

Approximation performance of the nonlinear hybrid fuzzy system based on variable universe

Guijun Wang¹ · Yu Li² · Xiaoping Li¹

Received: 8 June 2016 / Accepted: 1 September 2016 / Published online: 26 September 2016
© Springer International Publishing Switzerland 2016

Abstract Variable universe is through a group of nonlinear contraction expansion factors online timely to adjust the input domain, so that the input domain is subdivided as detailed as possible at surrounding the expected control points. It is characteristic of constant rules, quick response, and high stability precision. In this paper, a nonlinear hybrid fuzzy system was established by introducing adjusting parameters to combine Mamdani fuzzy system and Takagi–Sugeno (T–S) fuzzy system. The input–output expression of the nonlinear hybrid fuzzy system based on variable universe was deduced according to the hybrid inference rules and contraction expansion factors. Furthermore, the second-order approximation of the nonlinear hybrid fuzzy system based on variable universe on the second-order continuously differentiable function was proved by multiple Taylor formula and maximum norm, and a sufficient condition for the approximation of the nonlinear hybrid fuzzy system was disclosed when adjusting parameters and approximation accuracy are known. Finally, the simulation example confirmed that the established nonlinear hybrid fuzzy system based on variable universe has better approximation performance than other systems under different adjusting parameter meanings.

Keywords Contraction expansion factors · Variable universe · Fuzzy rules · Nonlinear hybrid fuzzy systems · Approximation

1 Introduction

Since linguistic variable and approximate reasoning were proposed for the first time by Zadeh in 1975, fuzzy system and its approximation research have become an important field of many scholars at home and abroad. It has been applied successfully in automatic control, communication engineering, and space technology fields (Buckley 1993; Castro 1995; Hassine et al. 2003; Kosko 1994; Wang and Mendel 1992; Zeng and Singh 1995). Generally, Mamdani fuzzy system and T–S fuzzy system are two most representative ones. Liu and Li (2000, 2001) studied the approximation performance of fuzzy system of Lebesgue integrable function for the first time by introducing piecewise linear function, proving T–S fuzzy system is a universal approximator with respect to p -integrable functions. Recently, Wang and Duan (2012) combines Mamdani and T–S fuzzy system by using adjusting parameters to establish a generalized hybrid fuzzy system and reduce its rules based on different hierarchical methods of input variables. In Wang et al. (2014), K -integral norm was redefined by quasi-subtraction operator, and the generalized Mamdani fuzzy system was proved still having approximation performance on one type of integrable function under the significance of K -integral norm. These research results play an important role in further exploration of fuzzy control and system modeling (Wang et al. 2015; Yang et al. 2013; Mendel 2016; Zeng et al. 2008). However, previous researches on the approximation of fuzzy system are mainly based on subdivision of refined universe. Total rules of the fuzzy system will achieve exponential growth as the input space dimension increases, which is easy to cause rules explosion. Hence, how to control total rules growth is an important issue that has to be considered.

✉ Guijun Wang
tjwgj@126.com

¹ School of Mathematics Science, Tianjin Normal University, Tianjin 300387, China

² The Second Middle School, Zhanjiang 524022, China

Pedrycz (1989) first put forward the idea of the variable universe of discourse to make an universe is subdivided better near the expected control point, and studied the nonlinear context adaptation questions in the calibration of fuzzy sets (Pedrycz et al. 1997). Li (1997) gave the concept of variable universe based on a group of nonlinear contraction expansion factors, and the design issues of high precision adaptive fuzzy controller are effectively discussed (see Li 1999; Li et al. 2002). In addition, Li et al. (2002) applied the fuzzy controller based on variable universe successfully to the simulation control experiment of level-4 inverted pendulum which was the theoretical basis for spacecraft control and research of chaotic system robustness. Long et al. (2008) proposed a latent genetic algorithm of fuzzy control based on variable universe for two inputs and one output fuzzy controller. The approximation of adaptive fuzzy system based on variable universe was studied by Long et al. (2010). Long et al. (2012) constructed the input output expression of Mamdani fuzzy system based on variable universe using singleton fuzzifier, product inference engine and central average defuzzifier. These useful results demonstrate that fuzzy controller based on variable universe can increase approximation accuracy of nonlinear fuzzy system, which lays foundation for following discussion on fuzzy control and system modeling of complicated fuzzy system. Actually, the nonlinear hybrid fuzzy system based on variable universe breaks previous limitation of independent research on Mamdani or T–S fuzzy system. It achieves approximation accuracy by initializing fine universe dissection and timely adjustment of contraction expansion factors.

The main purpose of this paper is to set up an analytical expression of the multiple input and one output nonlinear fuzzy system based on the variable universe approach, and the two types of common fuzzy systems (Mamdani and T–S fuzzy systems) are unified by introducing a control parameter, so as to establish the nonlinear hybrid fuzzy system based on the hybrid inference rules and variable universe method. The result shows that the nonlinear hybrid fuzzy system has better advantages than the separate Mamdani fuzzy system or T–S fuzzy systems.

This paper is organized as follows. After the introduction, the basic concepts of the variable universe and the structure of the nonlinear hybrid fuzzy system are briefly summarized in Sect. 2. In Sect. 3, the approximation performance of the nonlinear hybrid fuzzy system is discussed by multiple Taylor formula and maximum norm. Section 4 gives a sufficient condition for approximation of the nonlinear hybrid fuzzy system based on variable universe. In Sect. 5, the approximation of the nonlinear hybrid fuzzy system based on variable universe is realized through a simulation example.

2 Variable universe and hybrid fuzzy system

Usually, an adaptive fuzzy controller was designed by applying variable universe for the first time in Li (1999), which opens new research fields of adaptive fuzzy control and fuzzy system based on variable universe. In fact, general fuzzy system is the mapping relation between input and output. Generally speaking, whether the control function got from interpolation can approximate to actual control function effectively is determined by whether the distance between peak points of input fuzzy set is arbitrarily small? In other words, it requires adequate control rules, which is very difficult for designing fuzzy controller. According to the proposed contraction expansion factors in Li (1999). First, Mamdani fuzzy inference rules, T–S fuzzy inference rules, and hybrid fuzzy inference rules were given as follows.

Mamdani fuzzy inference rules:

$$R_{i_1 i_2 \dots i_n} : \text{if } x_1 \text{ is } \tilde{A}_{1i_1}, \\ x_2 \text{ is } \tilde{A}_{2i_2}, \dots, x_n \text{ is } \tilde{A}_{ni_n}, \text{ then } y \text{ is } \tilde{B}_{i_1 i_2 \dots i_n}. \quad (1)$$

T–S fuzzy inference rules:

$$T_{i_1 i_2 \dots i_n} : \text{if } x_1 \text{ is } \tilde{A}_{1i_1}, \\ x_2 \text{ is } \tilde{A}_{2i_2}, \dots, x_n \text{ is } \tilde{A}_{ni_n}, \text{ then } y \text{ is } \tilde{C}_{t(i_1 i_2 \dots i_n; x_1, x_2, \dots, x_n)}, \quad (2)$$

where $i_j = 1, 2, \dots, N_j$, $j = 1, 2, \dots, n$, $N_j \in \mathbb{N}$, and x_1, x_2, \dots, x_n are input variables; $\tilde{A}_{1i_1}, \tilde{A}_{2i_2}, \dots, \tilde{A}_{ni_n}$ are antecedent fuzzy sets; y is output variable; $\tilde{B}_{i_1 i_2 \dots i_n}$ and $\tilde{C}_{t(i_1 i_2 \dots i_n; x_1, x_2, \dots, x_n)}$ are consequent fuzzy sets; $t(\cdot)$ is a linear function shown as follows:

$$t(i_1 i_2 \dots i_n; x_1, x_2, \dots, x_n) = C_{i_1 i_2 \dots i_n}^0 + C_{i_1 i_2 \dots i_n}^1 x_1 \\ + C_{i_1 i_2 \dots i_n}^2 x_2 + \dots + C_{i_1 i_2 \dots i_n}^n x_n.$$

Let $\bar{y}_{i_1 i_2 \dots i_n}$ represent the center of fuzzy sets $\tilde{B}_{i_1 i_2 \dots i_n}$ in (1). In (2), the output value corresponding to the $i_1 i_2 \dots i_n$ rule is $t(i_1 i_2 \dots i_n; x_1, x_2, \dots, x_n) = \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i x_i$, where $x_0 = 1$.

$$\text{Based on single point fuzzifier } \tilde{C}_x(x') = \begin{cases} 1 & x = x' \\ 0 & x \neq x' \end{cases},$$

we can combine (1) and (2) to get the hybrid fuzzy inference rules:

$$G_{i_1 i_2 \dots i_n} : \text{IF } x_1 \text{ is } \tilde{A}_{1i_1}, \\ x_2 \text{ is } \tilde{A}_{2i_2}, \dots, x_n \text{ is } \tilde{A}_{ni_n}, \text{ THEN } y \text{ is } \lambda \tilde{A}_{t(i_1 i_2 \dots i_n; x_1, x_2, \dots, x_n)} \\ + (1 - \lambda) \tilde{B}_{i_1 i_2 \dots i_n} \quad (3)$$

where $\lambda \in [0, 1]$ is an adjusting parameter. Select step length $k = 0, 1, 2, \dots$. Let

$$X^k = [-U_1^k, U_1^k] \times [-U_2^k, U_2^k] \times \dots \times [-U_n^k, U_n^k] \subseteq \mathbb{R}^n$$

represent the input variable universe, then the output variable universe is $Y^k = [-V^k, V^k] \subset \mathbb{R}$, where $U_i^k, V^k \in \mathbb{R}^+$, $i = 1, 2, \dots, n$.

The following text introduces the concept of contraction expansion factor in variable universe proposed by Li (1999). The input and output universes of the given fuzzy system are $X = [-a, a]$ and $Y = [-b, b]$, respectively, where $a, b \in \mathbb{R}^+$, X and Y are taken as the initial input universes.

Definition 1 (Li 1999) A function $\alpha : X \rightarrow [0, 1]$ is called a contraction expansion factor on the universe X . If the following 1–4 axioms are satisfied: 1. $\alpha(x) = \alpha(-x)$, for arbitrary $x \in X$ (allelomorphism); 2. (zeroprotection); 3. $\alpha(0) = 0$ $\alpha(x)$ has strict monotone increasing on $[0, a]$ (monotonicity); 4. $|x| \leq \alpha(x)a$, for arbitrary $x \in X$ (coordination).

Let $X^0 = [-U_1^0, U_1^0] \times [-U_2^0, U_2^0] \times \dots \times [-U_n^0, U_n^0]$ be a given n -dimensional initial input universe, $\alpha_i : [-U_i^0, U_i^0] \rightarrow [0, 1]$ a contraction–expansion factor on $[-U_i^0, U_i^0]$, $i = 1, 2, \dots, n$. In the variable universe, the initial input universe is $X_i^0 = [-U_i^0, U_i^0]$ and the initial output universe is $x \in X$. Clearly, the contraction expansion factors Y are strictly increasing on $Y(y) \triangleq \beta(y)Y$, and $\alpha_i(0) = 0$, $\alpha_i(x_i^k) = \alpha_i(-x_i^k)$, $|x_i^k| \leq \alpha_i(x_i^k)U_i^0$, for any $x_i^k \in X_i^0$. If we agreed that

$$X_i^0(x_i^k) = \alpha_i(x_i^k)X_i^0 = [-\alpha_i(x_i^k)U_i^0, \alpha_i(x_i^k)U_i^0] \\ = \{\alpha_i(x_i^k)x' | x' \in X_i^0\}. \quad (4)$$

Then, $X_i^0(x_i^k)$ is called a variable universe on the initial universe X_i^0 for the i variable.

In the same way, $\beta : Y^0 \rightarrow [0, 1]$ and $Y^0(y^k) = \beta(y^k)Y^0$ on $x \in X$ can be defined. Based on contraction expansion factors and Definition 1 of variable universe, it is easy to deduce that both initial input universe and initial output universe meet the following boundary conditions:

$$\alpha_i(\pm U_i^0) = 1, \quad \beta(\pm V^0) = 1.$$

Suppose the parameter $\tau \in (0, 1)$, σ is a known sufficiently small positive number, which is to ensure nonzero denominator of $\alpha_i(x_i^k)$ and $\beta(y^k)$ during division. According to Definition 1 and above mentioned boundary conditions, the contraction expansion factors at the k th step during input variables into Eq. (4) are selected as:

$$\alpha_i(x_i^k) = \left(\frac{|x_i^k|}{U_i^0} \right)^\tau + \sigma, \quad \beta(y^k) = \left(\frac{|y^k|}{V^0} \right)^\tau + \sigma \quad (5)$$

Therefore, for any $i \in \{1, 2, \dots, n\}$ and $x_i^k \in X_i^0$, the input variable universe implementation of an hybrid fuzzy system can be implemented according to Eq. (5) under any

$k = 0, 1, 2, \dots$. In particular, we may select the initial variable $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X^0$ when $k = 0$.

Now, adjust the contraction expansion factors of variable universe continuously according to Eq. (5), so that the horizontal coordinates of a group of fuzzy sets on each input universes will decrease by $\alpha_i(x_i^k)$ times and the horizontal coordinates of a group of fuzzy sets on corresponding output universes will decrease by $\beta(y^k)$ times. As a result, when step k th increases continuously, the corresponding input–output variable universes are as follows:

$$[-U_i^k, U_i^k] = [-\alpha_i(x_i^k)U_i^0, \alpha_i(x_i^k)U_i^0]; \\ [-V^k, V^k] = [-\beta(y^k)V^0, \beta(y^k)V^0]$$

and

$$x_i^k \in [-U_i^k, U_i^k] \quad \text{iff} \quad \frac{x_i^k}{\alpha_i(x_i^k)} \in [-U_i^0, U_i^0]; \quad y^k \in [-V^k, V^k] \\ \text{iff} \quad \frac{y^k}{\beta(y^k)} \in [-V^0, V^0].$$

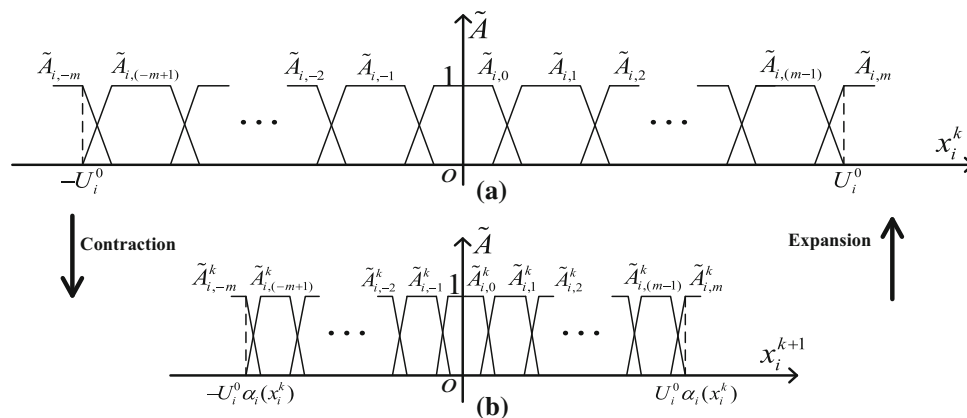
Fuzzy system based on variable universe can be established with approximate trend of rules, but does not need to much professional knowledge. In fact, whether the universe is divided equidistance or what membership function is selected during variable universe contraction makes no difference. The following text determines both antecedent fuzzy sets and consequent fuzzy sets as a group of two-phase trapezoidal waves, that is, a list of normal, consistent and complete trapezoid membership functions. A corresponding variable universe implementation is shown in Fig. 1.

The universe in Eq. (4) contracts continuously with the unceasing adjustment of contraction–expansion factor and the shrinkage of input variables. Of course, this agrees with the basic idea and characteristic of variable universe, and analytical expression of the nonlinear hybrid fuzzy system based on variable universe will be given in the following text according to contraction expansion factors.

Based on above-mentioned formulas of variable universe, let a component of the initial input variable be $x_i^0 = x_i^k / \alpha_i(x_i^k)$ and the center of the output fuzzy set be $\bar{y}_{i_1 i_2 \dots i_n}^0 = \bar{y}_{i_1 i_2 \dots i_n}^k / \beta(y^k)$ for any $\lambda \in [0, 1]$ and $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$. According to the triangular fuzzifier, product inference engine, center average defuzzifier, and the variable universe method, we can get the output expression of the nonlinear hybrid fuzzy system based on variable universe as follow:

where the adjustment coefficients $C_{i_1 i_2 \dots i_n}^i(x^k)$ and the center point $\bar{y}_{i_1 i_2 \dots i_n}^k(x^k)$ are functions about the input variable $x^k = (x_1^k, x_2^k, \dots, x_n^k)$, which may also be a constant, $k = 0, 1, 2, \dots$

Fig. 1 Contracting and expanding of a universe



Particularly, when $\lambda = 0$, Eq. (6) will degrade into a Mamdani fuzzy system; when $\lambda = 1$, Eq. (6) will degrade into a T–S fuzzy system.

adaptive fuzzy control system based on variable universe is an effective processing tool of nonlinear system, which is superior for short adjustment time, high stability pre-

$$y^{k+1} = f(x^k) = \frac{\sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji_j} \left(\frac{x_j^k}{\alpha_j(x_j^k)} \right) \times \left(\lambda \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(x^k) x_i^k + (1 - \lambda) y_{i_1 i_2 \dots i_n}^k(x^k) \right)}{\sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji_j} \left(\frac{x_j^k}{\alpha_j(x_j^k)} \right)}, \quad (6)$$

Note 1 Equation (6) is an output expression of the nonlinear hybrid fuzzy system based on variable universe which is gained by combining Mamdani and T–S fuzzy systems with λ . Since both $C_{i_1 i_2 \dots i_n}^i(x^k)$ and $\tilde{y}_{i_1 i_2 \dots i_n}^k(x^k)$ are functions about input variable x^k , Eq. (6) is not the simple linear combination of Mamdani and T–S fuzzy systems. The nonlinear hybrid fuzzy system based on variable universe has suffered fundamental structural changes, and its numerator is not linear functions any more. Actually, as the step k increases continuously, the universe of the nonlinear hybrid fuzzy system contracts with the decreasing of error, but expands with the increasing of error. In other words, a universe contraction is equivalent to increase the rules, or the interpolation node encryption, thus enabling to increasing the approximation accuracy of the system. Therefore, the concept of variable universe has important theoretical significance to further increasing response speed and stability precision of fuzzy system as well as optimizing design of adaptive fuzzy controller.

3 Approximation performance

Actually, the variable universe is characteristic of constant number of rules, quick response, and high stability precision, which are the key to increase approximation accuracy of general fuzzy systems. In addition, an

cision, strong robustness, and wide stability range. The approximation performance of the nonlinear hybrid fuzzy system based on variable universe on the second-order continuously differentiable function is further discussed in this section. The maximum norm is defined $\|f\|_\infty = \sup_{x \in U} |f(x)|$.

Theorem 1 Let $g : X^0 \rightarrow \mathbb{R}$ be a second-order continuously differentiable function, but its analytic expression is unknown, and α_i represent the given contraction expansion factor on $[-U_i^0, U_i^0]$ ($i = 1, 2, \dots, n$). Then, there's a group of adjusting parameters $C_{i_1 i_2 \dots i_n}^0(x^k)$, $C_{i_1 i_2 \dots i_n}^1(x^k), \dots, C_{i_1 i_2 \dots i_n}^n(x^k)$ under the significance of variable universe, so that

$$\left| \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(x^k) x_i^k - g(x^k) \right| \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty \alpha_i(x_i^k) \alpha_j(x_j^k) h_i^2, \quad (7)$$

where $k = 0, 1, 2, \dots$, and $h_i = \max_{1 \leq j \leq N_i-1} |\bar{e}_{i(j+1)}^0 - \bar{e}_{ij}^0|$ is the maximum distance between two adjacent centers of initial input fuzzy sets.

Proof Let $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in X^0$ represent the initial input variable, the $\tilde{A}_{1i_1}^k, \tilde{A}_{2i_2}^k, \dots, \tilde{A}_{ni_n}^k$ represent the input fuzzy set at the k th step of variable universe implementation. Coordinates of the corresponding center are recorded as $\bar{e}_{1i_1}^k(x_1^k), \bar{e}_{2i_2}^k(x_2^k), \dots, \bar{e}_{ni_n}^k(x_n^k)$.

According to the formulas of variable universe, $\bar{e}_{ji}^k(x_j^k) = \alpha_j(x_j^k)\bar{e}_{ji}^0$, $j = 1, 2, \dots, n$. For simplicity, $\bar{e}_{ji}^k(x_j^k)$ is simplified as \bar{e}_{ji}^k which is recorded uniformly as a n -vector $x_{i_1 i_2 \dots i_n}^k = (\bar{e}_{1i_1}^k, \bar{e}_{2i_2}^k, \dots, \bar{e}_{ni_n}^k)$. Its index set is $I = \{i_1 i_2 \dots i_n | i_j = 1, 2, \dots, N_j; j = 1, 2, \dots, n, \text{ so } \bar{e}_{ji}^k \in \text{Ker}(\tilde{A}_{ji}^k) \Leftrightarrow \tilde{A}_{ji}^k(\bar{e}_{ji}^k) \equiv 1$.

In addition, $g(x)$ supposed as a second-order continuously differentiable function, and for arbitrary $x = (x_1, x_2, \dots, x_n) \in X^0$ and $i_1, i_2, \dots, i_n \in I$. In accordance with the Taylor formula of multivariate function, $g(x)$ can be expanded to Taylor formula at the vertex $x_{i_1 i_2 \dots i_n}^k$ as follows:

$$g(x) = g(x_{i_1 i_2 \dots i_n}^k) + \sum_{j=1}^n \frac{\partial g}{\partial x_j} \Big|_{x=x_{i_1 i_2 \dots i_n}^k} (x_j - \bar{e}_{ji}^k) + \frac{1}{2!} (x - x_{i_1 i_2 \dots i_n}^k) (\nabla^2 g|_{x=\eta}) (x - x_{i_1 i_2 \dots i_n}^k)^T, \quad (8)$$

where $\nabla^2 g$ is the Hessian matrix of g , and η is an interior point of the hypersphere which is determined by moving point x and central point $x_{i_1 i_2 \dots i_n}^k$.

Specifically, the central point $(\bar{e}_{1i_1}^k, \bar{e}_{2i_2}^k, \dots, \bar{e}_{ni_n}^k)$ of the output fuzzy set at the k th step of variable universe implementation is as follows:

$$x_{i_1 i_2 \dots i_n}^k = (\bar{e}_{1i_1}^k, \bar{e}_{2i_2}^k, \dots, \bar{e}_{ni_n}^k) = (\alpha(x^k) \otimes x_{i_1 i_2 \dots i_n}^0) = (\alpha_1(x_1^k)\bar{e}_{1i_1}^0, \alpha_2(x_2^k)\bar{e}_{2i_2}^0, \dots, \alpha_n(x_n^k)\bar{e}_{ni_n}^0),$$

where $\alpha(x^k) = (\alpha_1(x_1^k), \alpha_2(x_2^k), \dots, \alpha_n(x_n^k))$, and $\alpha_i(x_i^k)$ is the contraction–expansion factor of each input component x_i^k on universe $X_i^k = [-U_i^k, U_i^k]$.

Substitute it into formula (8), and $g(x)$ at $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$ of k step can be represented as:

$$g(x^k) = \left(g(x_{i_1 i_2 \dots i_n}^k) - \sum_{j=1}^n \frac{\partial g}{\partial x_j} \Big|_{x=x_{i_1 i_2 \dots i_n}^k} \alpha_j(x_j^k) \bar{e}_{ji}^0 \right) + \sum_{j=1}^n \frac{\partial g}{\partial x_j} \Big|_{x=x_{i_1 i_2 \dots i_n}^k} x_j^k + \frac{1}{2!} \left(x^k - \alpha(x^k) x_{i_1 i_2 \dots i_n}^0 \right) \nabla^2 g|_{x=\eta} \left(x^k - \alpha(x^k) x_{i_1 i_2 \dots i_n}^0 \right)^T.$$

On the other hand, we may determine

$$C_{i_1 i_2 \dots i_n}^0(x^k), C_{i_1 i_2 \dots i_n}^1(x^k), \dots, C_{i_1 i_2 \dots i_n}^n(x^k)$$

when the step length of variable universe is k as follows:

$$\begin{cases} C_{i_1 i_2 \dots i_n}^0(x^k) = g(x_{i_1 i_2 \dots i_n}^k) - \sum_{j=1}^n C_{i_1 i_2 \dots i_n}^j(x^k) \alpha_j(x_j^k) \bar{e}_{ji}^0 \\ C_{i_1 i_2 \dots i_n}^j(x^k) = \frac{\partial g}{\partial x_j} \Big|_{x=x_{i_1 i_2 \dots i_n}^k}, \quad j = 1, 2, \dots, n. \end{cases} \quad (9)$$

Select parameters according to formulas (9) and then,

$$\sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(x^k) x_i^k = \left(g(x_{i_1 i_2 \dots i_n}^k) - \sum_{j=1}^n C_{i_1 i_2 \dots i_n}^j(x^k) \alpha_j(x_j^k) \bar{e}_{ji}^0 \right) + \sum_{j=1}^n \frac{\partial g}{\partial x_j} \Big|_{x=x_{i_1 i_2 \dots i_n}^k} x_j^k.$$

For arbitrary $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$, we can obtain that

where $h_i = \max_{1 \leq j \leq N_i-1} |\bar{e}_{i(j+1)}^0 - \bar{e}_{ij}^0|$ and $h_i^k = \alpha_i(x_i^k) h_i$, $i = 1, 2, \dots, n$.

Note 2 In Theorem 1, the maximum distance of variable universe dissection at the k th step is $h_i^k = \alpha_i(x_i^k) h_i$ (h_i is a constant determined by initial input universe). When the initial input universe of fuzzy system is divided equidistance, then $h_i = |\bar{e}_{i(j+1)}^0 - \bar{e}_{ij}^0|$. In fact, $\alpha_i(x_i^k)$ and $\alpha_j(x_j^k)$ can influence approximation performance of the nonlinear hybrid fuzzy system based on variable universe. More details of this will be introduced in the following discussion of approximation performance of fuzzy system on continuous function as well as corresponding sufficient conditions.

The approximation performance of the nonlinear hybrid fuzzy system based on variable universe on the second-order continuously differentiable function is studied based on Theorem 1 in the following text.

Theorem 2 Let $f(x^k)$ represent the nonlinear hybrid fuzzy system (6) based on variable universe, and $g(x)$ be a second-order continuously differentiable function on X^0 , but the analytical expression of $g(x)$ is unknown. Then, for arbitrary $\lambda \in [0, 1]$,

$$\|g - f\|_\infty \leq \frac{1-\lambda}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_\infty h_i^2 + \frac{\lambda}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty h_i^2,$$

where $k = 0, 1, 2, \dots$ and $h_i = \max_{1 \leq j \leq N_i-1} |\bar{e}_{i(j+1)}^0 - \bar{e}_{ij}^0|$.

Proof At the k th step of variable universe implementation, for any $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$, and $y^k \in Y^k$, $\lambda \in [0, 1]$. Substitute into Eq. (6), it is easy to get that:

Now, if we let the center of the output fuzzy sets be as:

$$\begin{aligned} \bar{y}_{i_1 i_2 \dots i_n}^k(x^k) &= \beta(y^k) \bar{y}_{i_1 i_2 \dots i_n}^0 \\ &= g(\alpha_1(x_1^k) \bar{e}_{1i_1}^0, \alpha_2(x_2^k) \bar{e}_{2i_2}^0, \dots, \alpha_n(x_n^k) \bar{e}_{ni_n}^0). \end{aligned}$$

Combine Theorem 1 and inequality (10), for all $\lambda \in [0, 1]$ and $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$,

$$\begin{aligned}
\left| \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(\mathbf{x}^k) x_i^k - g(\mathbf{x}^k) \right| &= \left| \frac{1}{2!} (\mathbf{x}^k - \alpha(\mathbf{x}^k) \mathbf{x}_{i_1 i_2 \dots i_n}^0) \nabla^2 g|_{\mathbf{x}=\eta} (\mathbf{x}^k - \alpha(\mathbf{x}^k) \mathbf{x}_{i_1 i_2 \dots i_n}^0)^T \right| \\
&= \frac{1}{2} \left| \begin{pmatrix} x_1^k - \alpha_1(x_1^k) \bar{e}_{1l_1}^0 \\ x_2^k - \alpha_2(x_2^k) \bar{e}_{2l_2}^0 \\ \vdots \\ x_n^k - \alpha_n(x_n^k) \bar{e}_{nl_n}^0 \end{pmatrix}^T \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2} \Big|_{\mathbf{x}=\eta} & \frac{\partial^2 g}{\partial x_1 \partial x_2} \Big|_{\mathbf{x}=\eta} & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n} \Big|_{\mathbf{x}=\eta} \\ \frac{\partial^2 g}{\partial x_2 \partial x_1} \Big|_{\mathbf{x}=\eta} & \frac{\partial^2 g}{\partial x_2^2} \Big|_{\mathbf{x}=\eta} & \cdots & \frac{\partial^2 g}{\partial x_2 \partial x_n} \Big|_{\mathbf{x}=\eta} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g}{\partial x_n \partial x_1} \Big|_{\mathbf{x}=\eta} & \frac{\partial^2 g}{\partial x_n \partial x_2} \Big|_{\mathbf{x}=\eta} & \cdots & \frac{\partial^2 g}{\partial x_n^2} \Big|_{\mathbf{x}=\eta} \end{pmatrix} \begin{pmatrix} x_1^k - \alpha_1(x_1^k) \bar{e}_{1l_1}^0 \\ x_2^k - \alpha_2(x_2^k) \bar{e}_{2l_2}^0 \\ \vdots \\ x_n^k - \alpha_n(x_n^k) \bar{e}_{nl_n}^0 \end{pmatrix} \right| \\
&= \frac{1}{2} \left| \sum_{j=1}^n \frac{\partial^2 g}{\partial x_1 \partial x_j} \Big|_{\mathbf{x}=\eta} (x_j^k - \alpha_j(x_j^k) \bar{e}_{jl_j}^0) (x_1^k - \alpha_1(x_1^k) \bar{e}_{1l_1}^0) + \sum_{j=1}^n \frac{\partial^2 g}{\partial x_2 \partial x_j} \Big|_{\mathbf{x}=\eta} (x_j^k - \alpha_j(x_j^k) \bar{e}_{jl_j}^0) (x_2^k - \alpha_2(x_2^k) \bar{e}_{2l_2}^0) \right. \\
&\quad \left. + \cdots + \sum_{j=1}^n \frac{\partial^2 g}{\partial x_n \partial x_j} \Big|_{\mathbf{x}=\eta} (x_j^k - \alpha_j(x_j^k) \bar{e}_{jl_j}^0) (x_n^k - \alpha_n(x_n^k) \bar{e}_{nl_n}^0) \right| \\
&= \frac{1}{2} \left| \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j} \Big|_{\mathbf{x}=\eta} (\alpha_i(x_i^k) x_i^0 - \alpha_i(x_i^k) \bar{e}_{il_i}^0) (\alpha_j(x_j^k) x_j^0 - \alpha_j(x_j^k) \bar{e}_{jl_j}^0) \right| \\
&\leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_{\infty} \alpha_i(x_i^k) \alpha_j(x_j^k) h_i^2,
\end{aligned}$$

$$\begin{aligned}
|f(\mathbf{x}^k) - g(\mathbf{x}^k)| &\leq \frac{1-\lambda}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_{\infty} \alpha_i^2(x_i^k) h_i^2 \\
&\quad + \frac{\lambda}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_{\infty} \alpha_i(x_i^k) \alpha_j(x_j^k) h_i^2.
\end{aligned} \tag{10}$$

On the other hand, according to Lemma 4 in Long et al. (2010), we have:

$$\|\bar{y}_{i_1 i_2 \dots i_n}^k(\mathbf{x}^k) - g(\mathbf{x}^k)\|_{\infty} \leq \frac{1}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_{\infty} \alpha_i^2(x_i^k) h_i^2. \tag{11}$$

Since α_i meets $|\alpha_i(x_i^k)| \leq 1$, $|\alpha_j(x_j^k)| \leq 1$. On the basis of the definition of maximum norm, we can obtain immediately that:

$$\begin{aligned}
\|f - g\|_{\infty} &= \sup_{\mathbf{x}^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k} |f(\mathbf{x}^k) - g(\mathbf{x}^k)| \\
&\leq \frac{1-\lambda}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_{\infty} h_i^2 + \frac{\lambda}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_{\infty} h_i^2.
\end{aligned} \tag{12}$$

Note 3 When input universes of the nonlinear hybrid fuzzy system contract continuously as k increases, the interval length $h_i^k = \alpha_i(x_i^k) h_i \rightarrow 0$ ($k \rightarrow \infty$), and the initial subdivision length h_i remains unchanged. This is different from previous increasing approximation accuracy by refining variable universe dissection. In fact, high approximation accuracy of the nonlinear hybrid fuzzy system based on variable universe can be achieved by timely adjustment with contraction expansion factors. This method not only can increase response speed and approximation accuracy of fuzzy system, but also would not change the total number of rules in the system. Therefore, it usually would not cause rules explosion.

4 A sufficient condition for approximation

In fact, Theorem 2 reflects that at the k th step of variable universe implementation, the smaller the approximation accuracy of the nonlinear hybrid fuzzy system on the second-order continuously differentiable function is, the smaller the

$$\begin{aligned}
|f(\mathbf{x}^k) - g(\mathbf{x}^k)| &= \left| \frac{\sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji_j} \left(x_j^k / \alpha_j(x_j^k) \right) \cdot \left[\lambda \left(\sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(\mathbf{x}^k) x_i^k - g(\mathbf{x}^k) \right) + (1-\lambda) \left(\bar{y}_{i_1 i_2 \dots i_n}^k(\mathbf{x}^k) - g(\mathbf{x}^k) \right) \right]}{\sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji_j} \left(x_j^k / \alpha_j(x_j^k) \right)} \right| \\
&\leq \lambda \max_{i_1 i_2 \dots i_n \in I} \left| \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(\mathbf{x}^k) x_i^k - g(\mathbf{x}^k) \right| + (1-\lambda) \max_{i_1 i_2 \dots i_n \in I} |\bar{y}_{i_1 i_2 \dots i_n}^k(\mathbf{x}^k) - g(\mathbf{x}^k)|.
\end{aligned}$$

contraction–expansion factor of its variable universe will be. This finds good accordance with previously stated timely adjustment of variable universe by using contraction expansion factor. As a result, arbitrary approximate accuracy can be achieved in practical application based on appropriate dissection of initial input universe. The smaller the approximation accuracy is, the smaller value the corresponding independent variable x_i^k will be and the closer the input variable x^k will be to the balance point. This is simpler than refining dissection of variable universe.

In addition, the nonlinear hybrid fuzzy system (6) based on variable universe is confirmed theoretically having approximation performance, but sufficient conditions for such approximation remain unknown. This is discussed in the following text.

Theorem 3 Let $g(x)$ be a second-order continuously differentiable function on X^0 , but its analytic expression is unknown. Then for arbitrary $\varepsilon > 0$ and $\lambda \in [0, 1]$, there is a natural number $m_0(\lambda) \in \mathbb{N}$, such that $\|f - g\|_\infty < \varepsilon$ whenever $m \geq m_0(\lambda)$.

Proof For simplicity, each initial input universe is supposed $X_1^0 = X_2^0 = \dots = X_n^0 = [-U_1^0, U_1^0]$, and it is divided equidistance into $2m$ sections. Therefore, the interval length between two adjacent sections is $h_1 = h_2 = \dots = h_n = U_1^0/m$.

Implement variable universe to each $[-U_1^0, U_1^0]$ according to formulas (5). Based on Theorem 2, for arbitrary $\varepsilon > 0$ and $\lambda \in [0, 1]$, if

$$\begin{aligned} \|g - f\|_\infty &\leq \frac{1-\lambda}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_\infty h_i^2 + \frac{\lambda}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty h_i^2 \\ &= \left(\frac{1-\lambda}{8} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_\infty + \frac{\lambda}{2} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty \right) \\ &\quad \times \left(\frac{U_1^0}{m} \right)^2 < \varepsilon. \end{aligned}$$

It is very easy to solve

$$m > U_1^0 \sqrt{\frac{1-\lambda}{8\varepsilon} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_\infty + \frac{\lambda}{2\varepsilon} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty}.$$

At this time, if let

$$m_0(\lambda) = \left\lceil U_1^0 \sqrt{\frac{1-\lambda}{8\varepsilon} \sum_{i=1}^n \left\| \frac{\partial^2 g}{\partial x_i^2} \right\|_\infty + \frac{\lambda}{2\varepsilon} \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial^2 g}{\partial x_i \partial x_j} \right\|_\infty} \right\rceil + 1. \quad (13)$$

Of course, $m_0(\lambda) \in \mathbb{N}$. Then, for arbitrary $\varepsilon > 0$ and $\lambda \in [0, 1]$, we have $\|f - g\|_\infty < \varepsilon$ when $m \geq m_0(\lambda)$.

Note 4 Theorem 3 shows how to select m_0 to design a hybrid fuzzy system with preset approximation accuracy, where m_0 is the number of subdivided initial input universe $[0, U_i^0]$. Generally speaking, it select minimum m_0 for known approximation accuracy ε . In other word, the smaller the m_0 is, the simpler the system structure will be. Therefore, Theorem 3 demonstrates that m_0 satisfying (13) is the sufficient condition for approximation of the nonlinear hybrid fuzzy system based on variable universe. On the other hand, approximation accuracy can be expanded or contracted through timely adjustment with contraction expansion factors according to Eq. (6).

5 Simulation analysis

First, for a given precision $\varepsilon = 0.2$, we may select a few adjusting parameters $\lambda = 0, 1$ and $1/3$. the number of subdivision m_0 can be calculated from Theorem 3, so the input–output expressions of three nonlinear hybrid fuzzy systems based on variable universe are set up.

For simplicity, we suppose $\sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji} (x_j^k / \alpha_j(x_j^k)) = 1$ under a group of known contraction expansion factors $\alpha_i(\cdot)$ and the antecedent fuzzy sets $\{\tilde{A}_{ji}\}$. It is easy to be satisfied, such as common two-phase triangular wave or two-phase trapezoidal wave (see Fig. 1). Then, Eq. (6) can be rewritten as follows:

$$f(x^k) = \begin{cases} \sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji} \left(\frac{x_j^k}{\alpha_j(x_j^k)} \right) \cdot y_{i_1 i_2 \dots i_n}^k(x^k), & \lambda = 0 \\ \sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji} \left(\frac{x_j^k}{\alpha_j(x_j^k)} \right) \cdot \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(x^k) x_i^k, & \lambda = 1 \end{cases}, \quad (14)$$

It indicates that when $\lambda = 0$, Eq. (6) degrades into Mamdani fuzzy system; when $\lambda = 1$, Eq. (6) degrades into T–S fuzzy system.

When $\lambda = 1/3$, for arbitrary $x^k = (x_1^k, x_2^k, \dots, x_n^k) \in X^k$ and at the k th step of variable universe implementation, the nonlinear hybrid fuzzy system (6) turns into

$$f(x^k) = \sum_{i_1 i_2 \dots i_n \in I} \prod_{j=1}^n \tilde{A}_{ji} \left(\frac{x_j^k}{\alpha_j(x_j^k)} \right) \cdot \left(\frac{1}{3} \sum_{i=0}^n C_{i_1 i_2 \dots i_n}^i(x^k) x_i^k + \frac{2}{3} y_{i_1 i_2 \dots i_n}^k(x^k) \right), \quad (15)$$

A simulation example Suppose a given second-order continuous differentiable function g is defined as $g(x, y) = x^2 y + \cos^2 y$, for any $(x, y) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] = X^0$.

The approximation accuracy is determined $\varepsilon = 0.2$, and $n = 2$. The approximation process of the nonlinear

hybrid fuzzy system (6) on g is analyzed in the following text.

First, the number of subdivision m_0 is calculated from Theorem 3, so three nonlinear hybrid fuzzy systems based on variable universe satisfying the given approximation accuracy is established:

$$m_0(1/3) = \left\lceil \frac{\pi}{2} \sqrt{\frac{1}{2.4} \left(\left\| \frac{\partial^2 g}{\partial x^2} \right\|_{\infty} + \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{\infty} \right) + \frac{1}{1.2} \left(\left\| \frac{\partial^2 g}{\partial x^2} \right\|_{\infty} + 2 \left\| \frac{\partial^2 g}{\partial x \partial y} \right\|_{\infty} + \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{\infty} \right)} \right\rceil + 1 = 6.$$

$$\begin{aligned} \left\| \frac{\partial^2 g}{\partial x^2} \right\|_{\infty} &= \|2y\|_{\infty} = \pi, \quad \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{\infty} = \|-2 \cos 2y\|_{\infty} = 2, \\ \left\| \frac{\partial^2 g}{\partial x \partial y} \right\|_{\infty} &= \|2x\|_{\infty} = \pi. \end{aligned}$$

According to formula (14), let $\lambda = 0$ and substitute $\varepsilon = 0.2$ into Theorem 3 to get $m_0(0)$ of Mamdani fuzzy system:

$$m_0(0) = \left\lceil \frac{\pi}{2} \sqrt{\frac{1}{1.6} \left(\left\| \frac{\partial^2 g}{\partial x^2} \right\|_{\infty} + \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{\infty} \right)} \right\rceil + 1 = 4.$$

Let $\lambda = 1$ and get $m_0(1)$ of T-S fuzzy system:

$$m_0(1) = \left\lceil \frac{\pi}{2} \sqrt{\frac{1}{0.4} \left(\left\| \frac{\partial^2 g}{\partial x^2} \right\|_{\infty} + 2 \left\| \frac{\partial^2 g}{\partial x \partial y} \right\|_{\infty} + \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{\infty} \right)} \right\rceil + 1 = 9.$$

Let $\lambda = 1/3$ and get $m_0(1/3)$ of the hybrid fuzzy system (6):

Without loss of generality, a nonlinear hybrid fuzzy system based on two-dimensional variable universe when $\lambda = 1/3$ and $m_0(1/3) = 6$ is established.

In fact, 11 equidistant subdivision points can be inserted into $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of x -axis when $m_0(1/3) = 6$, where $2m_0(1/3) - 1 = 11$ and all subdivision points are expressed by $\frac{\pi j}{12}$ ($j = 0, \pm 1, \pm 2, \dots, \pm 6$).

An equidistance fuzzy subdivision $\{\tilde{A}_{1,j}\}$ can be constructed by taking each subdivision point as a peak value, that is, $\tilde{A}_{1,j}(\frac{\pi j}{12}) = 1$, $j = 0, \pm 1, \pm 2, \dots, \pm 6$. Therefore, a triangular fuzzy set can be defined by the origin point of coordinates $(0, 0)$ as a peak value:

$$\tilde{A}(x) = \begin{cases} \frac{12}{\pi}x + 1, & -\frac{\pi}{12} \leq x \leq 0 \\ -\frac{12}{\pi}x + 1 & 0 < x \leq \frac{\pi}{12}. \end{cases} \quad (16)$$

Moving \tilde{A} horizontally for $\frac{\pi}{12}$ unit on $[-\frac{\pi}{2}, \frac{\pi}{12}]$ to left first and then to right, i.e., let

$$\tilde{A}_{1,j}(x) = \tilde{A}\left(x - \frac{\pi j}{12}\right), \quad j = 0, \pm 1, \pm 2, \dots, \pm 5. \quad (17)$$

The fuzzy sets at the left and right end points are separately defined as follows:

$$\begin{aligned} \tilde{A}_{1,6}(x) &= \begin{cases} \frac{12}{\pi}x - 5, & \frac{5}{12}\pi \leq x \leq \frac{\pi}{2}, \\ 0, & \text{Otherwise} \end{cases} \\ \tilde{A}_{1,-6}(x) &= \begin{cases} -\frac{12}{\pi}x - 5, & -\frac{\pi}{2} \leq x \leq -\frac{5\pi}{12} \\ 0, & \text{Otherwise.} \end{cases} \end{aligned}$$

Then, an equidistance fuzzy subdivision $\{\tilde{A}_{1,j}\}$ can be obtained.

Similarly, if let $\tilde{A}_{2,j}(y) = \tilde{A}(y - \frac{\pi j}{12})$ ($j = 0, \pm 1, \pm 2, \dots, \pm 5$) on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of y -axis. Another equidistance fuzzy subdivision $\{\tilde{A}_{2,j}\}$ can also be gained, and the interval length of each component of the coordinates (x, y) within the universes is $h = h_i = \frac{\pi}{12}$.

An equidistance fuzzy subdivision in the first quadrant $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ is shown in Fig. 2.

In fact, an equidistance fuzzy subdivision of the initial universe $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ is implemented according to Fig. 2. It is easy to know that the coordinates of central point of antecedent fuzzy sets is:

$$\begin{aligned} x_{i_1 i_2}^0 &= (\bar{e}_{1 i_1}^0, \bar{e}_{2 i_2}^0) = \left(\frac{\pi}{12} i_1, \frac{\pi}{12} i_2 \right), \\ i_1, i_2 &= 0, \pm 1, \pm 2, \dots, \pm 6. \end{aligned}$$

According to formula (5), we might as well choose $m = 10$ and f . At the k th step of variable universe, for any $(x_1^k, x_2^k) \in X^0$, the corresponding contraction-expansion

$$\text{factors are: } \alpha_1(x_1^k) = \sqrt{\frac{2|x_1^k|}{\pi}}, \quad \alpha_2(x_2^k) = \sqrt{\frac{2|x_2^k|}{\pi}}, \quad \alpha_j(x_j^k) = \sqrt{\frac{2|x_j^k|}{\pi}}, \quad j = 1, 2, \quad k = 0, 1, 2, \dots$$

According to the nonlinear hybrid fuzzy system (15), we will determine the coefficients of variable universe $C_{i_1 i_2}^k(x_1^k, x_2^k)$ and $\bar{y}_{i_1 i_2}^k(x_1^k, x_2^k)$. At the k th step of variable universe, the coordinate of central point of input fuzzy sets is $x_{i_1 i_2}^k = (\bar{e}_{1 i_1}^k(x_1^k), \bar{e}_{2 i_2}^k(x_2^k))$, with each component satisfied:

$$\begin{aligned}
\bar{e}_{1i_1}^k(x_1^k) &= \alpha_1(x_1^k)\bar{e}_{1i_1}^0 = \sqrt{\frac{2}{\pi}}|x_1^k|\frac{\pi i_1}{12} \\
&= \frac{i_1}{6}\sqrt{\frac{\pi}{2}}|x_1^k|, \quad i_1 = 0, \pm 1, \pm 2, \dots, \pm 6, \\
\bar{e}_{2i_2}^k(x_2^k) &= \alpha_2(x_2^k)\bar{e}_{2i_2}^0 = \sqrt{\frac{2}{\pi}}|x_2^k|\frac{\pi i_2}{12} \\
&= \frac{i_2}{6}\sqrt{\frac{\pi}{2}}|x_2^k|, \quad i_2 = 0, \pm 1, \pm 2, \dots, \pm 6.
\end{aligned}$$

Adjustment parameters at the k th step of variable universe can be calculated from (9) in Theorem 1:

$$\begin{aligned}
C_{i_1 i_2}^1(x_1^k, x_2^k) &= \frac{\partial g}{\partial x} \Big|_{x=x_{i_1 i_2}^k} = \frac{\pi i_1 i_2}{36} \sqrt{|x_1^k x_2^k|}, \\
C_{i_1 i_2}^2(x_1^k, x_2^k) &= \frac{\partial g}{\partial y} \Big|_{x=x_{i_1 i_2}^k} = \frac{\pi i_1^2}{72} |x_1^k| - \sin\left(\frac{i_2}{3} \sqrt{\frac{\pi}{2}} |x_2^k|\right), \\
C_{i_1 i_2}^0(x_1^k, x_2^k) &= g(\bar{e}_{1i_1}^k(x_1^k), \bar{e}_{2i_2}^k(x_2^k)) - C_{i_1 i_2}^1(x_1^k, x_2^k)\bar{e}_{1i_1}^k(x_1^k) \\
&\quad - C_{i_1 i_2}^2(x_1^k, x_2^k)\bar{e}_{2i_2}^k(x_2^k).
\end{aligned}$$

In addition, the central point of output fuzzy sets is:

$$\begin{aligned}
\bar{y}_{i_1 i_2}^k(x_1^k, x_2^k) &= g\left(\frac{i_1}{6}\sqrt{\frac{\pi}{2}}|x_1^k|, \frac{i_2}{6}\sqrt{\frac{\pi}{2}}|x_2^k|\right) \\
&= \frac{\pi i_1^2 i_2}{432} |x_1^k| \sqrt{\frac{\pi}{2}} |x_2^k| + \cos^2\left(\frac{i_2}{6}\sqrt{\frac{\pi}{2}} |x_2^k|\right),
\end{aligned}$$

where $i_1, i_2 = 0, \pm 1, \pm 2, \dots, \pm 6$. Specially, if let $x_1^k = x$, $x_2^k = y$, the nonlinear hybrid fuzzy system based on two-dimensional variable universe when $\lambda = 1/3$ can be represented as:

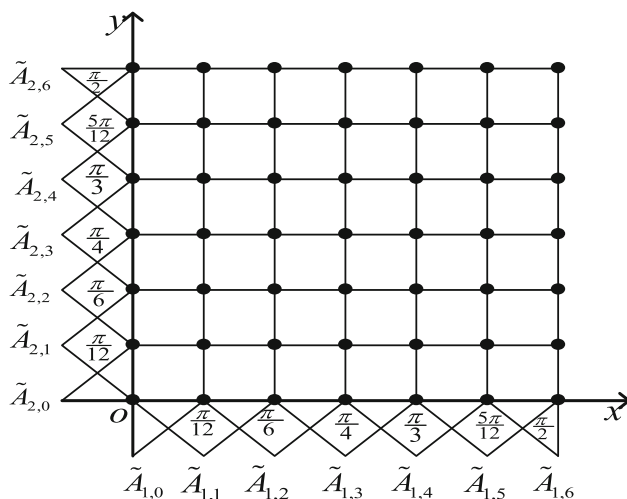


Fig. 2 Figure of an equidistance fuzzy subdivision on $[0, \pi/2] \times [0, \pi/2]$

$$\begin{aligned}
f_1(x, y) &= \sum_{i_1=-6}^6 \sum_{i_2=-6}^6 \tilde{A}_{1, i_1} \left(\sqrt{\frac{\pi}{2}} |x| \right) \tilde{A}_{2, i_2} \left(\sqrt{\frac{\pi}{2}} |y| \right) \\
&\quad \times \left(\frac{1}{3} (C_{i_1 i_2}^0(x, y) + C_{i_1 i_2}^1(x, y)x + C_{i_1 i_2}^2(x, y)y) + \frac{2}{3} \bar{y}_{i_1 i_2}^k(x, y) \right).
\end{aligned} \quad (18)$$

Similarly, when $\lambda = 0$ and $m_0 = 4$, the Mamdani fuzzy system based on two-dimensional variable universe is gained according to Eq. (14):

$$f_2(x, y) = \sum_{i_1=-4}^4 \sum_{i_2=-4}^4 \tilde{A}_{1, i_1} \left(\sqrt{\frac{\pi}{2}} |x| \right) \tilde{A}_{2, i_2} \left(\sqrt{\frac{\pi}{2}} |y| \right) \bar{y}_{i_1 i_2}^k(x, y),$$

where the central point

$$\bar{y}_{i_1 i_2}^k(x, y) = \frac{\pi i_1^2 i_2}{432} |x| \sqrt{\frac{\pi}{2}} |y| + \cos^2\left(\frac{i_2}{6}\sqrt{\frac{\pi}{2}} |y|\right).$$

When $\lambda = 1$ and $m_1 = 9$, the T-S fuzzy system based on two-dimensional variable universe can be gained according to Eq. (14):

$$\begin{aligned}
f_3(x, y) &= \sum_{i_1=-9}^9 \sum_{i_2=-9}^9 \tilde{A}_{1, i_1} \left(\sqrt{\frac{\pi}{2}} |x| \right) \\
&\quad \tilde{A}_{2, i_2} \left(\sqrt{\frac{\pi}{2}} |y| \right) (C_{i_1 i_2}^0(x, y) + C_{i_1 i_2}^1(x, y)x + C_{i_1 i_2}^2(x, y)y).
\end{aligned}$$

Apparently, these three nonlinear fuzzy systems based on variable universe can be viewed as bivariate continuous function about x and y . With MATLAB software programming, space curved faces of $f_1(x, y)$, $f_2(x, y)$ and $f_3(x, y)$ in the plane closed region $[-\pi/2, \pi/2] \times [-\pi/2, \pi/2]$ are drawn.

Figures 3, 4, 5 and 6 confirm the approximation performance of the nonlinear hybrid fuzzy system based on variable universe on given the second-order continuously differentiable function under different λ values. Actually, approximation accuracy of this nonlinear hybrid system can be achieved not only through the k th step of variable universe implementation, but also by adjusting the parameters λ and selecting appropriate number of subdivision m_0 , of course, the method of variable universe is one of the most important.

In addition, aiming at the given differentiable function $g(x, y) = x^2 y + \cos^2 y$, the nonlinear hybrid fuzzy system, Mamdani fuzzy system and T-S fuzzy system, we may randomly select some sample points to obtain their corresponding values and the error values by Fig. 2. Unfortunately, except $g(x, y)$, calculating the values of the nonlinear fuzzy system $f_1(x, y)$, $f_2(x, y)$, and $f_3(x, y)$ in the sample points is a more complex. This is because we must first find out the corresponding a few fuzzy sets $\tilde{A}_{1, j}$ and $\tilde{A}_{2, j}$, so that the variable universe coefficients

$C_{i_1 i_2}^0(x, y)$, $C_{i_1 i_2}^1(x, y)$, $C_{i_1 i_2}^2(x, y)$ and $\bar{y}_{i_1 i_2}^k(x, y)$ can be calculated.

Without loss of generality, we only compute the value $f_1(\frac{\pi}{4}, \frac{\pi}{3})$ of the vertex $(\frac{\pi}{4}, \frac{\pi}{3})$ as follows. In accordance with the Fig. 2, the corresponding fuzzy sets of the vertex $(\frac{\pi}{4}, \frac{\pi}{3})$ may be obtained by Eqs. (16) and (17), for example,

$$\tilde{A}_{1,3}(x) = \begin{cases} \frac{12}{\pi}x - 2, & \frac{\pi}{6} \leq x \leq \frac{\pi}{4} \\ -\frac{12}{\pi}x + 4, & \frac{\pi}{4} < x \leq \frac{\pi}{3} \end{cases} \dots$$

$$\tilde{A}_{2,4}(y) = \begin{cases} \frac{12}{\pi}y - 3, & \frac{\pi}{4} \leq y \leq \frac{\pi}{3} \\ \frac{12}{\pi}y + 5, & -\frac{\pi}{3} < y \leq \frac{5\pi}{12} \end{cases}$$

At this time, the hybrid fuzzy system based on variable universe contains at most four rules ($i_1 = 3, 4; i_2 = 4, 5$). According to Eq. (18) and MATLAB software programming, we may calculate that $f_1(\frac{\pi}{4}, \frac{\pi}{3}) \approx 1.395959402$.

Next, we randomly select a few sample points in Fig. 2, the corresponding values and errors of the three systems can be separately calculated, and the details please see following Table 1:

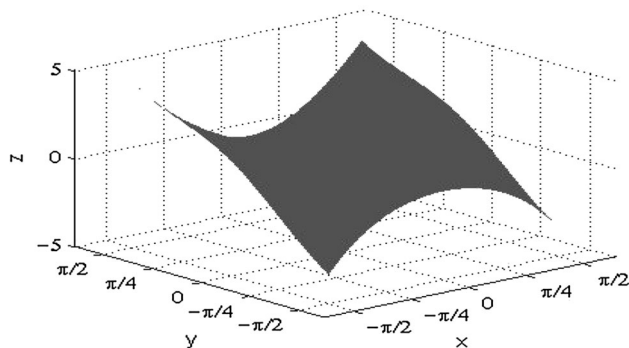


Fig. 3 Surface figure of the second-order continuous differentiable function g

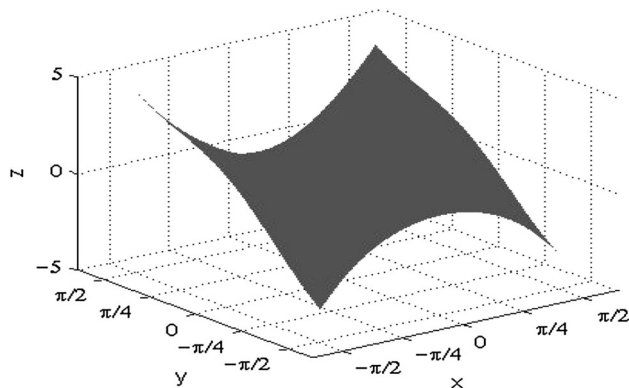


Fig. 4 Surface figure of nonlinear hybrid fuzzy system f_1 based on variable universe when $\lambda = 1/3$

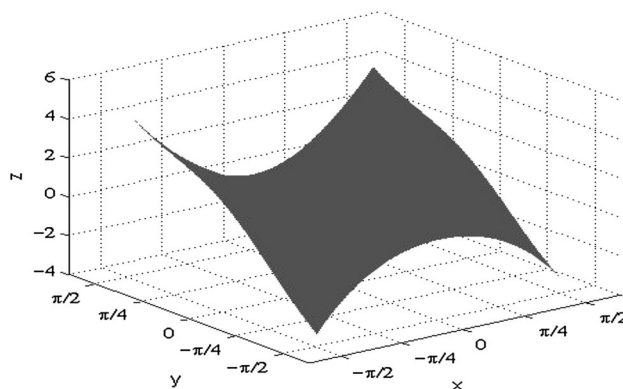


Fig. 5 Surface figure of Mamdani fuzzy system f_2 based on variable universe when $\lambda = 0$

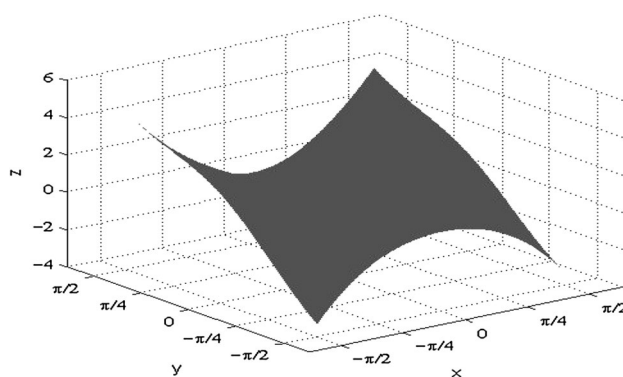


Fig. 6 Surface figure of T-S fuzzy system f_3 based on variable universe when $\lambda = 1$

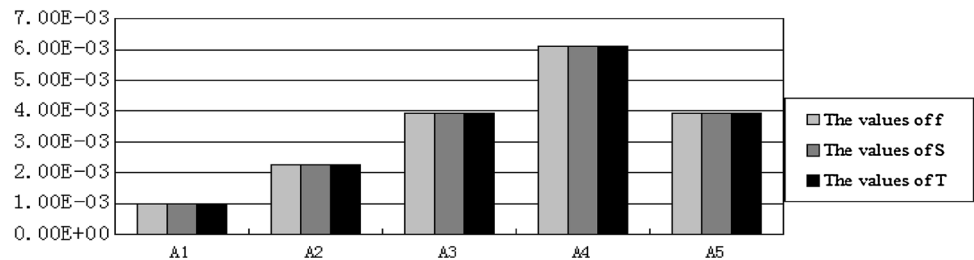
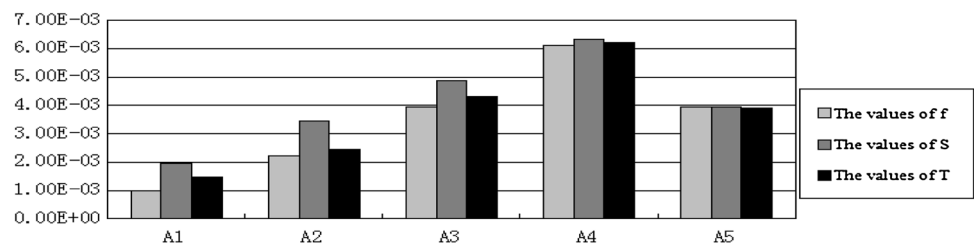
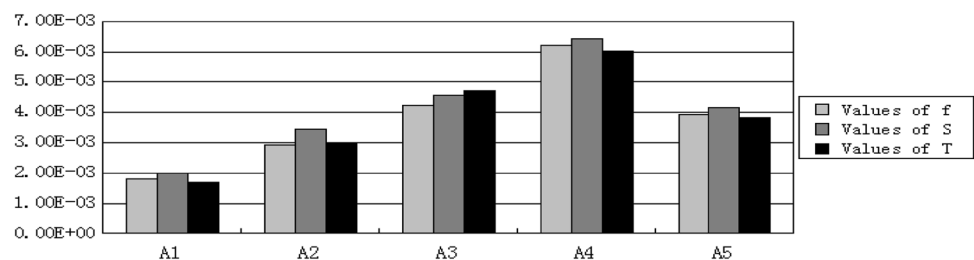
It is not hard to see from Table 1 that the approximation accuracy of the nonlinear hybrid fuzzy system f_1 to a given second-order continuous differentiable function g is relatively high. Usually, we do not need to know the specific analytical expression of g , but just need to obtain the values of g in some special points.

In addition, for describing the error values of the five points intuitively, if we write the sample points as $A1 = (\frac{\pi}{8}, \frac{2\pi}{9})$, $A2 = (\frac{\pi}{6}, \frac{\pi}{4})$, $A3 = (\frac{\pi}{4}, \frac{\pi}{3})$, $A4 = (\frac{\pi}{5}, \frac{\pi}{3})$, $A5 = (\frac{\pi}{9}, \frac{\pi}{3})$. The histograms of the error values can be expressed as follows:

Through the analysis of the above Table 1 and Figs. 7, 8, and 9, it is easy to get that the approximation performance of the nonlinear hybrid fuzzy system f_1 based on variable universe is the most effective, especially in the boundary points and vertices the nonlinear hybrid fuzzy system f_1 has better approximation. In fact, only from the five random sample points $A1, A2, A3, A4, A5$, the approximation performance of f_1 is better than f_3 , and f_3 is better than f_2 . Hence, the approximation performance of the nonlinear hybrid fuzzy system f_1 based on variable universe is the best.

Table 1 The corresponding values and error values of some random sample points

Sample Points X_i	$(\frac{2}{3}, \frac{2\pi}{3})$	$(\frac{2}{3}, \frac{\pi}{3})$	$(\frac{4}{3}, \frac{\pi}{3})$	$(\frac{4}{3}, \frac{2\pi}{3})$	$(\frac{5}{3}, \frac{\pi}{3})$
$g(X_i)$	0.767708076	0.840321366	1.395964098	0.836162774	0.436086736
$f_1(X_i)$	0.767707125	0.840325356	1.395959402	0.836157045	0.436079521
$f_2(X_i)$	0.767943521	0.840579665	1.396048678	0.836455462	0.436357231
$f_3(X_i)$	0.767845263	0.840531758	1.395874651	0.836275433	0.436192675
$ f_1(\cdot) - g(\cdot) $	9.51×10^{-7}	3.99×10^{-6}	4.70×10^{-6}	5.73×10^{-6}	7.22×10^{-6}
$ f_2(\cdot) - g(\cdot) $	2.35×10^{-4}	2.58×10^{-4}	8.46×10^{-5}	2.93×10^{-4}	2.70×10^{-4}
$ f_3(\cdot) - g(\cdot) $	1.37×10^{-4}	2.10×10^{-4}	8.94×10^{-5}	1.13×10^{-4}	1.06×10^{-4}

Fig. 7 The histograms of the hybrid fuzzy system f_1 when $\lambda = 1/3$ **Fig. 8** The histograms of Mamdani fuzzy system f_2 when $\lambda = 0$ **Fig. 9** The histograms of T–S fuzzy system f_3 when $\lambda = 1$ 

According to the simulation example, we can obviously see that the greater the adjusting parameters λ value, the smaller the subdivided number m_λ^0 value. Thus, it can lead to reduce the number of fuzzy rules and antecedent fuzzy sets; however, the approximation performance of the hybrid fuzzy system is poor. On the contrary, if the smaller the adjusting parameters λ value, the greater the subdivided number m_λ^0 value, then the number of fuzzy rules and antecedent fuzzy sets also will increase, of course, the approximation effect is better, but the corresponding fuzzy system is more complex. In particular, when $\lambda \rightarrow 0^+$ or $\lambda \rightarrow 1^-$, the nonlinear hybrid fuzzy system based on

variable universe will tend to be more independent Mamdani fuzzy system or T–S fuzzy system, respectively.

6 Conclusions

First of all, the hybrid inference rules of a general nonlinear hybrid fuzzy system are acquired by introducing an adjusting parameter Lambda in this paper. Second, Mamdani and T–S fuzzy systems are combined to establish the nonlinear hybrid fuzzy system based on hybrid inference rules and variable universe. Therefore,

the nonlinear hybrid fuzzy system has better advantages of both Mamdani and T–S fuzzy systems, thus having higher promotion value. In fact, implementation of variable universe to input variable can keep same total inference rules of the fuzzy system. This can make the new universe contract as error decreases (or expand as error increases). This indicates that the new universe contraction is equal to rules growth, thus increasing the approximation accuracy. Furthermore, what is the internal relationship between Lambda and approximation performance of fuzzy system under certain conditions of initial universe subdivision, and how to select the optimal contraction expansion factors for best approximation effect deserve key attentions of future researches.

Acknowledgments This work has been supported by National Natural Science Foundation China (Grant No. 61374009).

References

- Buckley JJ (1993) Sugeno type controllers are universal controllers. *Fuzzy Sets Syst* 53(3):299–303
- Castro JL (1995) Fuzzy logic controllers are universal approximators. *IEEE Trans Syst Man Cybern* 25(4):629–635
- Hassine R, Karray F, Alimi AM, Selmi M (2003) Approximation properties of fuzzy systems for smooth functions and their first-order derivative. *IEEE Trans Syst Man Cybern Part A* 33(2):160–168
- Kosko B (1994) Fuzzy systems as universal approximators. *IEEE Trans Comput* 43(11):1329–1333
- Li HX (1997) The essence of fuzzy control and a kind of fine fuzzy controller. *Control Theory and Application* 14(6):868–872
- Li HX (1999) Adaptive fuzzy control based on variable universe. *Sci China (Ser E)* 29(1):32–42
- Li HX, Miao ZH, Wang JY (2002a) Variable universe stable adaptive fuzzy control of nonlinear system. *Sci China (Ser E)* 45(3):211–223
- Li HX, Miao ZH, Wang JY (2002b) Variable universe adaptive fuzzy control on the quadruple inverted pendulum. *Sci China (Ser E)* 32(1):65–75
- Liu PY, Li HX (2000) Approximation of generalized fuzzy system to integrable function. *Sci China (Ser E)* 43(6):618–628
- Liu PY, Li HX (2001) Analyses for L_p (μ)-norm approximation capability of the generalized Mamdani fuzzy systems. *Inf Sci* 138(2):195–210
- Long ZQ, Liang XM, You KM (2008) Double-input and single-output fuzzy control algorithm with potentially-inherited variable universe and its convergence. *Control Theory Appl* 25(4):905–912
- Long ZQ, Liang XM, Yang LR (2010) Some approximation properties of adaptive fuzzy systems with variable universe of discourse. *Inf Sci* 180:2991–3005
- Long ZQ, Liang XM, Yan G (2012) Universal approximation properties of fuzzy controllers with variable universe of discourse and their approximation conditions. *J Central South Univ (Sci Technol)* 43(8):543–550
- Mendel JM (2016) A comparison of three approaches for estimating (synthesizing) an interval type-2 fuzzy set model of a linguistic term for computing with words. *Granul Comput* 1(1):59–69
- Pedrycz W (1989) *Fuzzy Control and Fuzzy Systems*. Research Studies Press/Wiley, Taunton/New York
- Pedrycz W, Gudwin RR, Gomide FAC (1997) Nonlinear context adaptation in the calibration of fuzzy sets. *Fuzzy Sets Syst* 88:91–97
- Wang GJ, Duan CX (2012) Generalized hierarchical hybrid fuzzy system and its universal approximation. *Control Theory and Application* 29(5):673–680
- Wang LX, Mendel J (1992) Fuzzy basis functions, universal approximation, and orthogonal least-squares learning. *IEEE Trans Neural Netw* 3(5):807–814
- Wang GJ, Li XP, Sui XL (2014) Universal approximation and its realization of generalized Mamdani fuzzy system based on K-integral norms. *Acta Autom Sinica* 40(1):143–148
- Wang DG, Song WY, Li HX (2015) Approximation properties of ELM-fuzzy systems for smooth functions and their derivatives. *Neurocomputing* 149:265–274
- Yang Y, Wang GJ, Yang YQ (2013) Reduction of inference rules for generalized hybrid fuzzy systems based on binary tree-type hierarchy. *Control Theory Appl* 30(6):765–772
- Zeng XJ, Singh MG (1995) Approximation theory of fuzzy systems-MIMO case. *IEEE Trans Fuzzy Syst* 3(2):219–235
- Zeng XJ, Goulermas JY, Liatsis P et al (2008) Hierarchical fuzzy systems for function approximation on discrete input spaces with application. *IEEE Trans Fuzzy Syst* 16(5):1197–1215