The Construction of Fraction Gamma Ring Using Noncommutative Gamma Ring

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Abstract

One of the first constructions of algebra is the quotient field of a commutative integral domain constructed as a set of fractions. The noncommutative case is restrictive. In this article, the researchers constructed a fraction gamma ring of noncommutative gamma ring. There should be an appropriate set X of elements in a gamma ring R to be used as denominators.

Keywords Γ -ring · Fraction Γ -ring · Noncommutative Γ -ring

1 Introduction

The notation of gamma ring was first introduced by Nobusawa (1964) as a generalization of a classical ring and afterward Barnes (1966) improved the concepts of Nobusawa's Γ -ring. The researchers have constructed fraction gamma ring of commutative integral domain, but this is very complicated in noncommutative gamma ring. Two Γ -ring of fractions are hypothesized, one with right-hand denominators and one with left-hand denominators (Goodearl and Warfield 2004).

Let R and Γ be two additive Abelian groups and there exists a mapping $(x, \gamma, y) \mapsto x\gamma y$ of $R \times \Gamma \times R \rightarrow R$, which satisfies the conditions:

- (i) $(x + y)\gamma z = x\gamma z + y\gamma z$,
 $x(\gamma_1 + \gamma_2) = x\gamma_1 y + x\gamma_2 y$,
 $x\gamma(y + z) = x\gamma y + x\gamma z$,
- (ii) $(x\gamma_1 y)\gamma_2 z = x\gamma_1(y\gamma_2 z)$,

for all $x, y, z \in R$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Then, R is called a gamma ring.

If there exists $1_R \in R$ and $\gamma_0 \in \Gamma$ such that for all $r \in R$, $1_R \gamma_0 r = r \gamma_0 1_R = r$, then $1 = 1_R$ is called identity element (Paul and Uddin 2010).

Let R be a Γ -ring with 1. An element $a \in R$ is called invertible if there exists $b \in R$ such that $a\gamma_0 b = b\gamma_0 a = 1$, also b is unique and called the multiplicative inverse of a and is denoted by a^{-1} .

An element $a \in R$ is said to be zero-divisor if there exists $b \neq 0$ such that $a\gamma_0 b = b\gamma_0 a = 0$.

For non-zero element x in R , the right annihilator of x and the left annihilator of x are defined to be

$$\text{ann}_r(x) = \{a \in R \mid x\gamma_0 a = 0\},$$

$$\text{ann}_l(x) = \{a \in R \mid a\gamma_0 x = 0\}.$$

A regular element in a Γ -ring R is any nonzero-divisor, i.e., any element $x \in R$ such that $\text{ann}_r(x) = \{0\}$ and $\text{ann}_l(x) = \{0\}$.

A multiplicatively closed subset of Γ -ring R is a subset X of R such that $1 \in X$ and $x_1 \Gamma x_2 \subseteq X$, for all $x_1, x_2 \in X$.

A Γ -ring homomorphism (Ullah and Chaudhry 2012) is a mapping f of Γ -ring R to Γ -ring R' such that

- (i) $f(x + y) = f(x) + f(y)$,
- (ii) $f(x\gamma y) = f(x)\gamma f(y)$,

for all $x, y \in R$ and $\gamma \in \Gamma$.

We consider the following condition:

$$(*) \quad x\alpha y\beta z = x\beta y\alpha z,$$

for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ (Dey and Paul 2014).

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Right P -condition: Let X be a multiplicatively closed set in Γ -ring R . Then, X satisfies the right P -condition, if for $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$, such that $r\gamma_0 y = x\gamma_0 s$, then X is called the right P -set. The left P -condition and the left P -set are defined in the same way.

Let R be a Γ -ring and $X \subseteq R$ be a multiplicative closed set of regular elements in R . A right Γ -ring of fractions for R with respect to X is any Γ -ring S ($R \subseteq S$) such that

- (a) Every element of X is invertible in S .
- (b) Every element of S can be expressed in the form $a\gamma_0 x^{-1}$, for some $a \in R$ and $x \in X$.

The left Γ -rings of fractions are defined similarly.

2 Fraction of Noncommutative Gamma Ring

Throughout this section, the word gamma ring means noncommutative gamma ring with 1.

Lemma 1 *Let X be a right P -set in a Γ -ring R . Then, for all elements $x_1, \dots, x_n \in X$, there exist $s_1, \dots, s_n \in R$ such that $x_1\gamma_0 s_1 = \dots = x_n\gamma_0 s_n$ and $x_1\gamma_0 s_1 \in X$, that is, $x_1\gamma_0 R \cap \dots \cap x_n\gamma_0 R \cap X \neq \emptyset$.*

Proof By induction. We prove for the cases $n = 2$ and $n = 3$. Since X is a right P -set, thus for every $x_1 \in X$ and $x_2 \in R$, there exist $y \in X$ and $s \in R$ such that $x_1\gamma_0 y = x_2\gamma_0 s$, and $x_1\gamma_0 y \in X$, because X is a multiplicatively closed set.

But we have $x_2\gamma_0 s \in X$ and $x_3 \in R$, thus there exist some $y' \in X$ and $s' \in R$ such that $x_2\gamma_0 s\gamma_0 y' = x_3\gamma_0 s'$ or $x_1\gamma_0 y\gamma_0 y'$, if we consider $y'' = y\gamma_0 y'$, then $x_1\gamma_0 y'' = x_3\gamma_0 s'$ or $x_1\gamma_0 y'' = x_2\gamma_0 y' = x_3\gamma_0 s'$. $\square$

Lemma 2 *Let R be a Γ -ring that satisfies the condition (*) and X be a multiplicatively closed set of regular elements in R , and assume that there exists a right Γ -ring of fractions, say S , for R with respect to X . Then*

- (i) X is a right P -set in R .
- (ii) For every $s_1, \dots, s_n \in S$, there exist $a_1, \dots, a_n \in R$ and $x \in X$ such that $s_i = a_i\gamma_0 x^{-1}$, for every $1 \leq i \leq n$.
- (iii) Let $a, b \in R$ and $x, y \in X$. Then, $a\gamma_0 x^{-1} = b\gamma_0 y^{-1}$ in S if and only if there exist $c, d \in R$ such that $a\gamma_0 c = b\gamma_0 d$ and $x\gamma_0 c = y\gamma_0 d \in X$.

Proof

- (i) Suppose S be a right Γ -ring of fractions for R with respect to X . For $x, y \in X$ and $a, b \in R$, using (*), we consider the product of fractions $a\gamma_0 x^{-1}$ and $b\gamma_0 y^{-1}$ in S

$$(a\gamma_0 x^{-1})\gamma(b\gamma_0 y^{-1}) = a\gamma(x^{-1}\gamma_0 b)\gamma_0 y^{-1}$$

Since every element of S has the form of a fraction with right-hand denominator, the left-hand fraction $x^{-1}\gamma_0 b$ must equal $c\gamma_0 z^{-1}$ for some $c \in R$ and $z \in X$, hence we have

$$\begin{aligned} (a\gamma_0 x^{-1})\gamma(b\gamma_0 y^{-1}) &= a\gamma(x^{-1}\gamma_0 b)\gamma_0 y^{-1} \\ &= a\gamma(c\gamma_0 z^{-1})\gamma_0 y^{-1} \\ &= a\gamma c\gamma_0(z^{-1}\gamma_0 y^{-1}) \\ &= a\gamma c\gamma_0(y\gamma_0 z)^{-1}. \end{aligned}$$

Note that we have a necessary condition: for $b \in R$ and $x \in X$, there must exist $c \in R$ and $z \in X$ such that $x^{-1}\gamma_0 b = c\gamma_0 z^{-1}$, that is, $b\gamma_0 z = x\gamma_0 c$. This is precisely the right P -condition.

- (ii) Since S is a right Γ -ring of fractions, for each $s_i \in S$ ($1 \leq i \leq n$), there exist $b_i \in R$ and $x_i \in X$ such that $s_i = b_i\gamma_0 x_i^{-1}$. Using Lemma 1, there exist $x \in X$ and $c_1, \dots, c_n \in R$ such that $x = x_i\gamma_0 c_i$, for all i . Since x and x_i are both invertible in S , so is c_i and $x^{-1} = c_i^{-1}\gamma_0 x_i^{-1}$. Thus, $x_i^{-1} = c_i\gamma_0 x^{-1}$ and we have $s_i = b_i\gamma_0 x_i^{-1} = b_i\gamma_0 c_i\gamma_0 x^{-1}$. If we consider $a_i = b_i\gamma_0 c_i$, then $s_i = a_i\gamma_0 x^{-1}$.
- (iii) $\Leftarrow$ Suppose that, there exist $c, d \in R$ such that $a\gamma_0 c = b\gamma_0 d$ and $x\gamma_0 c = y\gamma_0 d \in X$. Then

$$\begin{aligned} a\gamma_0 x^{-1} &= a\gamma_0 1\gamma_0 x^{-1} \\ &= a\gamma_0(c\gamma_0 c^{-1})\gamma_0 x^{-1} \\ &= (a\gamma_0 c)\gamma_0(x\gamma_0 c)^{-1} \\ &= (b\gamma_0 d)\gamma_0(y\gamma_0 d)^{-1} \\ &= b\gamma_0(d\gamma_0 d^{-1})\gamma_0 y^{-1} \\ &= b\gamma_0 y^{-1}. \end{aligned}$$

$\Rightarrow$ Suppose that, $a\gamma_0 x^{-1} = b\gamma_0 y^{-1}$. Using Lemma 1, there exist $c, d \in R$ such that $x\gamma_0 c = y\gamma_0 d \in X$. Hence

$$\begin{aligned} (a\gamma_0 c)\gamma_0(x\gamma_0 c)^{-1} &= a\gamma_0(c\gamma_0 c^{-1})\gamma_0 x^{-1} \\ &= a\gamma_0 x^{-1} \\ &= b\gamma_0 y^{-1} \\ &= b\gamma_0(d\gamma_0 d^{-1})\gamma_0 y^{-1} \\ &= (b\gamma_0 d)\gamma_0(y\gamma_0 d)^{-1} \\ &= (b\gamma_0 d)\gamma_0(x\gamma_0 c)^{-1}, \end{aligned}$$

and therefore, $a\gamma_0 c = b\gamma_0 d$. $\square$

Proposition 1 *Let X be a right P -set of regular elements in a Γ -ring R . We define a relation $\sim$ on $R \times X$ as follows:*

$$(a, x) \sim (b, y) \iff \exists c, d \in R, \quad a\gamma_0 c = b\gamma_0 d, \quad x\gamma_0 c = y\gamma_0 d.$$

Then, $\sim$ is an equivalence relation.

Proof We show that the relation $\sim$ is reflexive, symmetric and transitive.

For every $a \in R$ and $x \in X$, let $c = d = 1 \in R$. Then, $a\gamma_0 1 = a\gamma_0 1$ and $x\gamma_0 1 = x\gamma_0 1$, and therefore $(a, x) \sim (a, x)$.

If $(a, x) \sim (b, y)$, then there exist $c, d \in R$ such that $a\gamma_0 c = b\gamma_0 d$ and $x\gamma_0 c = y\gamma_0 d \in X$ and so $b\gamma_0 d = a\gamma_0 c$ and $y\gamma_0 d = x\gamma_0 c$. Thus, $(b, y) \sim (a, x)$.

If $(a, x) \sim (b, y)$ and $(b, y) \sim (c, z)$, then we have

$$(a, x) \sim (b, y) \Rightarrow \exists d, e \in R \quad \text{s.t.} \begin{cases} a\gamma_0 d = b\gamma_0 e \\ x\gamma_0 d = y\gamma_0 e \in X \end{cases} \quad (1)$$

$$(b, y) \sim (c, z) \Rightarrow \exists f, g \in R \quad \text{s.t.} \begin{cases} b\gamma_0 f = c\gamma_0 g \\ y\gamma_0 f = z\gamma_0 g \in X \end{cases} \quad (2)$$

Since y and $z\gamma_0 g$ are in X and all elements in X are invertible in S , thus f is invertible and so $b = c\gamma_0 g\gamma_0 f^{-1}$ and $y = z\gamma_0 g\gamma_0 f^{-1}$.

Using (1), we obtain

$$\begin{cases} a\gamma_0 d = b\gamma_0 e = c\gamma_0 g\gamma_0 f^{-1}\gamma_0 e \\ x\gamma_0 d = y\gamma_0 e = z\gamma_0 g\gamma_0 f^{-1}\gamma_0 e \end{cases}$$

We consider $h = g\gamma_0 f^{-1}\gamma_0 e \in R$, and so there exist $d, h \in R$ such that $a\gamma_0 d = c\gamma_0 h$ and $x\gamma_0 d = z\gamma_0 h$, that is $(a, x) \sim (c, z)$.

Hence, the proof is complete. $\square$

Definition 1 Let R be a Γ -ring that satisfies the condition (*) and X be a right P -set of regular elements in R , and assume that $[a, x]$ denote the equivalence class of (a, x) , and RX^{-1} denote the set of equivalence classes for all $a \in R$ and $x \in X$. We define addition and multiplication for $[a, x]$ and $[b, y]$ in RX^{-1} as follows:

- (i) Since $x, y \in X$ and X is a right P -set, thus using Lemma 1, there exist $c, d \in R$ such that $z = x\gamma_0 c = y\gamma_0 d \in X$ and we have

$$\begin{aligned} a\gamma_0 x^{-1} + b\gamma_0 y^{-1} &= a\gamma_0 c\gamma_0 z^{-1} + b\gamma_0 d\gamma_0 z^{-1} \\ &= (a\gamma_0 c + b\gamma_0 d)\gamma_0 z^{-1} \end{aligned}$$

We define $[a, x] + [b, y] = [a\gamma_0 c + b\gamma_0 d, x\gamma_0 c]$.

- (ii) Since $b \in R$, $x \in X$ and X is a right P -set, thus there exist $c \in R$ and $z \in X$ such that $b\gamma_0 z = x\gamma_0 c$ or $x^{-1}\gamma_0 b = c\gamma_0 z^{-1}$. Using (*), we have

$$\begin{aligned} (a\gamma_0 x^{-1})\alpha(b\gamma_0 y^{-1}) &= ax^{-1}\gamma_0 b\gamma_0 y^{-1} \\ &= ax(x^{-1}\gamma_0 b)\gamma_0 y^{-1} \\ &= ax(c\gamma_0 z^{-1})\gamma_0 y^{-1} \\ &= (axc)\gamma_0 (y\gamma_0 z)^{-1}. \end{aligned}$$

We define $[a, x]\alpha[b, y] = [axc, y\gamma_0 z]$, for all $\alpha \in \Gamma$.

These definitions are well defined, because let $[a, x] = [a', x']$. Then, there exist $c_1, d_1 \in R$ such that

$$a\gamma_0 c_1 = a'\gamma_0 d_1 \Rightarrow a = a'\gamma_0 d_1\gamma_0 c_1^{-1}, \quad (3)$$

$$x\gamma_0 c_1 = x'\gamma_0 d_1 \in X \Rightarrow x = x'\gamma_0 d_1\gamma_0 c_1^{-1}. \quad (4)$$

And let $[b, y] = [b', y']$. Then, there exist $c_2, d_2 \in R$ such that

$$b\gamma_0 c_2 = b'\gamma_0 d_2 \Rightarrow b = b'\gamma_0 d_2\gamma_0 c_2^{-1}, \quad (5)$$

$$y\gamma_0 c_2 = y'\gamma_0 d_2 \in X \Rightarrow y = y'\gamma_0 d_2\gamma_0 c_2^{-1}. \quad (6)$$

- (i) Using (3) and (4), we have

$$\begin{aligned} [a, x] + [b, y] &= [a\gamma_0 c + b\gamma_0 d, x\gamma_0 c] \\ &= [a'\gamma_0 d_1\gamma_0 c_1^{-1}\gamma_0 c \\ &\quad + b'\gamma_0 d_2\gamma_0 c_2^{-1}\gamma_0 d_2, x'\gamma_0 d_1\gamma_0 c_1^{-1}\gamma_0 c]. \end{aligned}$$

If we consider $c_3 = d_1\gamma_0 c_1^{-1}\gamma_0 c \in R$ and $d_3 = d_2\gamma_0 c_2^{-1}\gamma_0 d_2 \in R$, then

$$\begin{aligned} [a, x] + [b, y] &= [a'\gamma_0 c_3 + b'\gamma_0 d_3, x'\gamma_0 c_3] \\ &= [a', x'] + [b', y']. \end{aligned}$$

It is enough to prove that $x'\gamma_0 c_3 = y'\gamma_0 d_3$. Using (4), (6) and definition (i), we have

$$\begin{aligned} y'\gamma_0 d_3 &= y'\gamma_0 d_2\gamma_0 c_2^{-1}\gamma_0 d_2 \\ &= y\gamma_0 d \\ &= x\gamma_0 c \\ &= x'\gamma_0 d_1\gamma_0 c_1^{-1}\gamma_0 c \\ &= x'\gamma_0 c_3. \end{aligned}$$

Thus, the addition is well defined.

- (ii) Let $\alpha \in \Gamma$. Then using (3), (6) and (*), we have

$$\begin{aligned} [a, x]\alpha[b, y] &= [axb, y\gamma_0 z] \\ &= [(a'\gamma_0 d_1\gamma_0 c_1^{-1})\alpha c, y'\gamma_0 d_2\gamma_0 c_2^{-1}\gamma_0 z] \\ &= [a'\alpha(d_1\gamma_0 c_1^{-1}\gamma_0 c), y'\gamma_0 d_2\gamma_0 c_2^{-1}\gamma_0 z]. \end{aligned}$$

If we consider $c' = d_1\gamma_0 c_1^{-1}\gamma_0 c \in R$ and $z' = d_2\gamma_0 c_2^{-1}\gamma_0 z \in X$, then

$$\begin{aligned} [a, x]\alpha[b, y] &= [a'\alpha c', y'\gamma_0 z'] \\ &= [a', x']\alpha[b', y']. \end{aligned}$$

It is enough to prove that $b'\gamma_0 z' = x'\gamma_0 c'$. Using (5) and (4), we have

$$\begin{aligned}
b'\gamma_0z' &= b'\gamma_0d_2\gamma_0c_2^{-1}\gamma_0z \\
&= b\gamma_0z \\
&= x\gamma_0c \\
&= x'\gamma_0d_1\gamma_0c_1^{-1}\gamma_0c \\
&= x'\gamma_0c'.
\end{aligned}$$

Thus, multiplication is well defined.

Lemma 3 *If R is a Γ -ring that satisfies the condition (*) and X be a right P -set of regular elements in R , then RX^{-1} is an additive group.*

Proof It is clear that RX^{-1} is closed under addition.

For $[a_1, x_1], [a_2, x_2], [a_3, x_3] \in RX^{-1}$, there exist $c_1, d_1 \in R$ such that $z_1 = x_1\gamma_0c_1 = x_2\gamma_0d_2 \in X$ and there exist $c_2, d_2 \in R$ such that $z_2 = x_1\gamma_0c_1\gamma_0c_2 = x_3\gamma_0d_2 \in X$. We have

$$\begin{aligned}
([a_1, x_1] + [a_2, x_2]) + [a_3, x_3] \\
&= [a_1\gamma_0c_1 + a_2\gamma_0d_1, x_1\gamma_0c_1] + [a_3, x_3] \\
&= [(a_1\gamma_0c_1 + a_2\gamma_0d_1)\gamma_0c_2 + a_3\gamma_0d_2, x_1\gamma_0c_1\gamma_0c_2]. \quad (7)
\end{aligned}$$

Since $z_1 = x_1\gamma_0c_1 = x_2\gamma_0d_2$ and $z_2 = x_1\gamma_0c_1\gamma_0c_2 = x_3\gamma_0d_2$, we obtain

$$z_1^{-1} = c_1^{-1}\gamma_0x_1^{-1}, \quad (8)$$

$$x_1^{-1} = c_1\gamma_0z_1^{-1}, \quad (9)$$

$$x_2^{-1} = d_1\gamma_0z_2^{-1}, \quad (10)$$

$$c_1^{-1}\gamma_0x_1^{-1} = c_2\gamma_0z_2^{-1}, \quad (11)$$

$$x_3^{-1} = d_2\gamma_0z_2^{-1}. \quad (12)$$

And

$$\begin{aligned}
z_1 = x_1\gamma_0c_1 &\implies z_1\gamma_0c_2 = x_1\gamma_0c_1\gamma_0c_2 = z_2 \\
&\implies z_1^{-1} = c_2\gamma_0z_2^{-1}. \quad (13)
\end{aligned}$$

Using (8), (9), (11) and (12), we have

$$\begin{aligned}
&a_2\gamma_0x_2^{-1} + a_3\gamma_0x_3^{-1} \\
&= a_2\gamma_0d_1\gamma_0z_1^{-1} + a_3\gamma_0d_2\gamma_0z_2^{-1} \\
&= a_2\gamma_0d_1\gamma_0c_1^{-1}\gamma_0x_1^{-1} + a_3\gamma_0d_2\gamma_0z_2^{-1} \\
&= a_2\gamma_0d_1\gamma_0c_2\gamma_0z_2^{-1} + a_3\gamma_0d_2\gamma_0z_2^{-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&[a_1, x_1] + ([a_2, x_2] + [a_3, x_3]) \\
&= [a_1, x_1] + [a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2, x_1\gamma_0c_1\gamma_0c_2].
\end{aligned}$$

But using (9) and (13), we have

$$\begin{aligned}
&a_1\gamma_0x_1^{-1} + (a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2)\gamma_0(x_1\gamma_0c_1\gamma_0c_2)^{-1} \\
&= a_1\gamma_0c_1\gamma_0z_1^{-1} + (a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2)\gamma_0z_2^{-1} \\
&= a_1\gamma_0c_1\gamma_0c_2\gamma_0z_2^{-1} + (a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2)\gamma_0z_2^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&[a_1, x_1] + [a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2, x_1\gamma_0c_1\gamma_0c_2] \\
&= [a_1\gamma_0c_1\gamma_0c_2 + a_2\gamma_0d_1\gamma_0c_2 + a_3\gamma_0d_2, x_1\gamma_0c_1\gamma_0c_2] \\
&= [(a_1\gamma_0c_1 + a_2\gamma_0d_1)\gamma_0c_2 + a_3\gamma_0d_2, x_1\gamma_0c_1\gamma_0c_2]. \quad (14)
\end{aligned}$$

Using (7) and (14), $(RX^{-1}, +)$ is associative. Now, we prove $[0, 1] \in RX^{-1}$ is identity element. For every $a \in R$, $x \in X$ and $[a, x] \in RX^{-1}$, if we consider $c = 1 \in R$ and $d = x \in X$, then $x\gamma_0c = 1\gamma_0d$ and so

$$\begin{aligned}
[a, x] + [0, 1] &= [a\gamma_0c + 0\gamma_0d, x\gamma_0c] \\
&= [a, x].
\end{aligned}$$

And also, we consider $c' = x \in R$ and $d' = 1 \in R$, then we have

$$\begin{aligned}
[0, 1] + [a, x] &= [0\gamma_0c + a\gamma_0d', 1\gamma_0c'] \\
&= [0 + a\gamma_01, x] \\
&= [a, x].
\end{aligned}$$

For every $[a, x] \in RX^{-1}$, we know $[-a, x] \in RX^{-1}$. If we consider $c = d = x^{-1} \in R$, then $x\gamma_0c = x\gamma_0d = 1$ and so

$$\begin{aligned}
[a, x] + [-a, x] &= [a\gamma_0c - a\gamma_0d, x\gamma_0c] \\
&= [0, 1].
\end{aligned}$$

Therefore, $[-a, x]$ is inverse of $[a, x]$.

Now, we show that $(RX^{-1}, +)$ is Abelian group.

$$\begin{aligned}
[a, x] + [b, y] &= [a\gamma_0c + b\gamma_0d, x\gamma_0c] \quad (z = x\gamma_0c = y\gamma_0d \in X) \\
&= [b\gamma_0d + a\gamma_0c, y\gamma_0d] \quad (R \text{ is Abelian}) \\
&= [b, y] + [a, x].
\end{aligned}$$

Thus, the proof is complete. $\square$

Theorem 1 *Let R be a Γ -ring that satisfies the condition (*) and X be a right P -set of regular elements in R . Then, $(RX^{-1}, +, \cdot)$ is a Γ -ring with an identity element $[1, 1]$.*

Proof Using Lemma 3, $(RX^{-1}, +)$ is an Abelian group. For every $[a_1, x_1], [a_2, x_2]$ and $[a_3, x_3]$ in RX^{-1} and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
&([a_1, x_1] + [a_2, x_2])\alpha[a_3, x_3] \\
&= [a_1\gamma_0c_1 + a_2\gamma_0d_1, x_1\gamma_0c_1]\alpha[a_3, x_3] \\
&= [(a_1\gamma_0c_1 + a_2\gamma_0d_1)\alpha c_2, x_3\gamma_0z_1]. \quad (15)
\end{aligned}$$

There exist $c_1, d_1 \in R$ such that

$$x_1\gamma_0c_1 = x_2\gamma_0d_1. \quad (16)$$

There exist $c_2 \in R$ and $z_1 \in X$ such that

$$a_3\gamma_0z_1 = x_1\gamma_0c_1\gamma_0c_2. \quad (17)$$

Using (16) and (17), we obtain

$$\begin{aligned} x_1\gamma_0c_1 &= x_2\gamma_0d_1, \\ x_2 \in X &\implies \begin{cases} x_2^{-1}\gamma_0x_1\gamma_0c_1 = d_1, \\ x_2^{-1}\gamma_0x_1\gamma_0c_1\gamma_0c_2 = d_1\gamma_0c_2, \end{cases} \quad (18) \end{aligned}$$

$$a_3\gamma_0z_1 = x_1\gamma_0c_1\gamma_0c_2 \implies x_1^{-1}\gamma_0a_3 = c_1\gamma_0c_2\gamma_0z_1^{-1}. \quad (19)$$

Using (17) and (18), we have

$$x_2^{-1}\gamma_0a_3\gamma_0z_1 = d_1\gamma_0c_2 \implies x_2^{-1}\gamma_0a_3 = d_1\gamma_0c_2\gamma_0z_1^{-1}. \quad (20)$$

For $[a_1, x_1]\alpha[a_3, x_3] + [a_2, x_2]\alpha[a_3, x_3]$, by (*) and (19), we get

$$\begin{aligned} (a_1\gamma_0x_1^{-1})\alpha(a_3\gamma_0x_3^{-1}) &= a_1\alpha x_1^{-1}\gamma_0a_3\gamma_0x_3^{-1} \\ &= a_1\alpha c_1\gamma_0c_2\gamma_0z_1^{-1}\gamma_0x_3^{-1} \\ &= (a_1\alpha c_1\gamma_0c_2)\gamma_0(x_3\gamma_0z_1)^{-1}. \quad (21) \end{aligned}$$

Using (*) and (20), we have

$$\begin{aligned} (a_2\gamma_0x_2^{-1})\alpha(a_3\gamma_0x_3^{-1}) &= a_2\alpha x_2^{-1}\gamma_0a_3\gamma_0x_3^{-1} \\ &= a_2\alpha d_1\gamma_0c_2\gamma_0z_1^{-1}\gamma_0x_3^{-1} \\ &= (a_2\gamma_0d_1\alpha c_2)\gamma_0(x_3\gamma_0z_1)^{-1}. \quad (22) \end{aligned}$$

Using (21) and (22), we obtain

$$\begin{aligned} [a_1, x_1]\alpha[a_3, x_3] + [a_2, x_2]\alpha[a_3, x_3] \\ = [(a_1\gamma_0c_1)\alpha c_2, x_3\gamma_0z_1] + [(a_2\gamma_0d_1)\alpha c_2, x_3\gamma_0z_1]. \end{aligned}$$

If we consider $c = d = 1 \in R$, then $x_3\alpha z_1\gamma_0c = x_3\alpha z_1\gamma_0d$ and so

$$\begin{aligned} [(a_1\gamma_0c_1)\alpha c_2, x_3\gamma_0z_1] + [(a_2\gamma_0d_1)\alpha c_2, x_3\gamma_0z_1] \\ = [(a_1\gamma_0c_1)\alpha c_2\gamma_01 + (a_2\gamma_0d_1)\alpha c_2\gamma_01, x_3\gamma_0z_1\gamma_01] \\ = [(a_1\gamma_0c_1 + a_2\gamma_0d_1)\alpha c_2, x_3\gamma_0z_1]. \quad (23) \end{aligned}$$

Using (15) and (22), it is concluded that

$$\begin{aligned} ([a_1, x_1] + [a_2, x_2])\alpha[a_3, x_3] \\ = [a_1, x_1]\alpha[a_3, x_3] + [a_2, x_2]\alpha[a_3, x_3]. \end{aligned}$$

Now, we show that

$$[a_1, x_1](\alpha + \beta)[a_2, x_2] = [a_1, x_1]\alpha[a_2, x_2] + [a_1, x_1]\beta[a_2, x_2].$$

Since $x_1 \in X$ and X is a right P -set, so there exist $c \in R$ and $z \in X$ such that $a_2\gamma_0z = x_1\gamma_0c$. If we set $c' = d' = 1$, then we have

$$\begin{aligned} [a_1, x_1]\alpha[a_2, x_2] + [a_1, x_1]\beta[a_2, x_2] \\ = [a_1\alpha c, x_2\gamma_0z] + [a_1\beta c, x_2\gamma_0z] \\ = [a_1\alpha c\gamma_01 + a_1\beta c\gamma_01, x_2\gamma_0z\gamma_01] \\ = [a_1(\alpha + \beta)c, x_2\gamma_0z] \\ = [a_1, x_1](\alpha + \beta)[a_2, x_2]. \end{aligned}$$

For $([a_1, x_1]\alpha[a_2, x_2])\beta[a_3, x_3] = [a_1, x_1]\alpha([a_2, x_2]\beta[a_3, x_3])$, we have

$$\begin{aligned} ([a_1, x_1]\alpha[a_2, x_2])\beta[a_3, x_3] &= [a_1\alpha c_1, x_2\gamma_0z_1]\beta[a_3, x_3] \\ &= [a_1\alpha c_1\beta c_2, x_3\gamma_0z_2]. \quad (24) \end{aligned}$$

There exist $c_1 \in R$, $z_1 \in X$ such that

$$a_2\gamma_0z_1 = x_1\gamma_0c_1. \quad (25)$$

There exist $c_2 \in R$, $z_2 \in X$ such that

$$a_3\gamma_0z_1 = x_2\gamma_0z_1\gamma_0c_2. \quad (26)$$

Using (25) and (26), we obtain

$$\begin{aligned} a_2\gamma_0z_1 = x_1\gamma_0c_1 &\implies x_1^{-1}\gamma_0a_2\gamma_0z_1 = c_1, \quad (27) \\ a_3\gamma_0z_2 = x_2\gamma_0z_1\gamma_0c_2 &\implies x_2^{-1}\gamma_0a_3 = z_1\gamma_0c_2\gamma_0z_2^{-1}. \quad (28) \end{aligned}$$

Using (28) and (*), we have

$$\begin{aligned} (a_2\gamma_0x_2^{-1})\beta(a_3\gamma_0x_3^{-1}) &= a_2\beta(x_2^{-1}\gamma_0a_3)\gamma_0x_3^{-1} \\ &= a_2\beta(z_1\gamma_0c_2\gamma_0z_2^{-1})\gamma_0x_3^{-1} \\ &= a_2\beta z_1\gamma_0c_2\gamma_0(x_3\gamma_0z_2)^{-1} \\ &= (a_2\gamma_0z_1)\beta c_2\gamma_0(x_3\gamma_0z_2)^{-1}. \end{aligned}$$

Therefore

$$[a_1, x_1]\alpha([a_2, x_2]\beta[a_3, x_3]) = [a_1, x_1]\alpha((a_2\gamma_0z_1)\beta c_2, x_3\gamma_0z_2).$$

Using (27) and (*), we obtain

$$\begin{aligned} (a_1\gamma_0x_1^{-1})\alpha(a_2\gamma_0z_1)\beta c_2\gamma_0(x_3\gamma_0z_2)^{-1} \\ = a_1\alpha x_1^{-1}\gamma_0a_2\gamma_0z_1\beta c_2\gamma_0(x_3\gamma_0z_2)^{-1} \\ = a_1\alpha c_1\beta c_2\gamma_0(x_3\gamma_0z_2)^{-1}. \end{aligned}$$

And then

$$[a_1, x_1]\alpha([a_2, x_2]\beta[a_3, x_3]) = [a_1\alpha c_1\beta c_2, x_3\gamma_0z_2]. \quad (29)$$

Using (24) and (29), it is concluded that

$$([a_1, x_1]\alpha[a_2, x_2])\beta[a_3, x_3] = [a_1, x_1]\alpha([a_2, x_2]\beta[a_3, x_3]).$$

For all $[a, x] \in RX^{-1}$, we have

$$(a\gamma_0x^{-1})\gamma_0(1\gamma_01) = (a\gamma_0x^{-1})\gamma_01 = a\gamma_0x^{-1}.$$

Thus, $[a, x]\gamma_0[1, 1] = [a, x]$ and similarly $[1, 1]\gamma_0[a, x] = [a, x]$; therefore, $[1, 1] \in RX^{-1}$ is an identity element and, hence, the proof is complete. $\square$

Definition 2 The Γ -ring RX^{-1} is called the Γ -ring of right fraction of R with respect to X .

3 Homomorphisms of Noncommutative Gamma Rings

Theorem 2 Let R be a Γ -ring that satisfies the condition $(*)$ and RX^{-1} be a Γ -ring of right fraction. Then, the mapping $f : R \rightarrow RX^{-1}$ such that $f(r) = [r, 1]$ is a Γ -ring homomorphism.

Proof At first, we prove that f is well defined. If $r_1 = r_2$, since $r_1 = r_1\gamma_0 1$ and $r_2 = r_2\gamma_0 1$, then it is enough to set $c = d = 1$ and so $r_1\gamma_0 c = r_2\gamma_0 d$, $1\gamma_0 c = 1\gamma_0 d \in X$ and, therefore, $[r_1, 1] = [r_2, 1]$.

Now we prove that $f(r_1 + r_2) = f(r_1) + f(r_2)$ and $f(r_1\alpha r_2) = f(r_1)\alpha f(r_2)$, for all $r_1, r_2 \in R$ and $\alpha \in \Gamma$. We have $r_1\gamma_0 1^{-1} + r_2\gamma_0 1^{-1} = (r_1 + r_2)\gamma_0 1^{-1} = (r_1 + r_2)\gamma_0 1$ so let $c = d = 1$. Then

$$\begin{aligned} f(r_1) + f(r_2) &= [r_1, 1] + [r_2, 1] \\ &= [r_1\gamma_0 1 + r_2\gamma_0 1, 1\gamma_0 1] \\ &= [r_1 + r_2, 1] \\ &= f(r_1 + r_2). \end{aligned}$$

Also using $(*)$, we have $(r_1\gamma_0 1^{-1})\alpha(r_2\gamma_0 1^{-1}) = (r_1\alpha r_2)\gamma_0 1$ so let $z = 1 \in X$ and $c = r_2 \in R$. Then, $r_2\gamma_0 z = 1\gamma_0 c$, we get

$$\begin{aligned} f(r_1)\alpha f(r_2) &= [r_1, 1]\alpha[r_2, 1] = [r_1\alpha c, 1\gamma_0 z] \\ &= [r_1\alpha r_2, 1] = f(r_1\alpha r_2). \end{aligned}$$

Hence, the theorem is proved. $\square$

Lemma 4 Let R and R' be Γ -rings with identity elements and $f : R \rightarrow R'$ be a Γ -ring epimorphism. Then $f(1_R) = 1_{R'}$.

Proof We prove that $f(1_R)\gamma_0 r' = r'\gamma_0 f(1_R) = r'$, for all $r' \in R'$. Since f is surjective, there exists $r \in R$ such that $f(r) = r'$. We have

$$\begin{aligned} f(1_R)\gamma_0 r' &= f(1_R)\gamma_0 f(r) = f(1_R\gamma_0 r) = f(r) = r', \\ r'\gamma_0 f(1_R) &= f(r)\gamma_0 f(1_R) = f(r\gamma_0 1_R) = f(r) = r'. \end{aligned}$$

Hence $f(1_R) = 1_{R'}$. $\square$

Lemma 5 If R and R' are Γ -ring with identity elements, without zero-divisor and $f : R \rightarrow R'$ is a Γ -ring homomorphism, then $f(1_R) = 1_{R'}$.

Proof We have

$$\begin{aligned} f(1_R) &= f(1_R\gamma_0 1_R) = f(1_R)\gamma_0 f(1_R) \\ &\Rightarrow f(1_R) - f(1_R)\gamma_0 f(1_R) = 0 \\ &\Rightarrow f(1_R)\gamma_0 (1_{R'} - f(1_R)) = 0 \\ &\Rightarrow 1_{R'} - f(1_R) = 0. \end{aligned}$$

Hence $f(1_R) = 1_{R'}$. $\square$

Lemma 6 Let R and R' be Γ -rings with identity elements, without zero-divisor and $f : R \rightarrow R'$ is a Γ -ring homomorphism. Then, $f(a^{-1}) = (f(a))^{-1}$.

Proof Suppose $a \in R$ is invertible and a^{-1} is inverse of a . Using Lemma 5, we have

$$\begin{aligned} f(a\gamma_0 a^{-1}) &= f(1_R) = f(a^{-1}\gamma_0 a) \\ &\Rightarrow f(a)\gamma_0 f(a^{-1}) = 1_{R'} = f(a^{-1})\gamma_0 f(a) \\ &\Rightarrow f(a^{-1}) = (f(a))^{-1}. \end{aligned}$$

$\square$

Theorem 3 Suppose R and R' be Γ -rings and $g : R \rightarrow R'$ be a Γ -ring homomorphism such that $g(x)$ is invertible in R' for all $x \in X$. Then, there exists a unique Γ -ring homomorphism $h : RX^{-1} \rightarrow R'$ such that $g = h \circ f$, where $f : R \rightarrow RX^{-1}$ and $f(a) = [a, 1]$.

Proof We have $a\gamma_0 x^{-1} = a\gamma_0 (1^{-1}\gamma_0 1)\gamma_0 x^{-1} = (a\gamma_0 1^{-1})\gamma_0 (1\gamma_0 x^{-1})$ and so $[a, x] = [a, 1]\gamma_0 [1, x]$. Using Lemma 6, we define

$$\begin{aligned} h([a, x]) &= h([a, 1]\gamma_0 [1, x]) \\ &= h(f(a)\gamma_0 (f(x))^{-1}) \\ &= h(f(a))\gamma_0 h(f(x))^{-1} \\ &= g(a)\gamma_0 g(x)^{-1}. \end{aligned}$$

Therefore, $h([a, x]) = g(a)\gamma_0 g(x)^{-1}$. We prove that h is a Γ -ring homomorphism.

For all $[a_1, x_1], [a_2, x_2] \in RX^{-1}$ and $\alpha \in \Gamma$, since X is the right P -set using Lemma 1, there exist $c, d \in R$ such that $z = x_1\gamma_0 c = x_2\gamma_0 d$, we have

$$\begin{aligned} h([a_1, x_1] + [a_2, x_2]) &= h([a_1\gamma_0 c + a_2\gamma_0 d, x_1\gamma_0 c]) \\ &= g(a_1\gamma_0 c + a_2\gamma_0 d)\gamma_0 g(x_1\gamma_0 c)^{-1} \\ &= (g(a_1\gamma_0 c) + g(a_2\gamma_0 d))\gamma_0 g(x_1\gamma_0 c)^{-1} \\ &= g(a_1)\gamma_0 g(c)\gamma_0 g(x_1\gamma_0 c)^{-1} \\ &\quad + g(a_2\gamma_0 d)\gamma_0 g(x_2\gamma_0 d)^{-1} \\ &= g(a_1)\gamma_0 g(c)\gamma_0 g(c)^{-1}\gamma_0 g(x_1)^{-1} \\ &\quad + g(a_2)\gamma_0 g(d)\gamma_0 g(d)^{-1}\gamma_0 g(x_2)^{-1} \\ &= g(a_1)\gamma_0 g(x_1)^{-1} + g(a_2)\gamma_0 g(x_2)^{-1} \\ &= h([a_1, x_1]) + h([a_2, x_2]). \end{aligned}$$

For all $[a_1, x_1], [a_2, x_2] \in RX^{-1}$, since X is the right P -set, there exist $c \in R$ and $z \in X$ such that $a_2\gamma_0z = x_1\gamma_0c \in X$ or $c = x_1^{-1}\gamma_0a_2\gamma_0z$, we have

$$\begin{aligned} h([a_1, x_1]\alpha[a_2, x_2]) &= h([a_1\alpha c, x_2\gamma_0z]) \\ &= g(a_1\alpha c)\gamma_0g(x_2\gamma_0z)^{-1} \\ &= g(a_1)\alpha g(c)\gamma_0g(z)^{-1}\gamma_0g(x_2)^{-1} \\ &= g(a_1)\alpha g(x_1^{-1}\gamma_0a_2\gamma_0z)\gamma_0g(z)^{-1}\gamma_0g(x_2)^{-1} \\ &= g(a_1)\alpha g(x_1)^{-1}\gamma_0g(a_2)\gamma_0g(z)\gamma_0g(z)^{-1}\gamma_0 \\ &\quad g(x_2)^{-1} \\ &= g(a_1)\gamma_0g(x_1)^{-1}\alpha g(a_2)\gamma_0g(x_2)^{-1} \\ &= h([a_1, x_1])\alpha h([a_2, x_2]). \end{aligned}$$

□

Corollary 1 *If $g : R \longrightarrow R'$ is a Γ -ring homomorphism such that g has following properties:*

- (i) *For all $x \in X$, $g(x)$ in R' is invertible.*
- (ii) *If $g(a) = 0$, then there exists $t \in X$ such that $a\gamma_0t = 0$.*
- (iii) *Every element of R' is the form $g(a)\gamma_0g(x)^{-1}$, for some $a \in R$ and some $x \in X$, then the Γ -ring homomorphism $h : RX^{-1} \longrightarrow R'$ such that $h([a, x]) = g(a)\gamma_0g(x)^{-1}$ is isomorphism.*

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