Interval type-2 fuzzy Hamy mean operators and their application in multiple criteria decision making

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Abstract Interval type-2 fuzzy set (IT2FS) is a widely used tool for handling vagueness and imprecision in real-world problems. To aggregate the interval type-2 fuzzy information, some useful aggregation operators have been developed such as *IT2-OWA*, *IT2-FWA*, and *IT2-Frank operators*. In particular, some aggregation operators which can capture and reflect interrelationships among aggregated arguments are intensively studied, among which Hamy mean (HM) is an important alternative. However, the traditional HM does not focus on fusion of fuzzy information and the weighted vector of the aggregated arguments. In this study, we extend the HM to accommodate interval type-2 fuzzy environment and show its application to multiple criteria decision making (MADM). In order to simplify the computing, we first define the concept of symmetric triangular interval type-2 fuzzy set and propose some operational laws. Motivated by the idea of HM operator, we develop the symmetric triangular interval type-2 fuzzy Hamy mean (STIT2FHM) operator for aggregating the interval type-2 fuzzy information. Some useful properties such as *monotonicity*, *boundedness*, and *idempotency* are studied in detail. Furthermore, we discuss some special cases with respect to different parameter values of the STIT2FHM operator. For the situations where the input arguments have different importance, we further develop the weighted symmetric triangular interval type-2 fuzzy Hamy mean (WSTIT2FHM) operator to aggregate interval type-2 fuzzy information. Based on it, an approach to MCDM problems with symmetric triangular interval

type-2 fuzzy information is developed. Finally, a real-world example concerning tourism recommender system is provided to illustrate the practicality and effectiveness of the proposed method.

Keywords Multiple criteria analysis · Symmetric triangular interval type-2 fuzzy set · Hamy mean · Symmetric triangular interval type-2 fuzzy Hamy mean operator · Weighted symmetric triangular interval type-2 fuzzy Hamy mean operator · Tourism recommender system

1 Introduction

Multiple criteria decision making (MCDM) is a common human activity in our daily life, which aims to select the optimal alternative(s) from the feasible set and obtain their ranking by using aggregation technique the decision information of each alternative under several performance criteria, both qualitative and quantitative. In traditional MCDM methods (Brans and Vincke 1985; Doyle and Green 1993; Lai et al. 1994; Opricovic and Tzeng 2004; Saaty 1986; Srinivasan and Shocker 1973), the criteria value take the form of numeric, however, in many actual situations, due to the various complex factors, such as time pressure, lack of knowledge, and the experts' limited expertise about the problem domain, it is difficult for decision maker to express their preference by crisp number. To deal with this issue, a suitable approach is to use fuzzy number instead of numerical value to represent the decision maker's preference. Since Zadeh (1965) introduced the type-1 fuzzy set (TIFS) theory, the TIFS theory is widely used in management science, artificial intelligence, applied mathematics, mainly in decision making, many type-1 fuzzy techniques for dealing with MCDM problems have been developed during the last

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three decades (Bellman and Zadeh 1970; Pedrycz 1984; Liang 1999; Wang and Elhag 2006; Chu and Varma 2012; Wan and Li 2013; Herrera et al. 2003; Pedrycz et al. 2011; Wang and Xu 2016).

Due to the increasing complexity and uncertainty of social economic environment, when one has to cope with imprecise information where two or more sources of vagueness appear simultaneously, the traditional T1FS theory shows some limitations. Therefore, it has been extended to more general form. Type-2 fuzzy set (T2FS), as an extensive of type-1 fuzzy set, initially presented by Zadeh (1975), is a powerful tool for modeling vagueness and uncertainty. In the last decade, the T2FS theory has been one of the most popular artificial intelligence research topics that attracted the attention of many researchers. A large number of studies have been published (Karnik and Mendel 2001a, b; Mendel and John 2002; Mendel et al. 2006; Mendel 2007; Wu and Mendel 2009; Liu et al. 2012a; Zhai and Mendel 2012; Greenfield and Chiclana 2013a, b; Chiclana and Zhou 2013), mainly in interval type-2 MCDM field (Wu and Mendel 2010; Chen and Lee 2010a, b, c, d; Chen and Wang 2013; Chen et al. 2012; Chen 2013a, b, c; Qin and Liu 2015a, b; Qin et al. 2015, 2016; Mendel 2016). For example, Wu and Mendel (2010) developed a computing with word-based method for interval type-2 fuzzy hierarchical MCDM method, and then applied to evaluating a weapon system. Qin et al. (2016) developed an extension TODIM method within the context of IT2FS, and applied it to green supplier evaluation. Hu et al. (2013) proposed a new approach based on possibility degree to handle MCDM problems. Chen et al. (2013) extended the classical QUALIFLEX method based on sign distance for MCDM in the context of interval type-2 fuzzy information, and gave a case study of medicine decision making. Qin et al. (2015)

integrated VIKOR and the Prospect theory to propose a new interval type-2 fuzzy multiple criteria decision-making method. Chen and Lee (2010b, c) proposed a ranking method and some operation laws for interval type-2 fuzzy MCDM problems. Cheng et al. (2016) develop a novel autocratic decision-making method with the aid of group recommendations and ranking IT2FSs. Wang et al. (2012) investigated the MCDM problem and developed a new approach to handle the situations where the attribute values are characterized by IT2FSs. Celik et al. (2013) presented an integrated novel interval type-2 fuzzy MCDM method and applied it to evaluate the customer satisfaction in public transportation for Istanbul. In additional, some studies have focused on other interval type-2 fuzzy MCDM methods based on a variety of classical decision-making techniques. An overview of the previous literature on interval type-2 fuzzy decision making is provided in Table 1.

Type-2 fuzzy information aggregation is an interesting research topic in T2FS theory. Some authors have paid attention to this field and developed some useful aggregation operators for aggregating the interval type-2 fuzzy information during the past 5 years. For example, Zhou et al. (2010) developed a type-2 OWA operator to accommodate the environment in which the given arguments are IT2FSs. Wu and Mendel (2009) presented a linguistic weighted averaging operator for aggregating interval type-2 fuzzy information. Liu et al. (2012a, b) put forward an analytical solution method for the fuzzy weighted averaging (FWA), the method had some good mathematical analysis properties, which can be used for aggregating interval type-2 information. Recently, Chiclana and Zhou (2013) developed a type-reduction method based on alpha-level approach to type-1 OWA operations for IT2FS; the experimental results proved

Table 1 Summary of some relevant research on interval type-2 fuzzy decision making

Category	Method	Literature
Review paper	Overview	Celik et al. (2013)
Interval type-2 fuzzy MCDM model	AHP	Chen and Lee (2010d); Kahraman et al. (2014); Abdullah and Najib (2014)
	TOPSIS	Langroudi et al. (2013); Sang et al. (2015)
	PROMETHEE	Chen (2014b)
	ELECTRE	Chen (2014a)
	Linear assignment method/LINMAP	Chen (2013b); Qin et al. (2016)
Interval type-2 fuzzy applications	Aggregation operators	Merigó et al. 2014; Merigó et al. (2016)
	Social financial risk analysis	Bernardo et al. (2013)
	Supply chain management	Gong et al. (2014); Gong et al. (2015)
	Medical decision making	Chen (2013a, c)
	Water site selection	Wang and Chen (2014)
	Other methodologies	Chen et al. (2012); Naim and Hagrass (2012); Bilgin et al. (2013); Cabrerizo et al. (2013)

that this new method is effective for aggregating interval type-2 fuzzy information. Gong et al. (2015) extended the geometric Bonferroni mean (GBM) to accommodate trapezoidal interval type-2 fuzzy environment.

It is noted that the existing interval type-2 fuzzy aggregation operators ignore the interaction phenomenon among the aggregated information. In other words, there is a common assumption that the aggregated arguments are independent. However, due to the increasing complexity of socio-economic environment, the interaction phenomena among the decision-making criteria are commonly present. Therefore, this assumption is not reasonable and also unrealistic in many practical situations. To deal with this issue, the motivation of this paper is to develop some new interval type-2 fuzzy aggregation operators based on Hamy mean (HM) (Hara et al. 1998), which can capture the interactions among the aggregated decision information within the context of IT2FS. Hamy mean (HM), as a well-known aggregation technique, is widely used in information fusion. From the perspective of mathematics, it can be regarded as a simplified form of Maclaurin symmetric mean and its extensions (Qin and Liu 2014, 2015a, b). The prominent feature is that this operator can capture the overall interrelationships between the arguments and reflect the interaction phenomena. However, in the past, the HM was applied to the theory and application of inequality and resulted in many research results (Jiang 2007; Chu et al. 2012; Guan 2006; Guan and Guan 2011). Until now, we have not seen any research based on HM operator used for aggregating information under interval type-2 fuzzy environment. Motivated by this idea, we focus attention on extension of the HM to accommodate interval type-2 fuzzy environment and their application in multiple criteria decision making in this paper.

The paper is structured as follows: In Sect. 2, we briefly review some basic concepts of T2FS, IT2FS, and HM. In Sect. 3, we investigate the concept of symmetric triangular interval type-2 fuzzy set and propose some operational laws. In Sect. 4, we develop the symmetric triangular interval type-2 fuzzy Hamy mean (STIT2FHM) operator and the weighted symmetric triangular interval type-2 fuzzy Hamy mean (WSTIT2FHM) operator, respectively. A variety of desirable properties such as *idempotency*, *commutativity*, *monotonicity*, and *boundedness* and some special cases with respect to different parameters are discussed in detail. In Sect. 5, we develop the procedure based on WSTIT2FHM operator for MCDM problems. A practical example concerns that tourism recommender system is provided to demonstrate the decision-making application in Sect. 6. Finally, we present some conclusions and point out future directions in Sect. 7.

2 Preliminaries

In this section, we briefly review some basic concepts of type-2 fuzzy set (T2FS), interval type-2 fuzzy set (IT2FS), and Hamy mean (HM) which will be used in the next sections.

2.1 Type-2 fuzzy set

Definition 1 (Mendel et al. 2016) Let X be the universe of discourse. Then, a type-2 fuzzy set A can be represented by type-2 membership function $\mu_A(x, u)$ as follows:

$$A = \{((x, u), \mu_A(x, u)) | x \in X, u \in [0, 1]\}, \quad (1)$$

where J_x denotes an interval in $[0, 1]$. Moreover, the type-2 fuzzy set can also be expressed in the following form:

$$A = \int_{x \in X} \int_{u \in J_x} \mu_A(x, u) / (x, u) = \int_{x \in X} \left(\int_{u \in J_x} \mu_A(x, u) / u \right) / x \quad (2)$$

where $J_x = \{(x, u) | u \in [0, 1], \mu_A(x, u) > 0\}$, and $\int_{u \in J_x} \mu_A(x, u) / u$ indicates the second membership at x . For discrete spaces, the symbol $\int$ is replaced by $\sum$.

2.2 Interval type-2 fuzzy set

Definition 2 (Mendel et al. 2016) Let $\tilde{A}$ be a type-2 fuzzy set (T2FS) in the universe of discourse X represented by a type-2 membership function $\mu_{\tilde{A}}(x, u)$. If all $\mu_{\tilde{A}}(x, u) = 1$, then $\tilde{A}$ is called an interval type-2 fuzzy set (IT2FS). An interval type-2 fuzzy set can be regarded as a special case of the type-2 fuzzy set which is defined as follows:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1 / (x, u) = \int_{x \in X} \left(\int_{u \in J_x} 1 / u \right) / x. \quad (3)$$

It is obvious that the interval type-2 fuzzy set $\tilde{A}$ defined on X is completely determined by the primary membership which is called the domain of uncertainty (DOU), where the DOU can be expressed as follows:

$$\text{DOU}(\tilde{A}) = \bigcup_{x \in X} \{(x, u) \in X \times [0, 1] | \mu_{\tilde{A}}(x, u) > 0 \subseteq [0, 1]\}. \quad (4)$$

2.3 Hamy mean

The Hamy mean (HM) (Hara et al. 1998) is a very useful technique characterized by the ability to capture the

interrelationship among the multi-input arguments. Its definition is provided as follows:

Definition 3 (Hara et al. 1998) Let $x_j (j = 1, 2, \dots, n)$ be a collection of nonnegative real numbers, and parameter $k = 1, 2, \dots, n$. If

$$HM^{(k)}(x_1, x_2, \dots, x_n) = \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k x_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \quad (5)$$

then $HM^{(k)}$ is called the Hamy mean (HM), where $(i_1, i_2, \dots, i_k)$ traverses all the k -tuple combination of $(1, 2, \dots, n)$, $\binom{n}{k}$ is the binomial coefficient, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

From Eq. (5), it is clear that the HM satisfies the following desirable properties:

- (1) $HM^{(k)}(0, 0, \dots, 0) = 0$;
- (2) $HM^{(k)}(x, x, \dots, x) = x$;
- (3) $HM^{(k)}(x_1, x_2, \dots, x_n) \leq HM^{(k)}(y_1, y_2, \dots, y_n)$, if $x_i \leq y_i$ for all i ; and
- (4) $\min_i \{x_i\} \leq HM^{(k)}(x_1, x_2, \dots, x_n) \leq \max_i \{x_i\}$.

3 Symmetric triangular interval type-2 fuzzy set

Because of the operations on interval type-2 fuzzy sets are not complex, according to the decomposition theorem (Mendel and John 2002), the interval type-2 fuzzy sets are considered in applications. In this section, we propose a symmetric triangular interval type-2 fuzzy set to simplify computational overhead, which is defined as follows:

Definition 4 Let X be the universe of discourse. Then, a symmetric triangular interval type-2 fuzzy set $\tilde{A}$ can be represented as follows:

$$\tilde{A} = (c_{\tilde{A}}(x), \delta_{\tilde{A}}(x), \underline{h}_{\tilde{A}}(x), \bar{h}_{\tilde{A}}(x) | x \in X), \quad (6)$$

where $c_{\tilde{A}}(x), \delta_{\tilde{A}}(x), \underline{h}_{\tilde{A}}(x), \bar{h}_{\tilde{A}}(x)$ are the reference points of the symmetric triangular type-2 fuzzy set at point x , satisfying the inequalities $\delta_{\tilde{A}}(x) \leq c_{\tilde{A}}(x), 0 \leq \underline{h}_{\tilde{A}}(x) \leq \bar{h}_{\tilde{A}}(x) \leq 1$.

For convenience, we define $\tilde{A} = (c_{\tilde{A}}, \delta_{\tilde{A}}, \underline{h}_{\tilde{A}}, \bar{h}_{\tilde{A}})$ as a symmetric triangular interval type-2 fuzzy number (STIT2FN) (shown in Fig. 1). Let Θ be the set of all STIT2FNs. The

Fig. 1 A symmetric triangular interval type-2 fuzzy number

upper membership function (UMF) and the lower membership function (LMF) of $\tilde{A}$ are defined as follows:

$$UMF_{\tilde{A}}(x) = \begin{cases} \frac{\bar{h}_{\tilde{A}}}{\delta_{\tilde{A}}} (x - c_{\tilde{A}} + \delta_{\tilde{A}}) & c_{\tilde{A}} - \delta_{\tilde{A}} \leq x < c_{\tilde{A}} \\ \bar{h}_{\tilde{A}} & x = c_{\tilde{A}} \\ \frac{\bar{h}_{\tilde{A}}}{\delta_{\tilde{A}}} (c_{\tilde{A}} + \delta_{\tilde{A}} - x) & c_{\tilde{A}} < x \leq c_{\tilde{A}} + \delta_{\tilde{A}} \end{cases} \quad (7)$$

$$LMF_{\tilde{A}}(x) = \begin{cases} \frac{h_{\tilde{A}}}{\delta_{\tilde{A}}} (x - c_{\tilde{A}} + \delta_{\tilde{A}}) & c_{\tilde{A}} - \delta_{\tilde{A}} \leq x < c_{\tilde{A}} \\ h_{\tilde{A}} & x = c_{\tilde{A}} \\ \frac{h_{\tilde{A}}}{\delta_{\tilde{A}}} (c_{\tilde{A}} + \delta_{\tilde{A}} - x) & c_{\tilde{A}} < x \leq c_{\tilde{A}} + \delta_{\tilde{A}} \end{cases} \quad (8)$$

Definition 5 Let $\tilde{A} = (c_{\tilde{A}}, \delta_{\tilde{A}}, \underline{h}_{\tilde{A}}, \bar{h}_{\tilde{A}})$ be a STIT2FN.

Then, the score value of $\tilde{A}$ is defined as

$$s(\tilde{A}) = (s_x(\tilde{A}), s_y(\tilde{A})) = \left(c_{\tilde{A}} \frac{\underline{h}_{\tilde{A}} \bar{h}_{\tilde{A}}}{\underline{h}_{\tilde{A}} + \bar{h}_{\tilde{A}}}, \frac{\underline{h}_{\tilde{A}} + \bar{h}_{\tilde{A}}}{2} \right) \quad (9)$$

In order to rank any two STIT2FNs $\tilde{A} = (c_{\tilde{A}}, \delta_{\tilde{A}}, \underline{h}_{\tilde{A}}, \bar{h}_{\tilde{A}})$ and $\tilde{B} = (c_{\tilde{B}}, \delta_{\tilde{B}}, \underline{h}_{\tilde{B}}, \bar{h}_{\tilde{B}})$, we define a ranking order relation between two STIT2FNs $\tilde{A}, \tilde{B}$ as follows:

- If $s_x(\tilde{A}) > s_x(\tilde{B})$, then $\tilde{A} > \tilde{B}$.
- If $s_x(\tilde{A}) = s_x(\tilde{B})$, then

- (1) If $s_y(\tilde{A}) = s_y(\tilde{B})$, then $\tilde{A} = \tilde{B}$;
- (2) If $s_y(\tilde{A}) > s_y(\tilde{B})$, then $\tilde{A} > \tilde{B}$.

Definition 6 Let $\tilde{A} = (c_{\tilde{A}}, \delta_{\tilde{A}}, \underline{h}_{\tilde{A}}, \bar{h}_{\tilde{A}})$ and $\tilde{B} = (c_{\tilde{B}}, \delta_{\tilde{B}}, \underline{h}_{\tilde{B}}, \bar{h}_{\tilde{B}})$ be two STIT2FNs in the universe of discourse X and $n \geq 0$. Then the operations of STIT2FNs are defined as follows:

- (1) $\tilde{A} \oplus \tilde{B} = (c_{\tilde{A}} + c_{\tilde{B}}, \delta_{\tilde{A}} + \delta_{\tilde{B}}, h_{\tilde{A}\tilde{B}}, \bar{h}_{\tilde{A}} + \bar{h}_{\tilde{B}} - \bar{h}_{\tilde{A}}\bar{h}_{\tilde{B}});$
- (2) $\tilde{A} \otimes \tilde{B} = (c_{\tilde{A}}c_{\tilde{B}}, \delta_{\tilde{A}}\delta_{\tilde{B}}, h_{\tilde{A}} + h_{\tilde{B}} - h_{\tilde{A}}h_{\tilde{B}}, \bar{h}_{\tilde{A}}\bar{h}_{\tilde{B}});$
- (3) $n\tilde{A} = (nc_{\tilde{A}}, n\delta_{\tilde{A}}, h_{\tilde{A}}^n, 1 - (1 - \bar{h}_{\tilde{A}})^n);$ and
- (4) $\tilde{A}^n = (c_{\tilde{A}}^n, \delta_{\tilde{A}}^n, 1 - (1 - h_{\tilde{A}})^n, \bar{h}_{\tilde{A}}^n).$

4 The symmetric triangular interval type-2 fuzzy aggregation operators based on the Hamy mean

In this section, we shall explore the Hamy mean (HM) to accommodate a situation where the input arguments are symmetric triangular interval type-2 fuzzy numbers, and study some desirable properties in detail, which include *idempotency*, *commutativity*, *monotonicity*, and *boundness*.

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right) \quad (11)$$

First, we present a definition of symmetric triangular interval type-2 fuzzy Hamy mean operator, which is defined as follows:

Definition 7 Let $\tilde{A}_i = (c_{\tilde{A}_i}, \delta_{\tilde{A}_i}, h_{\tilde{A}_i}, \bar{h}_{\tilde{A}_i}) (i = 1, 2, \dots, n)$ be a collection of STIT2FNs, and parameter $k = 1, 2, \dots, n$. If

$$\text{STIT2HM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\otimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \quad (10)$$

then $\text{STIT2HM}^{(k)}$ is called the symmetric triangular interval type-2 fuzzy Hamy mean (STIT2FHM) operator.

Following the operations of STIT2FN described in Sect. 2, we derive the following Theorem 1.

Theorem 1 Let $\tilde{A}_i = (c_{\tilde{A}_i}, \delta_{\tilde{A}_i}, h_{\tilde{A}_i}, \bar{h}_{\tilde{A}_i}) (i = 1, 2, \dots, n)$ be a collection of STIT2FNs, and parameter $k = 1, 2, \dots, n$. Then the aggregated value, obtained by using the STIT2FHM operator, is also an STIT2FN, and

Proof In virtue of the operational laws of STIT2FNs, we have

$$\otimes_{j=1}^k \tilde{A}_{i_j} = \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}}, \prod_{j=1}^k \delta_{\tilde{A}_{i_j}}, 1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}), \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right). \quad (12)$$

This yields

$$\left(\otimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}} = \left(\left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}, \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}, 1 - \left(\prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}) \right)^{\frac{1}{k}}, \left(\prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}} \right) \quad (13)$$

and

$$\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigotimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}} = \left(\begin{array}{c} \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}, \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}, \\ \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - \frac{h_{\tilde{A}_{i_j}}}{\tilde{A}_{i_j}} \right) \right), 1 - \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}} \right) \end{array} \right) \quad (14)$$

Subsequently, we have

$$\frac{\bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\bigotimes_{j=1}^k \alpha_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} = \left(\begin{array}{c} \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \\ \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - \frac{h_{\tilde{A}_{i_j}}}{\tilde{A}_{i_j}} \right) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \end{array} \right) \quad (15)$$

Therefore,

$$\begin{aligned} & \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left(\begin{array}{c} \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \\ \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \left(1 - \frac{h_{\tilde{A}_{i_j}}}{\tilde{A}_{i_j}} \right) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \end{array} \right) \end{aligned} \quad (16)$$

which complete the proof of Theorem 1.

In what follows, we shall study some desirable properties of the STIT2FHM operator such as *idempotency*, *commutativity*, and *monotonicity*.

Theorem 2 (*Idempotency*) If all $\tilde{A}_i (i = 1, 2, \dots, n)$ are equal, i.e., $\tilde{A}_i = \tilde{A} = \left(c_{\tilde{A}}, \delta_{\tilde{A}}, \frac{h_{\tilde{A}}}{\tilde{A}}, \tilde{h}_{\tilde{A}} \right)$, then

$$\text{STIT2FHM}^{(k)}(\tilde{A}, \tilde{A}, \dots, \tilde{A}) = \tilde{A} \quad (17)$$

Proof Since $\tilde{A} = \left(c_{\tilde{A}}, \delta_{\tilde{A}}, \frac{h_{\tilde{A}}}{\tilde{A}}, \tilde{h}_{\tilde{A}} \right)$, based on Theorem 1, we have

$$\begin{aligned}
 & \text{STIT2FHM}^{(k)}(\tilde{A}, \tilde{A}, \dots, \tilde{A}) \\
 &= \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(c_{\tilde{A}}^k \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\delta_{\tilde{A}}^k \right)^{\frac{1}{k}}}{\binom{n}{k}}, \right. \\
 & \quad \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left((1 - h_{\tilde{A}})^k \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\bar{h}_{\tilde{A}}^k \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right) \\
 &= \left(\frac{\binom{n}{k} \left(c_{\tilde{A}}^k \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\binom{n}{k} \left(\delta_{\tilde{A}}^k \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left((1 - h_{\tilde{A}})^k \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}}, \right. \\
 & \quad \left. 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\bar{h}_{\tilde{A}}^k \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right) \\
 &= \left(c_{\tilde{A}}, \delta_{\tilde{A}}, \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} h_{\tilde{A}} \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} (1 - \bar{h}_{\tilde{A}}) \right)^{\frac{1}{\binom{n}{k}}} \right) \\
 &= \left(c_{\tilde{A}}, \delta_{\tilde{A}}, \left(\left(h_{\tilde{A}} \right)^{\binom{n}{k}} \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left((1 - \bar{h}_{\tilde{A}})^{\binom{n}{k}} \right)^{\frac{1}{\binom{n}{k}}} \right) \\
 &= \left(c_{\tilde{A}}, \delta_{\tilde{A}}, h_{\tilde{A}}, \bar{h}_{\tilde{A}} \right) \\
 &= \tilde{A}
 \end{aligned} \tag{18}$$

which completes the proof of Theorem 2.

Theorem 3 (Commutativity) Let $\tilde{A}_i = \left(c_{\tilde{A}_i}, \delta_{\tilde{A}_i}, h_{\tilde{A}_i}, \bar{h}_{\tilde{A}_i} \right)$ ($i = 1, 2, \dots, n$) be a collection of STIT2FNs, and $(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$ is any permutation of $(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$. Then

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n). \tag{19}$$

Proof Since $(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$ is any permutation of $(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$, based on the definition of STIT2FHM in Eq. (10), we have

$$\begin{aligned}
 & \text{STIT2FHM}^{(k)}(\tilde{A}, \tilde{A}, \dots, \tilde{A}) \\
 &= \frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\otimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} \\
 &= \frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(\otimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n}{k}} \\
 &= \text{STIT2FHM}^{(k)}(\tilde{A}, \tilde{A}, \dots, \tilde{A})
 \end{aligned} \tag{20}$$

which completes the proof of Theorem 3.

In order to discuss the monotonicity of the STIT2FHM operator, we first introduce two useful lemmas, which will be used in the following sections.

Lemma 1 (Hara et al. 1998) Let x_i ($i = 1, 2, \dots, n$) be a collection of nonnegative real numbers, and for $k = 1, 2, \dots, n$. Then

$$\begin{aligned}
 & \text{HM}^{(1)}(x_1, x_2, \dots, x_n) \geq \text{HM}^{(2)}(x_1, x_2, \dots, x_n) \\
 & \geq \dots \geq \text{HM}^{(n)}(x_1, x_2, \dots, x_n)
 \end{aligned} \tag{21}$$

with equality holding if and only if $x_1 = x_2 = \dots = x_n$.

Lemma 2 (Jiang 2007) Let $x_i, y_i > 0$ ($i = 1, 2, \dots, n$), and $\sum_{i=1}^n y_i = 1$. Then

$$\prod_{i=1}^n x_i^{y_i} \leq \sum_{i=1}^n x_i y_i \tag{22}$$

with equality holding if and only if $x_1 = x_2 = \dots = x_n$.

Theorem 4 (Monotonicity) Let $\tilde{A}_i = \left(c_{\tilde{A}_i}, \delta_{\tilde{A}_i}, h_{\tilde{A}_i}, \bar{h}_{\tilde{A}_i} \right)$ and $\tilde{B}_i = \left(c_{\tilde{B}_i}, \delta_{\tilde{B}_i}, h_{\tilde{B}_i}, \bar{h}_{\tilde{B}_i} \right)$ ($i = 1, 2, \dots, n$) be two collec-

Since $c_{\tilde{A}_i} \leq c_{\tilde{B}_i}$ for all i , we have

$$s_x(\alpha) = \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}} \leq \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{B}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}} = s_x(\beta) \quad (26)$$

tions of STIT2FHNs, and $k = 1, 2, \dots, n$. If $c_{\tilde{A}_i} \leq c_{\tilde{B}_i}$, $\delta_{\tilde{A}_i} \geq \delta_{\tilde{B}_i}$, $h_{\tilde{A}_i} \geq h_{\tilde{B}_i}$, $\bar{h}_{\tilde{A}_i} \leq \bar{h}_{\tilde{B}_i}$ for all i , then

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \leq \text{STIT2FHM}^{(k)}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n). \quad (23)$$

Proof Let $\alpha = \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$ and $\beta = \text{STIT2FHM}^{(k)}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n)$. Then according to Theorem 1, we have

Now we discuss the following two cases:

Case 1 If $s_x(\alpha) < s_x(\beta)$, then based on Definition 4, it can be easily obtained that

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) < \text{STIT2FHM}^{(k)}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n) \quad (27)$$

$$\begin{aligned} \alpha &= \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left[\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \right. \\ &\quad \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right] \end{aligned} \quad (24)$$

and

$$\begin{aligned} \beta &= \text{STIT2FHM}^{(k)}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n) \\ &= \left[\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{B}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{B}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \right. \\ &\quad \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{B}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right] \end{aligned} \quad (25)$$

Case 2 If $s_x(\alpha) = s_x(\beta)$, since $\underline{h}_{\tilde{A}_i} \leq \underline{h}_{\tilde{B}_i}, \bar{h}_{\tilde{A}_i} \leq \bar{h}_{\tilde{B}_i}$ for all i , then

Then by Definition 4, we obtain
 $\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \leq \text{STIT2FHM}^{(k)}(\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n)$
 which completes the proof of Theorem 4.

$$\begin{aligned}
 1 - \underline{h}_{\tilde{B}_{i_j}} &\leq 1 - \underline{h}_{\tilde{A}_{i_j}} \\
 \Rightarrow \prod_{j=1}^k (1 - \underline{h}_{\tilde{B}_{i_j}}) &\leq \prod_{j=1}^k (1 - \underline{h}_{\tilde{A}_{i_j}}) \\
 \Rightarrow 1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{B}_{i_j}}) &\geq 1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{A}_{i_j}}) \\
 \Rightarrow \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{B}_{i_j}}) \right) &\geq \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{A}_{i_j}}) \right) \\
 \Rightarrow \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{B}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}} &\geq \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}
 \end{aligned} \tag{28}$$

and

$$\begin{aligned}
 \bar{h}_{\tilde{B}_{i_j}} &\geq \bar{h}_{\tilde{A}_{i_j}} \\
 \Rightarrow \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} &\geq \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \\
 \Rightarrow 1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} &\leq 1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \\
 \Rightarrow \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} \right) &\leq \prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \\
 \Rightarrow \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} &\leq \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \\
 \Rightarrow 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} &\geq 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}}
 \end{aligned} \tag{29}$$

and thus one has

$$\begin{aligned}
 s_y(\alpha) &= \frac{\left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}} + 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}}}{2} \\
 &\leq \frac{\left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \underline{h}_{\tilde{B}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}} + 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{B}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}}}{2} \\
 &= s_y(\beta)
 \end{aligned} \tag{30}$$

Theorem 5 (Boundness) Let $\tilde{A}_i = (c_{\tilde{A}}, \delta_{\tilde{A}}, h_{\tilde{A}}, \bar{h}_{\tilde{A}})$ ($i = 1, 2, \dots, n$) be a collection of STIT2FNs and let

$$\tilde{A}^- = \left(\min_i \{c_{\tilde{A}_i}\}, \max_i \{\delta_{\tilde{A}_i}\}, \min_i \{h_{\tilde{A}_i}\}, \min_i \{\bar{h}_{\tilde{A}_i}\} \right)$$

$$\tilde{A}^+ = \left(\max_i \{c_{\tilde{A}_i}\}, \min_i \{\delta_{\tilde{A}_i}\}, \max_i \{h_{\tilde{A}_i}\}, \max_i \{\bar{h}_{\tilde{A}_i}\} \right)$$

Then

$$\tilde{A}^- \leq \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \leq \tilde{A}^+ \quad (31)$$

Proof Based on Theorems 3 and 4, we have

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \geq \text{STIT2FHM}^{(k)}(\tilde{A}^-, \tilde{A}^-, \dots, \tilde{A}^-) = \tilde{A}^- \quad (32)$$

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \geq \text{STIT2FHM}^{(k)}(\tilde{A}^+, \tilde{A}^+, \dots, \tilde{A}^+) = \tilde{A}^+. \quad (33)$$

Thus the proof has been completed.

Theorem 6 For given arguments $A_i \in \Theta$ ($i = 1, 2, \dots, n$), and $k = 1, 2, \dots, n$, the STIT2FHM is monotonically decreasing with respect to the parameter k .

Proof Based on Theorem 1, we have

$$\begin{aligned} & \text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left[\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \right. \\ & \quad \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right] \end{aligned} \quad (34)$$

and

$$\begin{aligned} & \text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left[\frac{\sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(\prod_{j=1}^{k+1} c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k+1}}}{\binom{n}{k+1}}, \frac{\sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(\prod_{j=1}^{k+1} \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{k+1}}}{\binom{n}{k+1}}, \right. \\ & \quad \left. \left(\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(1 - \prod_{j=1}^{k+1} (1 - h_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k+1}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(1 - \prod_{j=1}^{k+1} \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k+1}}} \right] \end{aligned} \quad (35)$$

Then, based on Definition 5 and Lemma 1, we obtain

$$\begin{aligned} & s_x(\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) \\ &= \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k}}}{\binom{n}{k}} \geq \frac{\sum_{1 \leq i_1 < \dots < i_{k+1} \leq n} \left(\prod_{j=1}^{k+1} c_{\tilde{A}_{i_j}} \right)^{\frac{1}{k+1}}}{\binom{n}{k+1}} \\ &= s_x(\text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) \end{aligned} \quad (36)$$

Now we discuss the following two cases:

Case 1 If $s_x(\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) > s_x(\text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n))$, then based Definition 4, it can be easily obtained that

$$\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) > \text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \quad (37)$$

Case 2 If $s_x(\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) = s_x(\text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n))$, Then

$$\begin{aligned} T(k) &= \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, \\ S(k) &= 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{k}}} \end{aligned} \quad (38)$$

In what follows, we prove that function $T(k)$ is monotonically decreasing with respect to the parameter k .

Based on Lemmas 1 and 2, we have

$$\begin{aligned} T(k) &= \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - \frac{h_{\tilde{A}_{i_j}}}{n}) \right) \right)^{\frac{1}{\binom{n}{k}}} \\ &\geq \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{1 - \prod_{j=1}^k (1 - \frac{h_{\tilde{A}_{i_j}}}{n})}{\binom{n}{k}} \\ &= 1 - \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{\prod_{j=1}^k (1 - \frac{h_{\tilde{A}_{i_j}}}{n})}{\binom{n}{k}} \end{aligned} \quad (39)$$

Then, the following proof is carried out by contradiction. Let us suppose that $T(k)$ is monotonically increasing with respect to the parameter k . Then it follows that

$$T(n) > T(n-1) > \dots > T(1) \quad (40)$$

also since

$$T(1) \geq 1 - \sum_{1 \leq i_1 \leq n} \frac{\prod_{j=1}^1 (1 - \frac{h_{\tilde{A}_{i_1}}}{n})}{\binom{n}{1}} = 1 - \frac{n - \sum_{i=1}^n h_{\tilde{A}_i}}{n} = \frac{\sum_{i=1}^n h_{\tilde{A}_i}}{n}. \quad (41)$$

Then, based on Eq. (41), we obtain

$$T(n) > T(1) = \frac{\sum_{i=1}^n h_{\tilde{A}_i}}{n} \Rightarrow \left(\prod_{i=1}^n h_{\tilde{A}_i} \right)^{\frac{1}{n}} > \frac{\sum_{i=1}^n h_{\tilde{A}_i}}{n}. \quad (42)$$

However, according to the elementary mean inequality (Jiang 2007), we have

$$\left(\prod_{i=1}^n \mu_i \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^n \mu_i}{n} \quad (43)$$

Clearly, it is a contradiction to this elementary mean inequality. Therefore, the function $T(k)$ is monotonically decreasing with respect to the parameter k . Similarly, we can also prove that function $S(k)$ is monotonically increasing with respect to the parameter k .

and thus

$$\begin{aligned} s_y(\text{STIT2FHM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) &= \frac{T(k) + S(k)}{2} > \\ \frac{T(k+1) + S(k+1)}{2} &= s_y(\text{STIT2FHM}^{(k+1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)) \end{aligned} \quad (44)$$

Thus, the STIT2FHM operator is monotonically decreasing with respect to the parameter k , which completes the proof of Theorem 5.

In the following, we shall discuss some special cases of the STIT2FHM operator by taking different values of the parameter k .

Case 1 If $k = 1$, based on the definition of STIT2FHM operator, we have

$$\begin{aligned} &\text{STIT2FHM}^{(1)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left[\frac{\sum_{1 \leq i_1 \leq n} \left(\prod_{j=1}^1 c_{\tilde{A}_{i_j}} \right)^{\frac{1}{1}}}{\binom{n}{1}}, \frac{\sum_{1 \leq i_1 \leq n} \left(\prod_{j=1}^1 \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{1}}}{\binom{n}{1}}, \right. \\ &\quad \left. \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 (1 - \frac{h_{\tilde{A}_{i_j}}}{n}) \right) \right)^{\frac{1}{\binom{n}{1}}}, 1 - \left(\prod_{1 \leq i_1 \leq n} \left(1 - \prod_{j=1}^1 \bar{h}_{\tilde{A}_{i_j}} \right) \right)^{\frac{1}{\binom{n}{1}}} \right] \\ &= \left(\frac{\sum_{i=1}^n c_{\tilde{A}_i}}{n}, \frac{\sum_{i=1}^n \delta_{\tilde{A}_i}}{n}, \left(\prod_{i=1}^n \left(1 - (1 - \frac{h_{\tilde{A}_i}}{n}) \right) \right)^{\frac{1}{n}}, 1 - \left(\prod_{i=1}^n (1 - \bar{h}_{\tilde{A}_i}) \right)^{\frac{1}{n}} \right) \\ &= \left(\frac{\sum_{i=1}^n c_{\tilde{A}_i}}{n}, \frac{\sum_{i=1}^n \delta_{\tilde{A}_i}}{n}, \left(\prod_{i=1}^n h_{\tilde{A}_i} \right)^{\frac{1}{n}}, 1 - \left(\prod_{i=1}^n (1 - \bar{h}_{\tilde{A}_i}) \right)^{\frac{1}{n}} \right) \end{aligned} \quad (45)$$

In this case, we call STIT2FHM a symmetric triangular interval type-2 fuzzy averaging (STIT2FA) operator.

Case 2 If $k = n$, based on the definition of STIT2FHM operator, we have

$$\begin{aligned} & \text{STIT2FHM}^{(n)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left(\frac{\sum_{1 \leq i_1 < \dots < i_n \leq n} \left(\prod_{j=1}^n c_{\tilde{A}_{i_j}} \right)^{\frac{1}{n}}}{\binom{n}{n}}, \frac{\sum_{1 \leq i_1 < \dots < i_n \leq n} \left(\prod_{j=1}^n \delta_{\tilde{A}_{i_j}} \right)^{\frac{1}{n}}}{\binom{n}{n}}, \right. \\ & \quad \left. \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n (1 - h_{\tilde{A}_{i_j}}) \right)^{\frac{1}{n}} \right), 1 - \left(\prod_{1 \leq i_1 < \dots < i_n \leq n} \left(1 - \prod_{j=1}^n \bar{h}_{\tilde{A}_{i_j}} \right)^{\frac{1}{n}} \right) \right) \\ &= \left(\left(\prod_{i=1}^n c_{\tilde{A}_i} \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n \delta_{\tilde{A}_i} \right)^{\frac{1}{n}}, \left(1 - \prod_{j=1}^n (1 - h_{\tilde{A}_j}) \right)^{\frac{1}{n}}, 1 - \left(1 - \prod_{j=1}^n \bar{h}_{\tilde{A}_j} \right)^{\frac{1}{n}} \right) \\ &= \left(\left(\prod_{i=1}^n c_{\tilde{A}_i} \right)^{\frac{1}{n}}, \left(\prod_{i=1}^n \delta_{\tilde{A}_i} \right)^{\frac{1}{n}}, \left(1 - \prod_{j=1}^n (1 - h_{\tilde{A}_j}) \right)^{\frac{1}{n}}, \prod_{j=1}^n \bar{h}_{\tilde{A}_j} \right) \end{aligned} \quad (46)$$

In this case, we call STIT2FHM a symmetric triangular interval type-2 fuzzy geometric (STIT2FG) operator.

the weighted symmetric triangular interval type-2 fuzzy Hamy mean (WSTIT2FHM) operator.

Definition 7 Let $\tilde{A}_i (i = 1, 2, \dots, n)$ be a collection of STIT2FNs, $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector

of $\tilde{A}_i (i = 1, 2, \dots, n)$, where ω_i indicates the importance degree of $\tilde{A}_i$, which satisfies $\omega_i \in [0, 1]$ and $\sum_{i=1}^n \omega_i = 1$. If

$$\text{WSTIT2FHM}_{\omega}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \begin{cases} \frac{\oplus_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \sum_{j=1}^k \omega_{i_j} \right) \left(\otimes_{j=1}^k \tilde{A}_{i_j} \right)^{\frac{1}{k}}}{\binom{n-1}{k}} & (1 \leq k < n) \\ \otimes_{j=1}^k \tilde{A}_j^{\frac{1-\omega_j}{n-1}} & (k = n) \end{cases} \quad (47)$$

From the definition of STIT2FHM operator above, we can see that the STIT2FHM operator does not consider the importance of the aggregated arguments. Nevertheless, in many real-world practical situations, especially in MCDM, the weights of attributes play an important role in the aggregation process. Therefore, in this section, we propose

then $\text{WSTIT2FHM}_{\omega}^{(k)}$ is called the weighted symmetric triangular interval type-2 fuzzy Hamy mean (WSTIT2FHM) operator,

According to the operations of STIT2FNs described in Sect. 2, we can derive the following Theorem 7.

Theorem 7 Let $1 \leq k \leq n$ ($k \in \mathbb{Z}$), and h_i ($i = 1, 2, \dots, n$) be a collection of STIT2FNs. Then the aggregated value, by using the WSTIT2FHM operator, is also an STIT2FN, and

Theorem 8 WSTIT2FHM operator is a special case of the STIT2FHM operator.

Proof Based on Theorem 4, when $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, we

$$\begin{aligned} & \text{WSTIT2FHM}_{\omega}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \begin{cases} \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \sum_{j=1}^k \omega_{i_j}\right) \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \sum_{j=1}^k \omega_{i_j}\right) \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \right. \\ \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}})\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n-1}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n-1}{k}}} \right) \right) \quad (1 \leq k < n) \\ \left(\prod_{j=1}^k c_{\tilde{A}_j}^{\frac{1-\omega_j}{n-1}}, \prod_{j=1}^k \delta_{\tilde{A}_j}^{\frac{1-\omega_j}{n-1}}, 1 - \prod_{j=1}^k (1 - h_{\tilde{A}_j})^{\frac{1-\omega_j}{n-1}}, \prod_{j=1}^k \tilde{h}_{\tilde{A}_j}^{\frac{1-\omega_j}{n-1}} \right) \quad (k = n) \end{cases} \quad (48) \end{aligned}$$

Proof The proof of Theorem 7 is similar to the proof of Theorem 1, so it is omitted here. consider two cases:

(1) If $1 \leq k < n$, we have

$$\begin{aligned} & \text{WSTIT2FHM}_{\omega}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \begin{cases} \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \frac{k}{n}\right) \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \frac{k}{n}\right) \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n-1}{k}}, \right. \\ \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}})\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n-1}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n-1}{k}}} \right) \right) \quad (49) \\ \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \frac{k}{n}\right) \left(\prod_{j=1}^k c_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k} \frac{n-k}{n}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \frac{k}{n}\right) \left(\prod_{j=1}^k \delta_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}}{\binom{n}{k} \frac{n-k}{n}}, \right. \\ \left. \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k (1 - h_{\tilde{A}_{i_j}})\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n}{k} \frac{n-k}{n}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \left(\prod_{j=1}^k \tilde{h}_{\tilde{A}_{i_j}}\right)^{\frac{1}{k}}\right)^{\frac{1}{\binom{n}{k} \frac{n-k}{n}}} \right) \right) \end{cases} \end{aligned}$$

$$= \left(\frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k c_{\tilde{A}_{ij}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\prod_{j=1}^k \delta_{\tilde{A}_{ij}} \right)^{\frac{1}{k}}}{\binom{n}{k}}, \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k (1 - h_{\tilde{A}_{ij}}) \right) \right)^{\frac{1}{\binom{n}{k}}}, 1 - \left(\prod_{1 \leq i_1 < \dots < i_k \leq n} \left(1 - \prod_{j=1}^k \tilde{h}_{\tilde{A}_{ij}} \right) \right)^{\frac{1}{\binom{n}{k}}} \right)$$

$$= \text{STIT2HM}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n)$$

(2) If $k = n$, we have

$$\begin{aligned} & \text{WSTIT2FHM}_{\omega}^{(k)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \\ &= \left(\prod_{j=1}^k c_{\tilde{A}_j}^{\frac{1-\frac{1}{n}}{n-1}}, \prod_{j=1}^k \delta_{\tilde{A}_j}^{\frac{1-\frac{1}{n}}{n-1}}, 1 - \prod_{j=1}^k (1 - h_{\tilde{A}_j})^{\frac{1-\frac{1}{n}}{n-1}}, \prod_{j=1}^k \tilde{h}_{\tilde{A}_j}^{\frac{1-\frac{1}{n}}{n-1}} \right) \\ &= \left(\prod_{j=1}^k c_{\tilde{A}_j}^{\frac{1}{n}}, \prod_{j=1}^k \delta_{\tilde{A}_j}^{\frac{1}{n}}, 1 - \prod_{j=1}^k (1 - h_{\tilde{A}_j})^{\frac{1}{n}}, \prod_{j=1}^k \tilde{h}_{\tilde{A}_j}^{\frac{1}{n}} \right) \\ &= \text{STIT2HM}^{(n)}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) \end{aligned} \quad (50)$$

which completes the proof of Theorem 8.

5 An approach to multiple criteria decision making based on the symmetric triangular interval type-2 fuzzy Hamy mean operator

In this section, we first apply the Spearman method to determine the weights of criteria. Then symmetric triangular interval type-2 fuzzy MCDM method based on the WSTIT2FHM operator is developed to rank the alternatives. Both of them are discussed as follows:

Table 2 Linguistic terms and their corresponding STIT2FN

Linguistic terms	Symmetric triangular interval type-2 fuzzy number
Very low (VL)	(0.2, 0.1, 0.9, 1)
Low (L)	(0.3, 0.1, 0.9, 1)
Medium low (ML)	(0.4, 0.1, 0.9, 1)
Medium (M)	(0.5, 0.1, 0.9, 1)
Medium high (MH)	(0.6, 0.1, 0.9, 1)
High (H)	(0.7, 0.1, 0.9, 1)
Very high (VH)	(0.8, 0.1, 0.9, 1)

Table 3 Complementary relations

Linguistic terms (L)	VL	L	ML	M	MH	H	VH
Complementary terms (L ^c)	VH	H	MH	M	ML	L	VL

5.1 The description of MCDM problems

For a MCDM problem, let $A = \{A_1, A_2, \dots, A_m\}$ be a discrete set of alternatives, and $C = \{C_1, C_2, \dots, C_n\}$ be the set of attributes, whose weight vector is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, satisfying $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$. Assume that $R = (r_{ij})_{m \times n}$ is the interval type-2 fuzzy decision matrix, where the attribute values $r_{ij} (i = 1, 2, \dots, m, j = 1, 2, \dots, n)$ are in the form of STIT2FNs. Based on these necessary conditions, we are able to rank the alternatives.

5.2 Spearman method

Spearman method (Spearman 1987) is a useful technique to determine the weights of criteria. When dealing with MCDM problems, a small sum of relative coefficient with respect to each criterion indicates that such a criterion has a big influence on the overall value of alternative. On the contrary, if sum of relative coefficient is small, it means that the criterion plays a less important role. Therefore, the big the sum of relative coefficient of alternative under a certain criteria, the smaller the corresponding weight of criteria. The Spearman method is summarized as follows:

- (1) For any two criteria C_k and C_j , the relative coefficient is computed as follows:

$$\Delta_{kj} = 1 - \frac{6 \sum_{i=1}^m (r_{ik} - r_{ij})^2}{n(n-1)} \quad (51)$$

- (2) Construct the relative coefficient matrix $\Delta_{n \times n} = (\Delta_{kj})_{n \times n}$

$$\Delta_{n \times n} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \dots & \Delta_{1n} \\ \Delta_{21} & \Delta_{22} & \dots & \Delta_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{n1} & \Delta_{n2} & \dots & \Delta_{nn} \end{pmatrix} \quad (52)$$

Table 4 The recommender information matrix

	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆	C ₇
A ₁	VL	L	M	MH	H	VH	H
A ₂	M	MH	H	VH	H	VH	VH
A ₃	L	VL	VH	M	MH	L	VL
A ₄	ML	VH	MH	H	VL	MH	H
A ₅	VL	L	VL	H	M	VL	L
A ₆	MH	VH	VH	M	VL	L	H

Table 5 Ordering of the alternatives by using different parameter k values in WSTIT2FM operator

Parameter value	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	Ranking order
$k = 1$	0.694	0.893	0.613	0.764	0.532	0.735	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
$k = 2$	0.638	0.847	0.573	0.725	0.469	0.684	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
$k = 3$	0.573	0.779	0.531	0.683	0.483	0.569	$A_2 > A_4 > A_1 > A_6 > A_3 > A_5$
$k = 4$	0.543	0.723	0.412	0.572	0.392	0.558	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
$k = 5$	0.513	0.692	0.372	0.662	0.376	0.535	$A_2 > A_4 > A_6 > A_1 > A_5 > A_3$
$k = 6$	0.469	0.667	0.365	0.654	0.353	0.462	$A_2 > A_4 > A_1 > A_6 > A_3 > A_5$
$k = 7$	0.379	0.642	0.347	0.623	0.354	0.374	$A_2 > A_4 > A_1 > A_6 > A_5 > A_3$

(3) Then we calculate the sum of relative coefficient under criteria C_j

$$\sigma_j = \frac{1}{\Delta_j} \quad (54)$$

$$\Delta_j = \sum_{\substack{k=1 \\ k \neq j}}^n \Delta_{jk} \quad (53)$$

(5) Calculate the weight of criteria

$$w_j = \frac{\sigma_j}{\sum_{j=1}^n \sigma_j} \quad (55)$$

Based on these steps, we can obtain the weights of criteria.

Fig. 2 Radar plot showing the result of the sensitivity analysis

5.3 Symmetry triangle interval type-2 fuzzy MCDM method based on WSTIT2FHM operator

The approach based on WSTIT2FHM operator to resolve the multiple criteria decision-making problems with symmetric triangular interval type-2 information mainly involves the following steps:

Step 1 Transform the decision matrix $R = (r_{ij})_{m \times n}$ into the normalization matrix $\bar{R} = (\bar{r}_{ij})_{m \times n}$ by the method given by Qin and Liu (2015).

Table 6 Comparisons with four interval type-2 fuzzy aggregation methods

Aggregation operator	Computational complexity	Ease in understanding	Ranking order
IT2FWA analytic solution method (Liu and Wang 2013)	Very high	Easy	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
Type-2 geometric Bonferroni mean method (Gong et al. 2015)	High	Difficult	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
AHP-based on information granular method (Pedrycz and Song 2014)	High	Difficult	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$
The proposed method	Low	Medium	$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$

$$\bar{r}_{ij} = \begin{cases} r_{ij} & \text{for benefit attribute } C_j \\ (r_{ij})^c & \text{for cost attribute } C_j \end{cases} \quad (56)$$

where $(r_{ij})^c$ is the complement of r_{ij} .

Step 2 Calculate the relative coefficient matrix based on Spearman method to obtain the criteria weights.

Step 3 Apply the decision information given in R , and the WSTIT2FHM operator

$$r_i = \text{WSTIT2FHM}_{\omega}^{(k)}(r_{i1}, r_{i2}, \dots, r_{im}) \quad i = 1, 2, \dots, m \quad (57)$$

to derive the overall preference values $r_i (i = 1, 2, \dots, m)$ of the alternative A_i , where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of criteria such that $\omega_j > 0$ and $\sum_{j=1}^n \omega_j = 1$.

Step 4 Calculate the score values $s(r_i) (i = 1, 2, \dots, m)$ of the overall values $r_i (i = 1, 2, \dots, m)$.

Step 5 Rank all the alternatives $A_i (i = 1, 2, \dots, m)$ and select the best one(s) according to $s(r_i)$. The greater the score values $s(r_i)$, the better the alternatives $A_i (i = 1, 2, \dots, m)$ will be.

Step 6 End.

6 A case study of WSTIT2HM operator in tourism recommender system

In this section, we apply the WSTIT2HM operator to a tourism recommender system, with which customers can explicitly express their general preferences by assigning the relative important level to the criteria. Then the system recommends the most relevant items according to customers' preferences on the user-defined multiple attributes of item taxonomy.

In this study, we assume that the customers expect to from linguistic terms (see Table 2) to give the linguistic value to express their decision preferences with symmetric triangular interval type-2 fuzzy information. Table 2 shows the linguistic terms "Very low" (VL), "Low" (L), "Medium low" (ML), "Medium" (M), "Medium high" (MH), "High" (H), "Very high" (VH), and their corresponding symmetric triangular interval type-2 fuzzy numbers, respectively.

In addition, the complementary relations corresponding to symmetric triangular interval type-2 fuzzy sets are shown in Table 3.

6.1 Description

Assume that a customer is planning his vacation and he decides to make a trip to another country (adapted from reference [Merigó et al. 2012]). After a general evaluation of different alternatives, they consider six alternatives (destinations):

- A_1 trip to Beijing (China),
- A_2 trip to Tokyo (Japan),
- A_3 trip to Granada (Spain),
- A_4 trip to New York (USA),
- A_5 trip to Edmonton (Canada), and
- A_6 trip to Cairo (Egypt)

Customers express their preferences according to the seven criteria: (1) C_1 : price of the trip, (2) C_2 : tourist activities, (3) C_3 : weather attractiveness, (4) C_4 : willingness for doing the trip, (5) C_5 : facilities of the place, (6) C_6 : peace and stability, and (7) C_7 : other factors. The recommender information matrix is shown in Table 4.

6.2 Illustration of decision-making steps

Then, we utilize the developed method based on WSTIT2HM operator to obtain the ranking order of the alternatives and select the most desirable one(s). The method involves the following steps:

Step 1 Consider that all the attributes $C_j (j = 1, 2, \dots, 7)$ are the benefit attribute, thus, the criteria values of the alternatives $A_i (j = 1, 2, \dots, 6)$ do not need normalization.

Step 2 Calculate the relative coefficient matrix based on Spearman method to obtain the criteria weights.

Based on Eq. (51), we can construct the relative coefficient matrix as follows:

$$\Delta_{7 \times 7} = \begin{pmatrix} 1 & 0.966 & 0.934 & 0.909 & 0.904 & 0.917 & 0.934 \\ 0.966 & 1 & 0.934 & 0.931 & 0.844 & 0.914 & 0.969 \\ 0.934 & 0.934 & 1 & 0.934 & 0.901 & 0.914 & 0.937 \\ 0.909 & 0.931 & 0.934 & 1 & 0.941 & 0.946 & 0.957 \\ 0.904 & 0.844 & 0.901 & 0.941 & 1 & 0.950 & 0.900 \\ 0.917 & 0.914 & 0.914 & 0.946 & 0.950 & 1 & 0.971 \\ 0.934 & 0.969 & 0.937 & 0.957 & 0.900 & 0.971 & 1 \end{pmatrix}$$

Then we calculate the sum of relative coefficient, and the obtained results are shown as follows:

$$\Delta_1 = 5.564, \Delta_2 = 5.558, \Delta_3 = 5.554, \Delta_4 = 5.618,$$

$$\Delta_5 = 5.440, \Delta_6 = 5.612, \Delta_7 = 5.668$$

Based on Eq. (55), we obtain the weights of criteria as follows:

$$w_1 = 0.1431, w_2 = 0.1432, w_3 = 0.1433, w_4 = 0.1417,$$

$$w_5 = 0.1463, w_6 = 0.1419, w_7 = 0.1405$$

Step 3 Utilize the WSTIT2FM operator to aggregate all the preference values $r_{ij} (j = 1, 2, \dots, 7)$ of the i th line and obtain the overall performance value r_i with respect to alternative A_i (without loss of generality, here we take $k=3$). Due to the limitation of space, trip line A_1 is provided as a representative example.

$$\begin{aligned}
r_1 &= \text{WSTIT2FHM}_\omega^{(3)}(r_{i1}, r_{i2}, \dots, r_{i6}) \\
&= \left(\frac{\sum_{1 \leq i_1 < i_2 < i_3 \leq 6} \left(1 - \sum_{j=1}^3 \omega_{i_j}\right) \left(\prod_{j=1}^3 c_{r_{i_j}}\right)^{\frac{1}{3}}}{\binom{5}{3}}, \frac{\sum_{1 \leq i_1 < i_2 < i_3 \leq 6} \left(1 - \sum_{j=1}^3 \omega_{i_j}\right) \left(\prod_{j=1}^3 \delta_{r_{i_j}}\right)^{\frac{1}{3}}}{\binom{5}{3}}, \right. \\
&\quad \left. \left(\prod_{1 \leq i_1 < i_2 < i_3 \leq 6} \left(1 - \left(\prod_{j=1}^3 (1 - \frac{h}{r_{i_j}})\right)^{\frac{1}{3}}\right)^{\left(1 - \sum_{j=1}^3 \omega_{i_j}\right)^{\frac{1}{3}} \left(\frac{5}{3}\right)}, 1 - \left(\prod_{1 \leq i_1 < i_2 < i_3 \leq 6} \left(1 - \left(\prod_{j=1}^3 \bar{h}_{r_{i_j}}\right)^{\frac{1}{3}}\right)^{\left(1 - \sum_{j=1}^3 \omega_{i_j}\right)^{\frac{1}{3}} \left(\frac{5}{3}\right)} \right) \right) \right) \\
&= (0.543, 0.114, 0.887, 1)
\end{aligned}$$

Similarly, we have

$$r_2 = (0.723, 0.114, 0.887, 1);$$

$$r_3 = (0.412, 0.114, 0.887, 1);$$

$$r_4 = (0.572, 0.114, 0.887, 1);$$

$$r_5 = (0.342, 0.114, 0.887, 1);$$

$$r_6 = (0.558, 0.114, 0.887, 1).$$

Step 4 Calculate the score values $s(r_i)$ ($i = 1, 2, \dots, 6$) of the overall values r_i ($i = 1, 2, \dots, 6$).

Based on Definition 5, we obtain

$$s(r_1) = (0.543, 0.943), s(r_2) = (0.723, 0.943), s(r_3) = (0.412, 0.943)$$

$$s(r_4) = (0.572, 0.943), s(r_5) = (0.342, 0.943), s(r_6) = (0.558, 0.943),$$

since

$$s_x(r_2) > s_x(r_4) > s_x(r_6) > s_x(r_1) > s_x(r_3) > s_x(r_5).$$

Therefore, we have

$$A_2 > A_4 > A_6 > A_1 > A_3 > A_5,$$

where the symbol “>” means “superior to.” Thus, the tourism destination is A_2 .

6.3 Parameter sensitivity analysis

In order to reflect the influence with the different values of the parameter k , we make a sensitivity analysis by taking the parameter k . By changing parameter k values from 1 to 7, we can obtain the changing ranking results of alternatives, which are listed in Table 5.

In order to visualize the influence of changing the value of k , we provide a radar diagram based on Table 5 to show the result of the sensitivity analysis, which is shown in Fig. 2.

From Table 5 and Fig. 2, it is clear to see that with the parameter k changes according to the decision maker's subjective preferences, the ranking orders are slightly different in this example, which indicates the WSTIT2FM operator

can reflect the decision maker's risk preferences. Furthermore, by further analysis, it is noted that the score values obtained by WHFMSM operator became smaller as the parameter k increases for the same alternative. In real-world practical decision-making situations, decision makers can choose the appropriate value in accordance with their risk preferences. In general, we take $k = [n/2]$ for computation in practical problems, where symbol $[]$ means round function and n is the number of attributes, which is not only intuitive and simple, but also in this case, the risk preference of decision maker's is neutral and the interrelationship of the individual arguments can be fully taken into account.

6.4 Comparative analysis and discussion

In order to verify the validity of our method, a comparative study was conducted to validate the results of the proposed method with those from other existing approaches. With the analysis on the same example, we select the analytic solution approach for interval type-2 fuzzy average operator proposed by Liu and Wang (2013), the type-2 geometric Bonferroni mean approach developed by Gong et al. (2015), and the AHP-based information granularity method (Pedrycz and Song 2014) to facilitate the comparative analysis.

- (1) According to Liu and Wang's IT2FWA approach (Liu and Wang 2013), we first transform the numerical weights to IT2FS weights of criteria. Then based on the definition of IT2FWA, by using the -cut analytic solution method for solving the IT2FWA problem with the KM iteration algorithm to calculate the centroid of the aggregation result, the final result is as follows:

$$\begin{aligned}
C(r_1) &= [0.604, 0.707], C(r_2) = [0.723, 0.824], C(r_3) = [0.583, 0.694], \\
C(r_4) &= [0.683, 0.724], C(r_5) = [0.523, 0.634], C(r_6) = [0.623, 0.724].
\end{aligned}$$

Then all the alternatives are ranked as follows:
 $A_2 > A_4 > A_6 > A_1 > A_3 > A_5$.

- (2) Using type-2 geometric Bonferroni mean approach developed by Gong et al. (2015), we first initialize the IT2FS weights of criteria and utilize the IT2FWGBM operator to aggregate the individual decision matrix into overall decision matrix:

$$\begin{aligned}\tilde{A}_k &= \text{IT2FWGBM}_{\omega}^{p,q}(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_m) \\ &= \frac{1}{p+q} \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^m (p(\tilde{A}_i)^{\omega_i} \oplus q(\tilde{A}_j)^{\omega_j}) \right)^{1/m(m-1)}\end{aligned}\quad (58)$$

where $k = 1, 2, \dots, m$. $w_i(w_j)$ is the weight of $A_i(A_j)$. Without any loss of generality, we take $p = q = 1$.

Due to the space limitation, we omit the computational process here, the final ranking result is as follows:
 $A_2 > A_4 > A_6 > A_1 > A_3 > A_5$.

- (3) In what follows, we use the AHP with the aid of information granular method which was proposed by Pedrycz and Song (2014) to conduct the comparative analysis. First, based on the decision maker's information, we establish the corresponding reciprocal preference matrix by using interval type-2 linguistic information. The progression of the optimization is quantified in terms of the fitness function obtained in successive generation labels. The PSO algorithm returns the optimal cutoff points interval of [0.12, 0.35], [0.15, 0.21], [0.23, 0.41], [0.31, 0.45], and [0.48, 0.64], respectively. The parameters of the PSO were set up as follows: the number of particles is 100, the number of iterations is set to 300, while $c_1 = c_2 = 2$. Then, based on aggregation phase and the exploitation phase, the reciprocal collective preference relation with the higher performance index Q is given below and the progression of the optimization is quantified in terms of the performance index fitness obtained in successive generations.

Then we can obtain the reciprocal collective preference relation with the higher performance index Q and use the quantifier guided dominance degree in accordance with average operator, and obtain the following overall ranking values:

$$R(A_1) = 0.48, R(A_2) = 0.73,$$

$$R(A_3) = 0.37, R(A_4) = 0.64, R(A_5) = 0.29, R(A_6) = 0.49$$

Therefore, the ranking order of the alternatives $A_1, A_2, A_3, A_4, A_5, A_6$ is as follows:

$$A_2 > A_4 > A_6 > A_1 > A_3 > A_5$$

The comparisons are shown in Table 6.

- (1) Compared with Liu and Wang's IT2FWA analytic solution approach (Liu and Wang 2013), the main advantage of our method is that computational complexity is reduced greatly, because our method only needs obtaining the aggregated result by using a simple formula, while the Liu and Wang's method needs using the α -cut with iteration algorithm by computer program. Moreover, our method can capture the relationship of the multiple aggregating arguments; however, the previous method can only reflect the individual importance, and it cannot measure the interactions among the multiple aggregating arguments.
- (2) The advantages of our method when compared with type-2 geometric Bonferroni mean method (Gong et al. 2015) are listed as follows: First, the computational complexity in our method is simple because the type-2 geometric Bonferroni mean show some complex operations with two parameters. The proposed method can not only capture the interrelationships of overall aggregation information, but also focus on the importance of individual information. In other words, the data mining ability of this aggregation operator is better than other operators. In addition, the interval type-2 fuzzy aggregation operator with the aid of geometric Bonferroni mean can only capture the interrelationships between two aggregation arguments. In practical decision-making problem, the decision makers are often difficult to capture the interrelationships among the multiple criteria interaction decision information and two parameters are difficult to set. Therefore, the proposed method is more general.
- (3) The advantages of the proposed method when compared with AHP-based information granularity method (Pedrycz and Song 2014) are shown as follows: First, the computation complexity in our method is relatively simple because the AHP-based information granularity method shows some complex computational intelligence algorithms, so it needs some software programming packages to obtain the optimal result. Second, it is more reliable to make a ranking result based on our method because the proposed Hamy mean aggregation operator method can eliminate the deviation, makes up for the defects of existing aggregation methods that do not take expert's utility or decision preference into consideration, and obtains interrelationships result that is more stable and creditable with smaller information loss, while the AHP-based information granularity method is mainly influenced by different dynamic initial parameter setting conditions, the final ranking result sometimes lack of consistency if the parameter selection is not appropriate. Third, the pro-

posed method not only considers the consistency of the alternatives, but also reflects the importance and interactions among any solutions of alternatives, while the AHP-based information granularity method can only calculate the optimal ranking value of the alternatives without relationships.

Based on the comparisons and analysis reported above, our proposed method is better than the other three methods for solving MCDM problem with interval type-2 fuzzy information.

7 Conclusions

In real practical MCDM problems, the criteria interaction phenomenon exists commonly. To deal with this issue, we have extended the HM to accommodate the interval type-2 fuzzy environment. To reduce the computational complexity of interval type-2 fuzzy set, we present a new concept called symmetric triangular interval type-2 fuzzy set, and define some operation laws. Motivated by the idea of HM, we have developed some new interval type-2 fuzzy aggregation operators such as symmetric triangular interval type-2 fuzzy Hamy mean (STIT2FHM) operator and weighted symmetric triangular interval type-2 fuzzy Hamy mean (WSTIT2FHM) operator, and discussed its desirable mathematical properties and special cases with respect to different parameters. To deal with the situation that criteria exist interaction phenomenon in multiple criteria decision making, an approach based on WSTIT2FHM operator is provided. Finally, an actual example concerns that tourism recommender system is provided to illustrate the practicality and effectiveness of the proposed method. The main contributions of this paper include the following three aspects:

- (1) Propose a simplified concept of interval type-2 fuzzy sets and obtain some desirable theoretical results. Some fundamental operational laws, ranking method, and a variety of properties are given in this study.
- (2) Develop some interval type-2 fuzzy aggregation operators with the aid of HM means. Compared with the existing interval type-2 fuzzy aggregation operators, the main advantage of the proposed operator is that it can capture the interrelationships among the aggregated information. Therefore, the proposed method has good mathematical structure and more general than other popular operators such as FWA, OWA, Bonferroni mean, and Heronian mean.
- (3) The proposed operators have some potential extensions with the aid of granular computing. Since the interval type-2 fuzzy HM mean is a family of parametric opera-

tor, we can use some granular computing methods (optimal allocation of information granular) to solve the optimal parameter setting problem, and also deal with the data-driven decision-making problems under type-2 fuzzy environment.

In the future research, we shall continue our work to extend the HM to accommodate the general type-2 fuzzy (GT2F) environment and develop some aggregation operators applicable to MCDM problems in many other real-life fields within the type-2 fuzzy information such as supply chain management, public transportation evaluation, social network analysis, water resource management, personalized (group) recommendation system, and others.

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