
&\left. \left[\frac{2 \prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}{\prod_{i=1}^n (2 + \sigma_{a_i}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}, \frac{2 \prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}}{\prod_{i=1}^n (2 + \sigma_{a_i}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}} \right] \right). \quad (17)
\end{aligned}$$

Case 3 If $g(t) = (\gamma+t)/t$, $\gamma > 0$, then $h(t) = \gamma t+1$, the IVIMWA operator reduces to

$$\begin{aligned}
\text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\frac{\prod_{i=1}^n (1 + \gamma\rho_{a_i}^-)^{\omega_i} - 1}{\gamma}, \frac{\prod_{i=1}^n (1 + \gamma\rho_{a_i}^+)^{\omega_i} - 1}{\gamma} \right], \right. \\
&\left. \left[\frac{\gamma \prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}{\prod_{i=1}^n (\gamma + \sigma_{a_i}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^-)^{\omega_i}}, \frac{\gamma \prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}}{\prod_{i=1}^n (\gamma + \sigma_{a_i}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_i}^+)^{\omega_i}} \right] \right). \quad (18)
\end{aligned}$$

Especially, if $\gamma = 1$, then Eq. (18) reduces to Eq. (16); if $\gamma = 2$, then Eq. (18) reduces to Eq. (17).

Motivated by the ordered weighted averaging (OWA) operator (Yager 1988), we can define the following aggregation operator:

Definition 4.2 (Yu et al. 2013) Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ ($i = 1, 2, \dots, n$) be a collection of IVIMNs, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ be the weight vector of $\tilde{a}_{\text{ind}(i)}$ such that $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, if

$$\text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{i=1}^n \omega_i \tilde{a}_{\text{ind}(i)} \quad (19)$$

then the function IVIMOWA is called the interval-valued intuitionistic multiplicative ordered weighted averaging (IVI-MOWA) operator, where $\text{ind}(i): \{i = 1, 2, \dots, n\} \rightarrow \{i = 1, 2, \dots, n\}$ is a non-increasing permutation, such that $\tilde{a}_{\text{ind}(i)} \geq \tilde{a}_{\text{ind}(i+1)}$. Without ambiguousness, $\text{ind}(i)$ can be simplified as (i) , such that $\tilde{a}_{(i)}$ is the i -th largest $\tilde{a}_i$. Similarly, here we call the IMIFOWA operator in (Yu et al. 2013) as the IVIMWA operator with regard to the homogeneity of the terms.

Especially, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the IVI-MOWA operator reduces to the IVIMA operator of IVIMNs (Eq. (8)).

In a similar way, based on Definition 4.2, we have

Theorem 4.2 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ ($i = 1, 2, \dots, n$) be a collection of IVIMNs, and $(i): \{i = 1, 2, \dots, n\} \rightarrow \{i = 1, 2, \dots, n\}$ be a non-increasing permutation such that $\tilde{a}_{(i)} \geq \tilde{a}_{(i+1)}$. Suppose that $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $\tilde{a}_{(i)}$, satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$, then the aggregated value by using the IVIMOWA operator is also an IVIMN, and

$$\begin{aligned} \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \sum_{i=1}^n \omega_i \tilde{a}_{(i)} \\ &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_{(i)}}^-) \right)^{\omega_i} \right), h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a_{(i)}}^+) \right)^{\omega_i} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_{(i)}}^-) \right)^{\omega_i} \right), g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a_{(i)}}^+) \right)^{\omega_i} \right) \right] \right). \end{aligned} \quad (20)$$

The proof is similar to that of Theorem 4.1, here we omit it for the space's limit.

Furthermore, we can easily get that the IVIMOWA operator satisfies Properties 1–6, and it also has the following property:

Property 7 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, if $\tilde{a}_{\xi(i)} (i = 1, 2, \dots, n)$ is any permutation of $\tilde{a}_i$, then

$$\begin{aligned} \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \\ = \text{IVIMOWA}(\tilde{a}_{\xi(1)}, \tilde{a}_{\xi(2)}, \dots, \tilde{a}_{\xi(n)}). \end{aligned} \quad (21)$$

If the function g is assigned different forms, then some specific IVIMOWA operators can be obtained as follows, where $\tilde{a}_{(i)}$ is a non-increasing permutation of $a_i (i = 1, 2, \dots, n)$.

Case 1' If $g(t) = (1+t)/t$, then $h(t) = t+1$, the IVIMOWA operator reduces to

$$\begin{aligned} \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \\ = \left(\left[\prod_{i=1}^n (1 + \rho_{a_{(i)}}^-)^{\omega_i} - 1, \prod_{i=1}^n (1 + \rho_{a_{(i)}}^+)^{\omega_i} - 1 \right], \right. \\ \left. \left[\frac{\prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}{\prod_{i=1}^n (1 + \sigma_{a_{(i)}}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}, \frac{\prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}}{\prod_{i=1}^n (1 + \sigma_{a_{(i)}}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}} \right] \right). \end{aligned} \quad (22)$$

Cased 1' is the same as the IMIFOWA operator in (Yu et al. 2013).

Case 2' If $g(t) = (2+t)/t$, then $h(t) = 2t+1$, the IVIMOWA operator reduces to

$$\begin{aligned} \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \\ = \left(\left[\frac{\prod_{i=1}^n (1 + 2\rho_{a_{(i)}}^-)^{\omega_i} - 1}{2}, \frac{\prod_{i=1}^n (1 + 2\rho_{a_{(i)}}^+)^{\omega_i} - 1}{2} \right], \right. \\ \left. \left[\frac{2 \prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}{\prod_{i=1}^n (2 + \sigma_{a_{(i)}}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}, \frac{2 \prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}}{\prod_{i=1}^n (2 + \sigma_{a_{(i)}}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}} \right] \right). \end{aligned} \quad (23)$$

Case 3' If $g(t) = (\gamma+t)/t, \gamma > 0$, then $h(t) = \gamma t + 1$, the IVIMOWA operator reduces to

$$\begin{aligned} \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \\ = \left(\left[\frac{\prod_{i=1}^n (1 + \gamma \rho_{a_{(i)}}^-)^{\omega_i} - 1}{\gamma}, \frac{\prod_{i=1}^n (1 + \gamma \rho_{a_{(i)}}^+)^{\omega_i} - 1}{\gamma} \right], \right. \\ \left. \left[\frac{\gamma \prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}{\prod_{i=1}^n (\gamma + \sigma_{a_{(i)}}^-)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^-)^{\omega_i}}, \frac{\gamma \prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}}{\prod_{i=1}^n (\gamma + \sigma_{a_{(i)}}^+)^{\omega_i} - \prod_{i=1}^n (\sigma_{a_{(i)}}^+)^{\omega_i}} \right] \right). \end{aligned} \quad (24)$$

Especially, if $\gamma = 1$, then Eq. (24) reduces to Eq. (22); if $\gamma = 2$, then Eq. (24) reduces to Eq. (23).

The characteristic of the IVIMOWA operator is that it first reorders the given IVIMNs in descending order, and then aggregates all the ordered IVIMNs. In other words, a particular input $\tilde{a}_i$ is not associated with a particular weight ω_i , whereas a particular ordered position (i) is associated with a particular weight ω_i . Thus, the IVIMWA operator emphasizes the IVIMNs themselves, while the IVIMOWA operator weights the ordered position of the IVIMNs.

We can easily find that the IMIFWA and IMIFOWA operators in (Yu et al. 2013) are two specific examples ($\gamma = 1$ which is the simplest case) of the IVIMWA and IVIMOWA operators in this paper.

Next, we give an example to illustrate the proposed aggregation operators:

Example 2 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, 3)$ be a collection of IVIMNs such that

$$\begin{aligned} \tilde{a}_1 &= ([3, 5], [1/6, 1/5]), \\ \tilde{a}_2 &= ([1/6, 1/3], [1/8, 1/7]), \\ \tilde{a}_3 &= ([2, 3], [1/8, 1/6]), \end{aligned}$$

and $\omega = (1/4, 1/2, 1/4)^T$ be the weight vector of $\tilde{a}_i$. If we utilize the IVIMWA operator to aggregate the three IVIMNs:

Case 1 Let $\gamma = 1$, then based on Eq. (16), we have

$$\begin{aligned} \tilde{a}_{\gamma=1} &= \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([1.0103, 1.5558], [0.1342, 0.1613]). \end{aligned}$$

Case 2 Let $\gamma = 2$, then based on Eq. (17), we have

$$\begin{aligned} \tilde{a}_{\gamma=2} &= \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([0.9043, 1.4121], [0.1343, 0.1614]). \end{aligned}$$

Case 3 Let $\gamma = 10$, then based on Eq. (18), we have

$$\begin{aligned} \tilde{a}_{\gamma=10} &= \text{IVIMWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([0.7249, 1.2126], [0.1343, 0.1615]). \end{aligned}$$

In order to observe the variations of the IVIMWA operator with respect to the parameter γ , we assign γ different values $0 < \gamma \leq 50$. Then the variations of the values

Fig. 1 ρ_{γ}^{-} , ρ_{γ}^{+} , σ_{γ}^{-} , and σ_{γ}^{+} of $\tilde{a}_{0 < \gamma \leq 50}$ with the IVIMWA operator

Fig. 2 The scores of $\tilde{a}_{0 < \gamma \leq 50}$ with the IVIMWA operator

ρ_{γ}^{-} , ρ_{γ}^{+} , σ_{γ}^{-} , and σ_{γ}^{+} can be found clearly in Fig. 1, respectively. For clear observation, the scores of $\tilde{a}_{0 < \gamma \leq 50}$ are presented in Fig. 2. If we utilize the IVIMOWA operator to aggregate these IVIMNs, then we need to rank the three IVIMNs according to the scores of $\tilde{a}_i$ which can be calculated:

$$s(\tilde{a}_1) = 21.2132, \quad s(\tilde{a}_2) = 1.7638, \quad s(\tilde{a}_3) = 16.9706$$

by which we can obtain the new non-decreasing ranking:

$$\tilde{a}_{(1)} = \tilde{a}_1, \quad \tilde{a}_{(2)} = \tilde{a}_3, \quad \tilde{a}_{(3)} = \tilde{a}_2$$

Case 1' Let $\gamma = 1$, then based on Eq. (22), we have

$$\begin{aligned} \tilde{a}_{\gamma=1} &= \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([1.5457, 2.3636], [0.1342, 0.1677]). \end{aligned}$$

Case 2' Let $\gamma = 2$, then based on Eq. (23), we have

$$\begin{aligned} \tilde{a}_{\gamma=2} &= \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([1.4542, 2.2373], [0.1343, 0.1678]). \end{aligned}$$

Case 3' Let $\gamma = 10$, then based on Eq. (24), we have

$$\begin{aligned} \tilde{a}_{\gamma=10} &= \text{IVIMOWA}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \\ &= ([1.2818, 2.0467], [0.1343, 0.1678]). \end{aligned}$$

Similarly, we assign the parameter γ different values $0 < \gamma \leq 50$ in order to observe the variations of the IVIMOWA operator with respect to the parameter γ . Then the variations of the values ρ_{γ}^{-} , ρ_{γ}^{+} , σ_{γ}^{-} , and σ_{γ}^{+} can be observed clearly in Fig. 3, respectively, and the scores of $\tilde{a}_{0 < \gamma \leq 50}$ are presented in Fig. 4 for clear observation.

Comparing Fig. 2 with Fig. 4, we can find that both the scores in Figs. 2 and 3 decrease as the value of the parameter γ increases, and the scores obtained by utilizing the IVIMOWA operator (see Fig. 4) are larger than the ones obtained by using the IVIMWA operator (see Fig. 2). Obviously, the scores in Fig. 2 concentrate between 6 and 10, whereas all the scores in Fig. 4 are larger than 10, and concentrate between 10 and 14. Actually, according to the characteristic of the IVIMOWA operator, those largest and smallest IVIMNs ($\tilde{a}_1$ and $\tilde{a}_2$) are assigned the smallest weights ($\omega_1 = \omega_3 = 1/4$), while the other IVIMN ($\tilde{a}_3$) is assigned the largest weight ($\omega_2 = 1/2$). Thus, the scores derived by the IVIMOWA operator are closer to the value of $s(\tilde{a}_3)$ ($s(\tilde{a}_3) = 16.9706$) than the ones calculated by the IVIMWA operator.

4.2 Some aggregation operators for the correlated inputs

In this section, we mainly propose some aggregation operators to reflect the correlations or connections of the aggregated input IVIMNs based on Choquet integral (Beliakov et al. 2007; Choquet 1953; Grabisch 1996; Yager 2004) and Sugeno measure (also called λ -fuzzy measure) (Beliakov et al. 2007; Sugeno 1974). Choquet integral is a useful class of aggregation functions to model interactions between inputs. Choquet Integration was defined with respect to not necessarily additive monotone measure, called fuzzy measures. Fuzzy measure is an offspring of classical measure, and the Sugeno measure is one typical non-additive of them which is suitable for comprehensive evaluation problems which exist interactions. Different types of fuzzy measures would lead to specific features. The following Choquet integral-based aggregation operator is the combination of Choquet integral and Sugeno measure.

Firstly, we introduce some basic concepts:

Let $N = \{1, 2, \dots, n\}$, $X = \{x_1, x_2, \dots, x_n\}$ and $E = \{e_1, e_2, \dots, e_n\}$ be three finite sets.

Fig. 3 ρ_γ^- , ρ_γ^+ , σ_γ^- , and σ_γ^+ of $\tilde{a}_{0 < \gamma \leq 50}$ with the IVIMOWA operator

Fig. 4 The scores of $\tilde{a}_{0 < \gamma \leq 50}$ with the IVIMOWA operator

Definition 4.3 (Sugeno 1974) A discrete fuzzy measure is a set function $\mu : 2^N \rightarrow [0, 1]$ which is monotonic and satisfies $\mu(\emptyset) = 0$ and $\mu(N) = 1$.

Definition 4.4 (Beliakov et al. 2007; Grabisch 1996) The discrete Choquet integral with respect to a fuzzy measure μ is given by

$$C_\mu(X) = \sum_{i=1}^n x_{(i)} [\mu(\{j | x_j \geq x_{(i)}\}) - \mu(\{j | x_j \geq x_{(i-1)}\})], \quad (25)$$

where $\{x_{(1)}, x_{(2)}, \dots, x_{(n)}\}$ is a non-increasing permutation of X , and $x_{(0)} = \infty$ by convention.

Definition 4.5 (Beliakov et al. 2007; Sugeno 1974) Given a parameter $\tau \in (-1, \infty)$, a Sugeno measure is a fuzzy measure m on the set E , for all $P, Q \subseteq E$, $P \cap Q = \emptyset$, satisfying

$$m(P \cup Q) = m(P) + m(Q) + \tau m(P)m(Q). \quad (26)$$

Suppose that $E_j (j = 1, 2, \dots, 2^n)$ is any subset of E , then all values $m(E_j)$ can be easily computed from n independent values $m(e_i) (i = 1, 2, \dots, n)$, by utilizing the following explicit formula given by Sugeno (Sugeno 1974):

$$m(E_j) = \begin{cases} \frac{1}{\tau} \left(\prod_{e_i \in E_j} (1 + \tau m(e_i)) - 1 \right), & \tau \neq 0 \\ \sum_{e_i \in E_j} m(e_i), & \tau = 0 \end{cases} \quad (27)$$

The value of τ can be uniquely determined from $m(E) = 1$, which can be written as

$$\tau + 1 = \prod_{i=1}^n (1 + \tau m(e_i)). \quad (28)$$

If we apply the Sugeno measure m to the Choquet integral when all input arguments are IVIMNs, then we can define the Choquet integral-based aggregation operator for IVIMNs:

Definition 4.6 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs defined on the set $E = \{e_1, e_2, \dots, e_n\}$, if

$$C_m\text{-IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \sum_{i=1}^n (m(E_i) - m(E_{i-1})) \cdot \tilde{a}_{(i)}, \quad (29)$$

then the function $C_m\text{-IVIM}$ is called as the Choquet integral-based aggregation operator of the IVIMNs ($C_m\text{-IVIM}$ operator), where (i) is the index of i -th largest $\tilde{a}_i$, implying $\tilde{a}_{(i)} \geq \tilde{a}_{(i+1)}$, $E_i = \{e_{(1)}, \dots, e_{(i)}\}$ and $m(E_i)$ is the Sugeno measure of E_i . We let $E_0 = \emptyset$ by convention.

The $C_m\text{-IVIM}$ operator has the following properties:

Property 8 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i})$ and $\tilde{a}'_i = (\tilde{\rho}_{a'_i}, \tilde{\sigma}_{a'_i}) (i = 1, 2, \dots, n)$ be two collections of IVIMNs, if $\tilde{a}_i \leq \tilde{a}'_i$, then $C_m\text{-IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) \leq C_m\text{-IVIM}(\tilde{a}'_1, \tilde{a}'_2, \dots, \tilde{a}'_n)$

namely, the $C_m\text{-IVIM}$ operator is a monotonically increasing function with respect to input argument variable.

Proof This proposition can be obtained by taking partial derivative of the $C_m\text{-IVIM}$ operator:

$$\begin{aligned} \frac{\partial C_m\text{-IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)}{\partial \tilde{a}_i} &= \frac{\partial \sum_{i=1}^n (m(E_i) - m(E_{i-1})) \cdot \tilde{a}_{(i)}}{\partial \tilde{a}_{(i)}} \\ &= m(E_i) - m(E_{i-1}). \end{aligned}$$

Since $m(E) = 1$, $m(\emptyset) = 0$, and $E_i \subseteq E (i = 1, 2, \dots, n)$, then $0 < m(E_i) \leq 1$ and $0 < m(E_{i-1}) \leq 1$. Therefore, $m(E_i) - m(E_{i-1}) > 0$, then we have

$$\frac{\partial C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)}{\partial \tilde{a}_i} > 0.$$

Furthermore, the following properties can be easily obtained:

Property 9 If all IVIMNs $\tilde{a}_i (i = 1, 2, \dots, n)$ are equal, i.e., $\tilde{a}_i = \tilde{a} = (\tilde{\rho}_a, \tilde{\sigma}_a)$ for all i , then

$$C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \tilde{a}. \quad (30)$$

Property 9 can be called the idempotence law of the C_m -IVIM operator.

Property 10 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs, if $\tilde{a}_{\xi(i)}$ is any permutation of $\tilde{a}_i$, then

$$C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = C_m - \text{IVIM}(\tilde{a}_{\xi(1)}, \tilde{a}_{\xi(2)}, \dots, \tilde{a}_{\xi(n)}). \quad (31)$$

Based on the IVIMOWA operator (Eq. (20)), we can have the following theorem:

Theorem 4.3 Let $\tilde{a}_i = (\tilde{\rho}_{a_i}, \tilde{\sigma}_{a_i}) (i = 1, 2, \dots, n)$ be a collection of IVIMNs defined on the set $E = \{e_1, e_2, \dots, e_n\}$, then the aggregated value by using the C_m -IVIM operator is also an IVIMN, and

$$\begin{aligned} C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a(i)}) \right)^{m(E_i) - m(E_{i-1})} \right), \right. \right. \\ &\quad \left. \left. h^{-1} \left(\prod_{i=1}^n \left(h(\rho_{a(i)}^+) \right)^{m(E_i) - m(E_{i-1})} \right) \right], \right. \\ &\quad \times \left[g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a(i)}^-) \right)^{m(E_i) - m(E_{i-1})} \right), \right. \\ &\quad \left. \left. g^{-1} \left(\prod_{i=1}^n \left(g(\sigma_{a(i)}^+) \right)^{m(E_i) - m(E_{i-1})} \right) \right] \right), \end{aligned} \quad (32)$$

where (i) is the index of i -th largest $\tilde{a}_i$, $E_i = \{e_{(1)}, \dots, e_{(i)}\}$, and $m(E_i)$ is the Sugeno measure of E_i . Let $E_0 = \emptyset$ by convention.

Proof Based on Property 8, we have $m(E_i) - m(E_{i-1}) > 0$. Since $m(E) = 1$ and $m(\emptyset) = 0$, then $0 < m(E_i) - m(E_{i-1}) \leq 1$ and $\sum_{i=1}^n (m(E_i) - m(E_{i-1})) = m(E) = 1$, satisfying the conditions of the weight vector ω in Theorem 4.2 (Eq. (20)). The remaining steps of the proof are similar as that of Theorem 4.1; here, we omit it for the space's limit.

Actually, if we extend the Choquet integral to the IVIMOWA operator proposed in Sect. 4.1 by $\omega_i = m(E_i) - m(E_{i-1})$, we can obtain Eq. (32) as well.

Similar to Cases 1–3 and Cases 1'–3', we assign the function g in Eq. (32) different forms, then some specific C_m -IVIM operators can be got, where $\tilde{a}_{(i)}$ is the i -th largest $\tilde{a}_i$, $E_i = \{e_{(1)}, \dots, e_{(i)}\}$, and $m(E_i)$ is the Sugeno measure of $E_i (i = 1, 2, \dots, n)$:

Case 1'' If $g(t) = (1 + t)/t$, then $h(t) = t + 1$, the C_m -IVIM operator reduces to

$$\begin{aligned} C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\frac{\prod_{i=1}^n (1 + \gamma \rho_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - 1}{\gamma}, \frac{\prod_{i=1}^n (1 + \gamma \rho_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - 1}{\gamma} \right], \right. \\ &\quad \left[\frac{\gamma \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (\gamma + \sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}, \right. \\ &\quad \left. \frac{\gamma \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (\gamma + \sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}} \right] \right). \end{aligned} \quad (33)$$

Case 2'' If $g(t) = (2 + t)/t$, then $h(t) = 2t + 1$, and the C_m -IVIM operator reduces to

$$\begin{aligned} C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\frac{\prod_{i=1}^n (1 + 2\rho_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - 1}{2}, \frac{\prod_{i=1}^n (1 + 2\rho_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - 1}{2} \right], \right. \\ &\quad \left[\frac{2 \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (2 + \sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}, \right. \\ &\quad \left. \frac{2 \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (2 + \sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}} \right] \right). \end{aligned} \quad (34)$$

Case 3'' If $g(t) = (\gamma + t)/t$, $\gamma > 0$, then $h(t) = \gamma t + 1$, and the C_m -IVIM operator reduces to

$$\begin{aligned} C_m - \text{IVIM}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) &= \left(\left[\frac{\prod_{i=1}^n (1 + \gamma \rho_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - 1}{\gamma}, \frac{\prod_{i=1}^n (1 + \gamma \rho_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - 1}{\gamma} \right], \right. \\ &\quad \left[\frac{\gamma \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (\gamma + \sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^-)^{m(E_i) - m(E_{i-1})}}, \right. \\ &\quad \left. \frac{\gamma \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}}{\prod_{i=1}^n (\gamma + \sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})} - \prod_{i=1}^n (\sigma_{a(i)}^+)^{m(E_i) - m(E_{i-1})}} \right] \right). \end{aligned} \quad (35)$$

Especially, if $\gamma = 1$, then Eq. (35) reduces to Eq. (33); if $\gamma = 2$, then Eq. (35) reduces to Eq. (34).

Generally speaking, it is not easy to determine fuzzy measure. Sometimes, it is even much more difficult than determining the weights in the weighted average method, due to the non-additive of fuzzy measures. Many scholars have devoted themselves to constructing fuzzy measures and some methods have been developed. For example, Wang and Wang (1997) used a neural network algorithm to optimize the inverse problem of synthetic evaluation to determine the Sugeno measures by the Choquet integral and statistics of given data. Combarro and Miranda (2010) studied the vertices of polytopes on the structure of the k -additive measures and provided an algorithm to compute the vertices described by the authors. How to determine the fuzzy measure is recognized as a challenging problem. Because it is not the essence of this paper, we will not elaborate it here and assign n values in advance in the following examples, which can be determined by using the existing method (Wang and Wang 1997).

Next, we use a simple example adapted from (Beliakov et al. 2007) to illustrate the proposed aggregation operators:

Example 3 Consider a problem involving the evaluation of two students $X = \{x_1, x_2\}$ in a same class with respect to three subjects $E = \{e_1, e_2, e_3\}$: mathematics, physics, and literature. Their marks expressed by IVIMNs are listed in Table 2.

Usually, this is done by using a weighted arithmetic mean, whose weights are interpreted as the importance degrees of different subjects. Let $g(t) = (\gamma + t)/t$, $h(t) = \gamma t + 1$, $\gamma > 0$, then based on Eq. (18), we can easily obtain the aggregated IVIMNs by using the IVIMWA operator.

If $\gamma = 1$, Eq. (18) reduces to Eq. (16), then

$$\tilde{a}_1 = ([6.7460, 11.9615], [0.0438, 0.0577])$$

$$\tilde{a}_2 = ([8.7980, 7.4853], [0.0450, 0.0597]).$$

By calculating the scores $s(\tilde{a}_i)$ of $\tilde{a}_i (i = 1, 2)$, we have $s(\tilde{a}_1) = 8.9829$, $s(\tilde{a}_2) = 8.1151$.

Thus, $\tilde{a}_1 > \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_1 \succ x_2$.

Similarly, we can obtain the same ranking $x_1 \succ x_2$ when $\gamma = 2$ and $\gamma = 10$.

However, it is noticed that students who are good at mathematics are usually also good at physics and vice versa, as these

disciplines present some overlap. Thus, the evaluation by a weighted arithmetic mean would be either overestimated or underestimated for the students who are good at mathematics and physics. In what follows, we use the Sugeno measure to obtain the correlated importance of subjects. Assume that

$$m(\emptyset) = 0, \quad m(\{e_1\}) = 0.3,$$

$$m(\{e_2\}) = 0.2, \quad m(\{e_3\}) = 0.4.$$

By Eqs. (27) and (28), we have

$$m(\{e_1, e_2\}) = 0.5223, \quad m(\{e_1, e_3\}) = 0.7446,$$

$$m(\{e_2, e_3\}) = 0.6297, \quad m(\{e_1, e_2, e_3\}) = 1.$$

By calculating the scores of $\tilde{a}_{ij} (i = 1, 2 \text{ and } j = 1, 2, 3)$:

$$s(a_{11}) = 36.6606, \quad s(a_{12}) = 21.2132,$$

$$s(a_{13}) = 16.9706$$

$$s(a_{21}) = 10.9545, \quad s(a_{22}) = 12.0000,$$

$$s(a_{23}) = 63.4980$$

we can re-rank the IVIMNs:

$$\tilde{a}_{1(1)} = \tilde{a}_{11}, \quad \tilde{a}_{1(2)} = \tilde{a}_{12}, \quad \tilde{a}_{1(3)} = \tilde{a}_{13}$$

$$\tilde{a}_{2(1)} = \tilde{a}_{23}, \quad \tilde{a}_{2(2)} = \tilde{a}_{22}, \quad \tilde{a}_{2(3)} = \tilde{a}_{21}$$

Then we can compute the weights of subjects with respect to $x_i (i = 1, 2)$, respectively, which are listed in Table 3 as follows:

Then we use the C_m -IVIM operator to obtain the average value $\tilde{a}_i$ of the students $x_i (i = 1, 2)$:

Case 1'' Let $\gamma = 1$, then based on Eq. (33), we have

$$\tilde{a}_1 = ([2.7278, 4.1775], [0.1331, 0.1655]),$$

$$\tilde{a}_2 = ([3.6085, 3.6060], [0.1437, 0.1750]).$$

Calculating the scores $s(\tilde{a}_i) (i = 1, 2)$ of $\tilde{a}_i (i = 1, 2)$:

$$s(\tilde{a}_1) = 3.3757, \quad s(\tilde{a}_2) = 3.6072.$$

Thus, $\tilde{a}_1 < \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_2 \succ x_1$.

Case 2'' Let $\gamma = 2$, then based on Eq. (34), we have

$$\tilde{a}_1 = ([2.7137, 4.1597], [0.1332, 0.1656]),$$

$$\tilde{a}_2 = ([3.5568, 3.5017], [0.1441, 0.1755]).$$

Then the scores $s(\tilde{a}_i) (i = 1, 2)$ of $\tilde{a}_i (i = 1, 2)$ can be calculated as

$$s(\tilde{a}_1) = 3.3598, \quad s(\tilde{a}_2) = 3.5291.$$

Table 2 Marks of students with IVIMNs

	Mathematics	Physics	Literature
x_1	$([4, 6], [1/8, 1/7])$	$([3, 5], [1/6, 1/5])$	$([2, 3], [1/8, 1/6])$
x_2	$([2, 3], [1/5, 1/4])$	$([3, 4], [1/8, 1/6])$	$([7, 8], [1/9, 1/8])$

Table 3 The weights of subjects

	$m(E_1)$	$m(E_2) - m(E_1)$	$m(E_3) - m(E_2)$
$\tilde{a}_{1(j)}$	0.3000	0.2223	0.4777
$\tilde{a}_{2(j)}$	0.4000	0.2297	0.3703

Therefore, $\tilde{a}_1 < \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_2 \succ x_1$.

Case 3'' Let $\gamma = 10$, then based on Eq. (35), we have

$$\tilde{a}_1 = ([2.6989, 4.1424], [0.1332, 0.1657]),$$

$$\tilde{a}_2 = ([3.5056, 3.3871], [0.1443, 0.1761]).$$

Then the scores $s(\tilde{a}_i)(i = 1, 2)$ of $\tilde{a}_i(i = 1, 2)$ can be calculated as

$$s(\tilde{a}_1) = 3.3437, \quad s(\tilde{a}_2) = 3.4458.$$

Thus, $\tilde{a}_1 < \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_2 \succ x_1$.

In the above three cases, we obtain the same ranking: $x_2 \succ x_1$. We can find that x_1 has advantages over x_2 in both mathematics and physics, whereas x_2 has advantages over x_1 only in literature. Even so, it does not mean that x_1 is synthetically better than x_2 . The importance of mathematics and physics presents some overlap, which means that the weight of the pair (e_1, e_2) is less than the sum of the weights of e_1 and e_2 . Thus, after aggregating the individual IVIMs by the C_m -IVIM operator, we obtain the ranking of the students' marks: $x_2 \succ x_1$, which is in accordance with the reality.

In what follows, we discuss the performance of the C_m -IVIM operator according to the variations of $m(\{e_i\})(j = 1, 2, 3)$:

Example 4 We still consider the problem in Example 3 under similar conditions but the Sugeno measure.

(I) Assume that $m(\{e_2\})$ and $m(\{e_3\})$ remain unchanged, while $m(\{e_1\})$ decreases:

$$m(\emptyset) = 0, \quad m(\{e_1\}) = 0.2,$$

$$m(\{e_2\}) = 0.2, \quad m(\{e_3\}) = 0.4.$$

By Eqs. (27) and (28), we have

$$m(\{e_1, e_2\}) = 0.4327, \quad m(\{e_1, e_3\}) = 0.6745,$$

$$m(\{e_2, e_3\}) = 0.6745, \quad m(\{e_1, e_2, e_3\}) = 1.$$

Compared with the results in Example 3, the value of $m(\{e_2, e_3\})$ increases which means the degree of the correlated importance becomes larger. In addition, both the values of $m(\{e_1, e_2\})$ and $m(\{e_1, e_3\})$ decrease which means the subject e_3 maintains the high correlated importance.

Then we can compute the weights of subjects with respect to $x_i(i = 1, 2)$, respectively, which are listed in Table 4 as follows:

Then we use the C_m -IVIM operator to obtain the average value $\tilde{a}_i$ of the students $x_i(i = 1, 2)$:

Case 1'' Let $\gamma = 1$, then based on Eq. (33), we have

$$\tilde{a}_1 = ([2.5527, 3.9164], [0.1335, 0.1685]),$$

$$\tilde{a}_2 = ([3.4676, 3.3059], [0.1445, 0.1773]).$$

Table 4 The weights of subjects (I)

	$m(E_1)$	$m(E_2) - m(E_1)$	$m(E_3) - m(E_2)$
$\tilde{a}_{1(j)}$	0.2000	0.2327	0.5673
$\tilde{a}_{2(j)}$	0.4000	0.2745	0.3255

Calculating the scores $s(\tilde{a}_i)(i = 1, 2)$ of $\tilde{a}_i(i = 1, 2)$:

$$s(\tilde{a}_1) = 3.1619, \quad s(\tilde{a}_2) = 3.3858.$$

Thus, $\tilde{a}_1 < \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_1 \prec x_2$.

Case 2''–Case 3'' Similarly, when $\gamma = 2$ and $\gamma = 10$, we also can have the same result: $x_1 \prec x_2$, which is in accordance with Example 3.

(II) Assume that $m(\{e_1\})$ and $m(\{e_2\})$ remain unchanged, while $m(\{e_3\})$ increases:

$$m(\emptyset) = 0, \quad m(\{e_1\}) = 0.3,$$

$$m(\{e_2\}) = 0.2, \quad m(\{e_3\}) = 0.6.$$

Here, the algebraic sum of the Sugeno measure of three subjects is more than one. By Eqs. (27) and (28), we have

$$m(\{e_1, e_2\}) = 0.4823, \quad m(\{e_1, e_3\}) = 0.8485,$$

$$m(\{e_2, e_3\}) = 0.7657, \quad m(\{e_1, e_2, e_3\}) = 1.$$

Then we can compute the weights of subjects with respect to $x_i(i = 1, 2)$, respectively, which are listed in Table 5 as follows:

Then we use the C_m -IVIM operator to obtain the average value $\tilde{a}_i$ of the students $x_i(i = 1, 2)$:

Case 1'' Let $\gamma = 1$, then based on Eq. (33), we have

$$\tilde{a}_1 = ([2.6851, 4.0942], [0.1316, 0.1644]),$$

$$\tilde{a}_2 = ([2.9594, 3.3122], [0.1604, 0.1973]).$$

Calculating the scores $s(\tilde{a}_i)(i = 1, 2)$ of $\tilde{a}_i(i = 1, 2)$:

$$s(\tilde{a}_1) = 3.3156, \quad s(\tilde{a}_2) = 3.1308.$$

Thus, $\tilde{a}_1 > \tilde{a}_2$, by which we can get the ranking of the students' marks: $x_1 \succ x_2$.

Case 2''–Case 3'' Similarly, when $\gamma = 2$ and $\gamma = 10$, we also can have the same result: $x_1 \succ x_2$, which is contrary to the above two situations.

Table 5 The weights of subjects (II)

	$m(E_1)$	$m(E_2) - m(E_1)$	$m(E_3) - m(E_2)$
$\tilde{a}_{1(j)}$	0.3000	0.1823	0.5177
$\tilde{a}_{2(j)}$	0.6000	0.1657	0.2343

Obviously, the value of $m(\{e_3\})$ increases, and both the values of $m(\{e_1, e_3\})$ and $m(\{e_2, e_3\})$ increase correspondingly. Thus, the correlations between e_3 and another two subjects enhance which leads to the reduction of the correlated importance of subject e_3 .

5 An approach for GDM based on interval-valued intuitionistic multiplicative preference relations

Suppose that there are n alternatives $x_i (i = 1, 2, \dots, n)$ to be evaluated, and t correlated experts $e_k (k = 1, 2, \dots, t)$ are authorized to compare these alternatives and give their preferences with IVIMPRs, then t IVIMPRs $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n \times n} (k = 1, 2, \dots, t)$ can be constructed. To get the ranking of the alternatives, the following steps are exploited:

Step 1 Utilize the IVIMWA or IVIMOWA operator to calculate the average value $\tilde{a}_i^{(k)}$ of the alternative x_i for the expert e_k :

$$\begin{aligned} \tilde{a}_i^{(k)} &= \text{IVIMWA} \left(\tilde{a}_{1j}^{(k)}, \tilde{a}_{2j}^{(k)}, \dots, \tilde{a}_{nj}^{(k)} \right) \\ &= \left(\left[h^{-1} \left(\prod_{j=1}^n (h(\rho_{a_{ij}}^-))^{w_j} \right), h^{-1} \left(\prod_{j=1}^n (h(\rho_{a_{ij}}^+))^{w_j} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{j=1}^n (g(\sigma_{a_{ij}}^-))^{w_j} \right), g^{-1} \left(\prod_{j=1}^n (g(\sigma_{a_{ij}}^+))^{w_j} \right) \right] \right) \end{aligned} \quad (36)$$

or

$$\begin{aligned} \tilde{a}_i^{(k)} &= \text{IVIMOWA} \left(\tilde{a}_{1j}^{(k)}, \tilde{a}_{2j}^{(k)}, \dots, \tilde{a}_{nj}^{(k)} \right) \\ &= \left(\left[h^{-1} \left(\prod_{j=1}^n (h(\rho_{a_{i(j)}}^-))^{w_i} \right), h^{-1} \left(\prod_{j=1}^n (h(\rho_{a_{i(j)}}^+))^{w_i} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{j=1}^n (g(\sigma_{a_{i(j)}}^-))^{w_i} \right), g^{-1} \left(\prod_{j=1}^n (g(\sigma_{a_{i(j)}}^+))^{w_i} \right) \right] \right). \end{aligned} \quad (37)$$

Step 2 Use the C_m -IVIM operator to obtain the average value $\tilde{a}_i$ of the alternative x_i :

$$\begin{aligned} \tilde{a}_i &= C_m\text{-IVIM}(\tilde{a}_i^{(1)}, \tilde{a}_i^{(2)}, \dots, \tilde{a}_i^{(t)}) = \sum_{k=1}^t (m(E_i) - m(E_{i-1})) \cdot \tilde{a}_i^{\text{ind}(k)} \\ &= \left(\left[h^{-1} \left(\prod_{k=1}^t (h(\rho_{a_i^{\text{ind}(k)}}^-))^{m(E_i) - m(E_{i-1})} \right), h^{-1} \left(\prod_{k=1}^t (h(\rho_{a_i^{\text{ind}(k)}}^+))^{m(E_i) - m(E_{i-1})} \right) \right], \right. \\ &\quad \left. \left[g^{-1} \left(\prod_{k=1}^t (g(\sigma_{a_i^{\text{ind}(k)}}^-))^{m(E_i) - m(E_{i-1})} \right), g^{-1} \left(\prod_{k=1}^t (g(\sigma_{a_i^{\text{ind}(k)}}^+))^{m(E_i) - m(E_{i-1})} \right) \right] \right), \end{aligned} \quad (38)$$

where $\tilde{a}_i^{\text{ind}(k)}$ is a non-increasing permutation of $\tilde{a}_i^{(k)} (k = 1, 2, \dots, t)$, such that $\tilde{a}_i^{\text{ind}(k)}$ is the k -th largest $\tilde{a}_i^{(k)} (i = 1, 2, \dots, n)$.

Step 3 Calculate the score function $s(\tilde{a}_i)$ and the accuracy degree $r(\tilde{a}_i)$ of $\tilde{a}_i$, and obtain the ranking of the alternatives according to Definition 2.2.

Next, we use a numerical example to illustrate our developed method:

Example 5 It is well known that communication is closely related to social development. Especially, satellite communication has wide coverage area for communication, without geographical restrictions and being less susceptible to the impact of land disasters. There is a unique advantage of satellite communication in many application places, such as remote areas, mountains, islands, ocean-going vessels, and voyage aircrafts. Therefore, satellite communication not only effectively supplements the lack of other means of communication, but also has an irreplaceable role as the primary method of communication in the mass media, especially to military. In modern warfare, it is very important to quantify the quality of the communication service. Suppose that a communication joint department in China plans to design a satellite communication system. Thus, a new test methodology for the synthetic communication system is required to be investigated, in order to provide an evidence for building satellite earth stations in future. According to expeditions, there are four possible testing venues $x_i (i = 1, 2, 3, 4)$ to choose from (Xichang x_1 , Chengdu x_2 , Nanjing x_3 , and Lhasa x_4). However, due to funds, resources, and other factors, only the most suitable one of them would be selected from. A decision-making group composed of five experts $e_k (k = 1, 2, 3, 4, 5)$ (meteorology expert e_1 , topography expert e_2 , communication expert e_3 , military tactical expert e_4 and military strategist e_5) gives the judgments with five individual IVIMPRs as follows:

$$\begin{aligned}
\tilde{A}^{(1)} &= \begin{pmatrix} ([1, 1], [1, 1]) & ([4/3, 2], [1/9, 1/7]) & ([9/2, 5], [1/6, 1/5]) & ([14/3, 6], [1/8, 1/7]) \\ ([1/9, 1/7], [4/3, 2]) & ([1, 1], [1, 1]) & ([16/3, 6], [1/8, 1/6]) & ([9/2, 5], [1/6, 1/6]) \\ ([1/6, 1/5], [9/2, 5]) & ([1/8, 1/6], [16/3, 6]) & ([1, 1], [1, 1]) & ([8/3, 5], [1/7, 1/6]) \\ ([1/8, 1/7], [14/3, 6]) & ([1/6, 1/6], [9/2, 5]) & ([1/7, 1/6], [8/3, 5]) & ([1, 1], [1, 1]) \end{pmatrix}_{4 \times 4} \\
\tilde{A}^{(2)} &= \begin{pmatrix} ([1, 1], [1, 1]) & ([5, 21/3], [1/9, 1/7]) & ([3, 4], [1/6, 1/4]) & ([2, 3], [1/8, 1/4]) \\ ([1/9, 1/7], [5, 21/3]) & ([1, 1], [1, 1]) & ([10/3, 5], [1/6, 1/5]) & ([7/2, 4], [1/6, 1/5]) \\ ([1/6, 1/4], [3, 4]) & ([1/6, 1/5], [10/3, 5]) & ([1, 1], [1, 1]) & ([3/2, 2], [1/6, 1/3]) \\ ([1/8, 1/4], [2, 3]) & ([1/6, 1/5], [7/2, 4]) & ([1/6, 1/3], [3/2, 2]) & ([1, 1], [1, 1]) \end{pmatrix}_{4 \times 4} \\
\tilde{A}^{(3)} &= \begin{pmatrix} ([1, 1], [1, 1]) & ([3/4, 4/3], [1/3, 1/2]) & ([5/2, 3], [1/5, 1/3]) & ([2, 3], [2/9, 1/4]) \\ ([1/3, 1/2], [3/4, 4/3]) & ([1, 1], [1, 1]) & ([3, 5], [1/9, 1/6]) & ([8/3, 4], [1/6, 1/5]) \\ ([1/5, 1/3], [5/2, 3]) & ([1/9, 1/6], [3, 5]) & ([1, 1], [1, 1]) & ([4, 5], [1/8, 1/6]) \\ ([2/9, 1/4], [2, 3]) & ([1/6, 1/5], [8/3, 4]) & ([1/8, 1/6], [4, 5]) & ([1, 1], [1, 1]) \end{pmatrix}_{4 \times 4} \\
\tilde{A}^{(4)} &= \begin{pmatrix} ([1, 1], [1, 1]) & ([5/2, 3], [1/5, 1/4]) & ([5/3, 3], [1/5, 1/3]) & ([2/3, 3], [2/9, 1/4]) \\ ([1/5, 1/4], [5/2, 3]) & ([1, 1], [1, 1]) & ([1/6, 1/2], [1/3, 1/2]) & ([2, 4], [1/6, 1/5]) \\ ([1/5, 1/3], [5/3, 3]) & ([1/3, 1/2], [1/6, 1/2]) & ([1, 1], [1, 1]) & ([7/2, 4], [1/7, 1/5]) \\ ([2/9, 1/4], [2/3, 3]) & ([1/6, 1/5], [2, 4]) & ([1/7, 1/5], [7/2, 4]) & ([1, 1], [1, 1]) \end{pmatrix}_{4 \times 4} \\
\tilde{A}^{(5)} &= \begin{pmatrix} ([1, 1], [1, 1]) & ([4, 6], [1/9, 1/8]) & ([2/3, 3], [1/8, 1/6]) & ([3/2, 2], [2/9, 1/4]) \\ ([1/9, 1/8], [4, 6]) & ([1, 1], [1, 1]) & ([1/3, 1/2], [2/5, 1/2]) & ([3, 4], [1/9, 1/6]) \\ ([1/8, 1/6], [2/3, 3]) & ([2/5, 1/2], [1/3, 1/2]) & ([1, 1], [1, 1]) & ([2, 8/3], [2/9, 2/7]) \\ ([2/9, 1/4], [3/2, 2]) & ([1/9, 1/6], [3, 4]) & ([2/9, 2/7], [2, 8/3]) & ([1, 1], [1, 1]) \end{pmatrix}_{4 \times 4}
\end{aligned}$$

Suppose that the weight vector of venues is $\omega = (1/4, 1/4, 1/4, 1/4)^T$.

Case I Let $\gamma = 1$, in order to obtain the ranking of the alternative, we give the following steps:

Step 1 Due to $\omega_i = 1/4$ (for all i) in this example, we are free to choose the IVIMWA or IVIMOWA operator. We here take the IVIMWA operator for example, Eq. (36) is equal to

$$\tilde{a}_3^{(1)} = ([0.7614, 1.0245], [0.8367, 0.9057]),$$

$$\tilde{a}_4^{(1)} = ([0.3161, 0.3281], [2.3733, 2.8257]),$$

$$\tilde{a}_1^{(2)} = ([2.4641, 3.2295], [0.2017, 0.2880]),$$

$$\tilde{a}_2^{(2)} = ([1.5657, 1.8776], [0.4361, 0.4971]),$$

$$\tilde{a}_3^{(2)} = ([0.6152, 0.7321], [0.8200, 1.1612]),$$

$$\begin{aligned}
\tilde{a}_i^{(k)} &= \text{IVIMWA}(\tilde{a}_{1j}^{(k)}, \tilde{a}_{2j}^{(k)}, \dots, \tilde{a}_{nj}^{(k)}) \\
&= \left(\left[\prod_{j=1}^n \left(1 + \rho_{a_{ij}}^- \right)^{\omega_i} - 1, \prod_{j=1}^n \left(1 + \rho_{a_{ij}}^+ \right)^{\omega_i} - 1 \right], \left[\frac{\prod_{j=1}^n \left(\sigma_{a_{ij}}^- \right)^{\omega_i}}{\prod_{j=1}^n \left(1 + \sigma_{a_{ij}}^- \right)^{\omega_i} - \prod_{j=1}^n \left(\sigma_{a_{ij}}^- \right)^{\omega_i}}, \frac{\prod_{j=1}^n \left(\sigma_{a_{ij}}^+ \right)^{\omega_i}}{\prod_{j=1}^n \left(1 + \sigma_{a_{ij}}^+ \right)^{\omega_i} - \prod_{j=1}^n \left(\sigma_{a_{ij}}^+ \right)^{\omega_i}} \right] \right). \quad (39)
\end{aligned}$$

Then the average values $\tilde{a}_i^{(k)} (i = 1, 2, 3, 4)$ of the alternatives $x_i (i = 1, 2, 3, 4)$ for the experts $e_k (k = 1, 2, 3, 4, 5)$ are

$$\tilde{a}_1^{(1)} = ([2.4728, 2.9843], [0.2017, 0.2345]),$$

$$\tilde{a}_2^{(1)} = ([1.9662, 2.1302], [0.3505, 0.4029]),$$

$$\tilde{a}_4^{(2)} = ([0.3229, 0.4142], [1.6883, 2.0188]),$$

$$\tilde{a}_1^{(3)} = ([1.4621, 1.9396], [0.3299, 0.4329]),$$

$$\tilde{a}_2^{(3)} = ([1.5008, 2.0801], [0.3076, 0.4029]),$$

$$\tilde{a}_3^{(3)} = ([0.9109, 1.0786], [0.7104, 0.8507]),$$

$$\begin{aligned}\tilde{a}_4^{(3)} &= ([0.3384, 0.3678], [1.9728, 2.4142]), \\ \tilde{a}_1^{(4)} &= ([1.3617, 2.3636], [0.2889, 0.3622]), \\ \tilde{a}_2^{(4)} &= ([0.7024, 1.0809], [0.5062, 0.6127]), \\ \tilde{a}_3^{(4)} &= ([0.9480, 1.1147], [0.3761, 0.6127]), \\ \tilde{a}_4^{(4)} &= ([0.3436, 0.3774], [1.3120, 2.3325]), \\ \tilde{a}_1^{(5)} &= ([1.5407, 2.6002], [0.2170, 0.2494]), \\ \tilde{a}_2^{(5)} &= ([0.8554, 1.0268], [0.4858, 0.6076]), \\ \tilde{a}_3^{(5)} &= ([0.7533, 0.8927], [0.4467, 0.6899]), \\ \tilde{a}_4^{(5)} &= ([0.3498, 0.3916], [1.6478, 1.9728]).\end{aligned}$$

Step 2 Assume that the decision makers have correlations with each other and

$$\begin{aligned}m(\emptyset) &= 0, \quad m(\{e_1\}) = 0.3, \quad m(\{e_2\}) = 0.2, \\ m(\{e_3\}) &= 0.4, \quad m(\{e_4\}) = 0.5, \quad m(\{e_5\}) = 0.3.\end{aligned}$$

By Eqs. (27) and (28), we have

$$\begin{aligned}m(\{e_1, e_2\}) &= 0.4518, \quad m(\{e_1, e_3\}) = 0.6035, \\ m(\{e_4, e_5\}) &= 0.6794, \quad m(\{e_1, e_2, e_3\}) = 0.7065, \\ m(\{e_1, e_2, e_5\}) &= 0.6428, \quad m(\{e_2, e_4, e_5\}) = 0.7702, \\ m(\{e_3, e_4, e_5\}) &= 0.8609, \quad m(\{e_1, e_2, e_3, e_5\}) = 0.8361, \\ m(\{e_1, e_2, e_4, e_5\}) &= 0.8844, \quad m(\{e_1, e_3, e_4, e_5\}) = 0.9533, \\ m(\{e_2, e_3, e_4, e_5\}) &= 0.9225, \quad m(\{e_1, e_2, e_3, e_4, e_5\}) = 1.\end{aligned}$$

Moreover, based on Definition 2.2, we obtain

$$\begin{aligned}\tilde{a}_1^{\text{ind}(1)} &= \tilde{a}_1^{(1)}, \quad \tilde{a}_1^{\text{ind}(2)} = \tilde{a}_1^{(2)}, \quad \tilde{a}_1^{\text{ind}(3)} = \tilde{a}_2^{(5)}, \\ \tilde{a}_1^{\text{ind}(4)} &= \tilde{a}_1^{(4)}, \quad \tilde{a}_1^{\text{ind}(5)} = \tilde{a}_1^{(3)}, \\ \tilde{a}_2^{\text{ind}(1)} &= \tilde{a}_2^{(1)}, \quad \tilde{a}_2^{\text{ind}(2)} = \tilde{a}_2^{(3)}, \quad \tilde{a}_2^{\text{ind}(3)} = \tilde{a}_1^{(2)}, \\ \tilde{a}_2^{\text{ind}(4)} &= \tilde{a}_3^{(5)}, \quad \tilde{a}_2^{\text{ind}(5)} = \tilde{a}_2^{(4)}, \\ \tilde{a}_3^{\text{ind}(1)} &= \tilde{a}_3^{(4)}, \quad \tilde{a}_3^{\text{ind}(2)} = \tilde{a}_3^{(5)}, \quad \tilde{a}_3^{\text{ind}(3)} = \tilde{a}_3^{(3)}, \\ \tilde{a}_3^{\text{ind}(4)} &= \tilde{a}_2^{(1)}, \quad \tilde{a}_3^{\text{ind}(5)} = \tilde{a}_3^{(2)}, \\ \tilde{a}_4^{\text{ind}(1)} &= \tilde{a}_4^{(4)}, \quad \tilde{a}_4^{\text{ind}(2)} = \tilde{a}_4^{(5)}, \quad \tilde{a}_4^{\text{ind}(3)} = \tilde{a}_4^{(2)}, \\ \tilde{a}_4^{\text{ind}(4)} &= \tilde{a}_4^{(3)}, \quad \tilde{a}_4^{\text{ind}(5)} = \tilde{a}_4^{(1)}.\end{aligned}$$

Then we can calculate the values of $m(E_i) - m(E_{i-1})$ in the C_m -IVIM operator (Eq. (38)) respectively, which are listed in Table 6 as follows:

We utilize the C_m -IVIM operator (Eq. (38)) to obtain the average value $\tilde{a}_i$ of x_i ($i = 1, 2, 3, 4$):

$$\begin{aligned}\tilde{a}_1 &= ([1.8633, 2.6546], [0.2353, 0.2903]), \\ \tilde{a}_2 &= ([1.3839, 1.7299], [0.3797, 0.4625]), \\ \tilde{a}_3 &= ([0.8708, 1.0392], [0.4781, 0.7046]), \\ \tilde{a}_4 &= ([0.3419, 0.3824], [1.6635, 2.1507]).\end{aligned}$$

Step 3 Calculate the scores $s(\tilde{a}_i)$ ($i = 1, 2, 3, 4$) of $\tilde{a}_i$ ($i = 1, 2, 3, 4$):

$$\begin{aligned}s(\tilde{a}_1) &= 8.5097, \quad s(\tilde{a}_2) = 3.6923, \\ s(\tilde{a}_3) &= 1.6390, \quad s(\tilde{a}_4) = 0.1912.\end{aligned}$$

Thus, $\tilde{a}_1 > \tilde{a}_2 > \tilde{a}_3 > \tilde{a}_4$, by which we can get the ranking of the alternatives: $x_1 \succ x_2 \succ x_3 \succ x_4$.

Case II–Case III Let $\gamma = 2$ and $\gamma = 10$. Similarly, we can compute the average value $\tilde{a}_i$ of x_i ($i = 1, 2, 3, 4$) and the scores $s(\tilde{a}_i)$ of $\tilde{a}_i$ ($i = 1, 2, 3, 4$). For comparison, the results are shown in Table 7.

In the cases above, we obtain the same result: Xichang is the most suitable venue for building the satellite earth station. This indicates that the approach proposed offers good stability.

Next, we select the GDM method developed by Yu (2015a, b) to aggregate the given interval-valued intuitionistic multiplicative information and compare the results with our approach. In Yu (2015a, b), the correlations of inputs have not been taken into account in the GDM method. Although some aggregation operators for the correlated inputs have been given, 2^n values of fuzzy measures should be known in advance which is huge in quantity and not easy to realize. Thus, they are not suitable to solve the GDM problem and the correlations of inputs are ignored in GDM. While only n values of fuzzy measures need to be determined in this paper, which is more suitable for GDM. Then instead of the C_m -IVIM operator in Step 2, we use the IVIMWA operator to obtain the average value $\tilde{a}_i$ of the alternative x_i . In order to maintain unity, here we let the weights of five experts be $(3/17, 2/17, 4/17, 5/17, 3/17)^T$. The results in different cases are shown in Table 8 under other invariable conditions.

From Table 8, we find that the most suitable venue for building the satellite earth station is still Xichang. But different ranking results are obtained without considering correlations of the given interval-valued intuitionistic multiplicative information. In practical applications, designing a good alternative is also critical. Take this venue-selection problem as an example, if we use the GDM approach proposed in this paper, we should take x_2 as an

Table 6 $m(E_i) - m(E_{i-1})$ in Eq. (38)

	$m(E_1)$	$m(E_2) - m(E_1)$	$m(E_3) - m(E_2)$	$m(E_4) - m(E_3)$	$m(E_5) - m(E_4)$
$\tilde{a}_1$	0.3000	0.1518	0.1910	0.2416	0.1156
$\tilde{a}_2$	0.3000	0.3035	0.1030	0.1296	0.1639
$\tilde{a}_3$	0.5000	0.1794	0.1815	0.0924	0.0467
$\tilde{a}_4$	0.5000	0.1794	0.0908	0.1523	0.0775

Table 7 Average value $\tilde{a}_i$ of x_i and the scores $s(\tilde{a}_i)$ of $\tilde{a}_i$ using the C_m -IVIM operator ($i = 1, 2, 3, 4$)

γ	Average values	Scores	Ranking
Case I	$\tilde{a}_1 = ([1.8633, 2.6546], [0.2353, 0.2903])$	$s(\tilde{a}_1) = 8.5097$	$x_1 \succ x_2 \succ x_3 \succ x_4$
$\gamma = 1$	$\tilde{a}_2 = ([1.3839, 1.7299], [0.3797, 0.4625])$	$s(\tilde{a}_2) = 3.6923$	
	$\tilde{a}_3 = ([0.8708, 1.0392], [0.4781, 0.7046])$	$s(\tilde{a}_3) = 1.6390$	
	$\tilde{a}_4 = ([0.3419, 0.3824], [1.6635, 2.1507])$	$s(\tilde{a}_4) = 0.1912$	
Case II	$\tilde{a}_1 = ([1.8095, 2.5935], [0.2427, 0.2982])$	$s(\tilde{a}_1) = 8.0526$	$x_1 \succ x_2 \succ x_3 \succ x_4$
$\gamma = 2$	$\tilde{a}_2 = ([1.2661, 1.5845], [0.4051, 0.4952])$	$s(\tilde{a}_2) = 3.1620$	
	$\tilde{a}_3 = ([0.7935, 0.9561], [0.5167, 0.7621])$	$s(\tilde{a}_3) = 1.3888$	
	$\tilde{a}_4 = ([0.3233, 0.3665], [1.6996, 2.2119])$	$s(\tilde{a}_4) = 0.1775$	
Case III	$\tilde{a}_1 = ([1.7457, 2.5242], [0.2516, 0.3077])$	$s(\tilde{a}_1) = 7.5443$	$x_1 \succ x_2 \succ x_3 \succ x_4$
$\gamma = 10$	$\tilde{a}_2 = ([1.0776, 1.3625], [0.4429, 0.5492])$	$s(\tilde{a}_2) = 2.4569$	
	$\tilde{a}_3 = ([0.6681, 0.8351], [0.5831, 0.8541])$	$s(\tilde{a}_3) = 1.0585$	
	$\tilde{a}_4 = ([0.2942, 0.3256], [1.7703, 2.3412])$	$s(\tilde{a}_4) = 0.1520$	

Table 8 Average value $\tilde{a}_i$ of x_i and the scores $s(\tilde{a}_i)$ of $\tilde{a}_i$ using the IVIMWA operator ($i = 1, 2, 3, 4$)

γ	Average values	Scores	Ranking
Case I	$\tilde{a}_1 = ([1.4947, 2.0920], [0.3088, 0.3815])$	$s(\tilde{a}_1) = 5.1515$	$x_1 \succ x_3 \succ x_2 \succ x_4$
$\gamma = 1$	$\tilde{a}_2 = ([0.6192, 0.7511], [0.9320, 1.0886])$	$s(\tilde{a}_2) = 0.6771$	
	$\tilde{a}_3 = ([0.9885, 1.2116], [0.4639, 0.6261])$	$s(\tilde{a}_3) = 2.0307$	
	$\tilde{a}_4 = ([0.5336, 0.5923], [1.0545, 1.4138])$	$s(\tilde{a}_4) = 0.4604$	
Case II	$\tilde{a}_1 = ([1.3981, 1.9547], [0.3351, 0.4111])$	$s(\tilde{a}_1) = 4.4538$	$x_1 \succ x_3 \succ x_4 \succ x_2$
$\gamma = 2$	$\tilde{a}_2 = ([0.5117, 0.6144], [1.1466, 1.3289])$	$s(\tilde{a}_2) = 0.4543$	
	$\tilde{a}_3 = ([0.9324, 1.1494], [0.4828, 0.6586])$	$s(\tilde{a}_3) = 1.8360$	
	$\tilde{a}_4 = ([0.5323, 0.6015], [0.9924, 1.3608])$	$s(\tilde{a}_4) = 0.4869$	
Case III	$\tilde{a}_1 = ([1.1857, 1.6525], [0.4038, 0.4908])$	$s(\tilde{a}_1) = 3.1440$	$x_1 \succ x_3 \succ x_4 \succ x_2$
$\gamma = 10$	$\tilde{a}_2 = ([0.3034, 0.3611], [1.9654, 2.2582])$	$s(\tilde{a}_2) = 0.1571$	
	$\tilde{a}_3 = ([0.8921, 1.1429], [0.4625, 0.6547])$	$s(\tilde{a}_3) = 1.8350$	
	$\tilde{a}_4 = ([0.6127, 0.7337], [0.7129, 1.0448])$	$s(\tilde{a}_4) = 0.7768$	

alternative. While, if we choose the GDM method in (Yu 2015a, b) without considering the interrelationships of inputs, we shall make x_3 an alternative to x_1 . Thus, to capture the interrelationships among the aggregated information is important in the GDM problems.

6 Concluding Remarks

In this paper, we have introduced some interval-valued intuitionistic multiplicative aggregation operators and presented an approach for GDM where the correlated decision

makers provide their preference information over alternatives by using IVIMPRs. To aggregate the interval-valued intuitionistic multiplicative information, several aggregation operators have been developed and their properties also have been discussed and proven. The contributions of this paper are as follows: (1) the IVIMWA and IVIMOWA operators based on the pseudo-multiplication can be utilized to aggregate a collection of IVIMNs which is more general. Based on different aggregation operations, several specific forms of the IVIMWA and IVIMOWA operators can be obtained. (2) The C_m -IVIM operator, based on the Choquet integral and Sugeno measure, can be used for aggregating the

correlated IVIMNs whose weights can be easily calculated. (3) The proposed approach for GDM can deal with the situations where the preference information is constructed by IVIMPRs and there exist the correlations or the connections among the decision makers. Of course, the proposed approach has its limitations: (1) the formulas of the aggregation operators proposed in this paper seem to be a little complex. (2) How to determine the fuzzy measure has not been discussed deeply, but is important and cannot be avoided with the deepening of the research. (3) Other families of fuzzy measures still need to be investigated, such as k -additive fuzzy measures, the p -symmetric measures, the decomposable measures and so on. Their advantages and disadvantages are also worth studying. In the future, we will deepen our research and apply our approach to solve more practical GDM problems with IVIMPRs.

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