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The relative class number one problem for function fields, I

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Abstract

We reduce the classification of finite extensions of function fields (of curves over finite fields) with the same class number to a finite computation; complete this computation in all cases except when both curves have base field $\mathbb{F}_2$ and genus > 1; and give a conjectural answer in the remaining cases. The conjecture will be resolved in subsequent papers.

Keywords: Function fields, Class number one, Weil polynomials

1 Introduction

The *relative class number one problem* for function fields (of curves over finite fields) is to classify finite extensions for which the relative class number equals 1, or equivalently the class numbers of the two function fields coincide. In this paper, we solve this problem in all cases except where both function fields have base field $\mathbb{F}_2$, and to reduce that case to a *feasible* finite computation. This extends work of numerous authors [2–6] but our arguments are independent of these.

For comparison, the relative class number one problem for number fields was formulated by Stark [7] only for *CM fields*, viewed as totally imaginary quadratic extensions of totally real fields. This restriction is quite natural: outside of this case, the relative unit rank is nonzero and the relative class number behaves erratically (e.g., it is not generally integral). Odlyzko [8] established conditionally on GRH that there are only finitely many CM fields with relative class number one. The complete set of *normal* CM fields with relative class number one has been determined recently by Hoffman–Sircana [9].

Before continuing, we introduce some terminology and notation. By a *function field*, we mean the field of rational functions on a curve over some finite field. Given a finite extension F'/F of function fields, we write C, C' for the curves corresponding to F, F'; $q_F, q_{F'}$ for the orders of the base fields of C, C'; $g_F, g_{F'}$ for the genera of C, C'; and $h_F, h_{F'}$ for the class numbers of F, F'. We write $J(C), J(C')$ for the Jacobians of C, C', so that $\#J(C)(\mathbb{F}_{q_F}) = h_F$ and $\#J(C')(\mathbb{F}_{q_{F'}}) = h_{F'}$.

The *relative class number* $h_{F'/F}$ is the ratio $h_{F'}/h_F$; this can be interpreted as the order of a certain finite group (see below), and hence is an integer. This implies the following reduction: for $E = F \cdot \mathbb{F}_{q_{F'}}$, $h_{F'/F} = 1$ if and only if $h_{E/F} = h_{F'/E} = 1$. We may thus focus

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on the cases where $F' = E$, in which case we say the extension F'/F is *constant*, and where $E = F$, in which case we say F'/F is *purely geometric*.

In the case of a constant extension, the equality $h_{F'/F} = 1$ holds for trivial reasons when $F' = F$ and when $g_F = g_{F'} = 0$ (as in this case $h_F = h_{F'} = 1$). Excluding these, we have the following result; see §3 for the proof.

Theorem 1.1 *Let F'/F be a constant extension of degree $d > 1$ of function fields with $g_F > 0$, $q_{F'} > q_F$, and $h_{F'/F} = 1$. Then $(q_F, d, g_F, J(C))$ is one of*

$(2, 2, 1, \mathit{1.2.c})$, $(2, 2, 2, \mathit{2.2.c\_c})$, $(2, 2, 2, \mathit{2.2.d\_f})$, $(2, 2, 3, \mathit{3.2.e\_j\_p})$,
$(2, 3, 1, \mathit{1.2.b})$, $(2, 3, 1, \mathit{1.2.c})$, $(3, 2, 1, \mathit{1.3.ad})$, $(4, 2, 1, \mathit{1.4.ae})$,

where $J(C)$ is specified up to isogeny by an LMFDB label.

In the case of a purely geometric extension, the equality $h_{F'/F} = 1$ holds for trivial reasons when $F' = F$ and when $g_F = g_{F'} \in \{0, 1\}$. Moreover, when $g_F \in \{0, 1\}$, for any fixed pair of isomorphism classes of F and F', the existence of a single finite morphism $F \to F'$ implies the existence of infinitely many more. It is thus natural to separate the cases $g_F \leq 1$ and $g_F > 1$; see §6 and §8 for the proofs.

Theorem 1.2 *Let F'/F be a purely geometric extension of degree d of function fields with $g_F \leq 1$, $g_{F'} > g_F$, and $h_{F'/F} = 1$. Then $(q_F, g_F, g'_F, J(C), J(C'))$ appears in Table 3. (Note that the tuple does not always uniquely determine F'.)*

When $g_F = 0$, Theorem 1.2 recovers the solution of the *absolute class number one problem* for function fields [10–13].

Theorem 1.3 *Let F'/F be a purely geometric extension of degree d of function fields with $g_{F'} > g_F > 1$ and $h_{F'/F} = 1$.*

(a) *If $q_F > 2$, then $q_F \in \{3, 4\}$, $(g_F, g'_F) \in \{(2, 3), (2, 4), (3, 5)\}$, F'/F is (Galois) cyclic, and $(q_F, g_F, g_{F'}, F)$ appears in Table 4. In each listed case, the tuple uniquely determines F'.*
(b) *If $q_F = 2$, then $g_F \leq 7$ and $g_{F'} \leq 13$. The isogeny classes of $J(C)$ and the Prym variety A (see below) form one of 208 pairs listed in Table 7.*
(c) *If $q_F = 2$, then assuming that F'/F is cyclic, there are exactly 61 tuples $(d, g_F, g_{F'}, F)$ with $g_F \notin \{6, 7\}$, and at least 3 with $g_F \in \{6, 7\}$; see Tables 5 and 6. In each listed case, the tuple uniquely determines F'.*

In Theorem 1.3(c), there are only two cases (3.2.ab_a_c and 5.2.b_c_e_i_i) where F is not uniquely specified by $d, g_F, g_{F'}, J(C)$. The scarcity of such examples reflects that curves with isogenous Jacobians can typically be distinguished by the L-functions of their abelian covers [14, 15].

By our earlier reduction, we recover the following corollary.

Corollary 1.4 *Let F'/F be an extension of degree d of function fields with $g_{F'} > g_F$ and $h_{F'/F} = 1$ which is neither constant nor purely geometric. Then $q_F = 2$, $q_{F'} = 4$, and $(g_F, g_{F'}, J(C), J(C')) \in \{(0, 1, 0, \mathit{1.4.ae}), (1, 2, \mathit{1.2.c}, \mathit{2.4.ae\_i})\}$.*

We now summarize the techniques used to prove Theorems 1.1, 1.2, and 1.3. The extension F'/F induces an injective morphism f from $J(C)$ to the Weil restriction of $J(C')$

from $\mathbb{F}_{q_{F'}}$ to $\mathbb{F}_{q_F}$, and $h_{F'/F}$ can be interpreted as the order of the group $A(\mathbb{F}_{q_F})$ where A is the cokernel of f; we call A the *Prym variety* of the covering $C' \to C$. We restrict options for C and C' using the structure of simple abelian varieties of order 1 over $\mathbb{F}_q$: for $q \geq 5$ there are none; for $q = 3, 4$ there are only elliptic curves; for $q = 2$ there is an infinite series described in work of Madan–Pal [16] and Robinson [17].

The severe restrictions on A impose constraints in turn on the number of rational points on C and C' over various finite extensions of their base fields. In the constant case, the restrictions lead quickly to Theorem 1.1 because the zeta function of C' is uniquely determined by the zeta function of C and the degree of the extension. By contrast, in the purely geometric case there is no obvious way to predict the zeta function of C' from that of C; we instead argue that C is forced to have many rational points, which for $g_F \gg 0$ will violate a "linear programming" bound [18, Part II]. This yields effective upper bounds on g_F and $g_{F'}$; we then obtain a list of candidates for the Weil polynomials of F and F' by an exhaustion in SageMath (as described in [19], and later used in LMFDB as per [20]). There is a loose parallel here with the Serre–Lauter method for refining upper bounds on rational points on curves over finite fields [21].

To complete the proofs, we identify candidates for C with a given zeta function using data from LMFDB [22], which includes a table of genus-4 curves by Xarles [23], plus a similar table of genus-5 curves computed by Dragutinović [24]. We then make a computation of abelian extensions of function fields in Magma.

The relative class number one problem is now reduced to the following.

Conjecture 1.5 *Let F'/F be a purely geometric extension of degree $d > 1$ with $q_F = 2$, $g_F > 1$, and $h_{F'/F} = 1$. Then F appears in one of Tables 5 or 6*

By Theorem 1.3, this further reduces to the following two logically independent statements, which will be addressed in subsequent work [25,26].

- Any extension as in Conjecture 1.5 is cyclic. This will follow from Theorem 1.3(b) by extending the argument for $q_F > 2$ (see Lemma 8.1).
- Table 6 is complete in genera 6 and 7. This will follow from a limited census based on Mukai's descriptions of canonical curves of these genera [27,28]; the entries in Table 6 come from a preliminary version of this census.

We have not considered the relative class number m problem for $m > 1$, as in [10]. This would require adapting Lemma 5.2 to abelian varieties over $\mathbb{F}_2$ of order m. For each m it is known that there are infinitely many simple abelian varieties of order m over $\mathbb{F}_2$ [29], but it seems hopeless to give a complete classification; a better approach might be modeled on the use of resultants to prove statements about small algebraic integers (see [30] for recent progress in this direction).

All computations in SageMath [31] and Magma [32] are documented in Jupyter notebooks available from a GitHub repository [33]; the computations take under 2 hours on a single CPU (Intel i5-1135G7@2.40GHz) and generate an Excel spreadsheet of the 208 pairs of Weil polynomials in Theorem 1.3(b). We use LMFDB labels for isogeny classes of abelian varieties over finite fields, formatted as links into the site.

2 Abelian varieties of order 1

We say that an abelian variety A over a finite field $\mathbb{F}_q$ has *order* 1 if we have $\#A(\mathbb{F}_q) = 1$; that is, the group of $\mathbb{F}_q$-rational points of A is trivial. Recall that $\#A(\mathbb{F}_q) = P(1)$ where $P(T) \in \mathbb{Z}[T]$ is the Weil polynomial associated to A.

Lemma 2.1 *Let A be a simple abelian variety of order 1 over some finite field $\mathbb{F}_q$.*

(a) *We must have $q \leq 4$.*
(b) *If $q \in \{3, 4\}$, then A is an elliptic curve with Weil polynomial $T^2 - qT + q$.*
(c) *If $q = 2$, then each root α of the Weil polynomial of A satisfies*

$$\alpha^2 + (\eta - 1)\alpha - 2\eta = 0 \tag{2.2}$$

for some root of unity η. The roots of unity η of order n give rise to two irreducible Weil polynomials if $n = 7, 30$ and one otherwise. The resulting A is ordinary unless n is a power of 2, in which case it has p-rank 0.

Proof This follows from [16, Theorem 4], [17] (for the second assertion of (c)), [34, Lemma 5.1] (for the description in (2.2)), and [34, Lemma 4.3] (for the p-rank). □

We deduce some consequences for the Frobenius traces of abelian varieties of order 1; for $q = 2$ we establish a stronger result later (Lemma 5.2). For A an abelian variety over a finite field $\mathbb{F}_q$ and n a positive integer, let T_{A,q^n} be the trace of the q^n-power Frobenius on A; we also write T_{C,q^n} in case $A = J(C)$.

Lemma 2.2 *Let A be a simple abelian variety of order 1 over $\mathbb{F}_2$. Choose α, η as in (2.2) and assume that the order of η is not in $\{1, 2, 7, 30\}$. For*

$$t_i = \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(\eta^i) = \frac{\phi(n)}{\phi(n/\gcd(n,i))}\mu\left(\frac{n}{\gcd(n,i)}\right) \tag{2.3}$$

(where μ is the Möbius function), we have

$$\begin{aligned}
T_{A,2} &= \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(1 - \eta) = \phi(n) - t_1 \\
T_{A,4} &= \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(1 + 2\eta + \eta^2) = \phi(n) + 2t_1 + t_2 \\
T_{A,8} &= \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(1 + 3\eta - 3\eta^2 - \eta^3) = \phi(n) + 3t_1 - 3t_2 - t_3 \\
T_{A,16} &= \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(1 + 4\eta - 2\eta^2 + 4\eta^3 + \eta^4) = \phi(n) + 4t_1 - 2t_2 + 4t_3 + t_4.
\end{aligned}$$

Proof Our assumption on n ensures that $\mathbb{Q}(\alpha)$ is a quadratic extension of $\mathbb{Q}(\eta)$. From (2.2), we see that

$$T_{A,2} = \mathrm{Trace}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha) = \mathrm{Trace}_{\mathbb{Q}(\eta)/\mathbb{Q}}(1 - \eta) = \phi(n) - t_1.$$

Similarly, from (2.2) we deduce that

$$\begin{aligned}
0 &= \alpha^4 + (-1 - 2\eta - \eta^2)\alpha^2 + 4\eta^2 \\
&= \alpha^6 + (-1 - 3\eta + 3\eta^2 + \eta^3)\alpha^3 - 8\eta^3 \\
&= \alpha^8 + (-1 - 4\eta + 2\eta^2 - 4\eta^3 - \eta^4)\alpha^4 + 16\eta^4,
\end{aligned}$$

from which we read off the expressions for $T_{A,4}$, $T_{A,8}$, $T_{A,16}$. □

Lemma 2.3 *Let A be an abelian variety of order 1 and dimension g over $\mathbb{F}_q$.*

(a) *If $q = 4$, then $T_{A,q} = 4g$, $T_{A,q^2} = 8g$.*
(b) *If $q = 3$, then $T_{A,q} = 3g$, $T_{A,q^2} = 3g$.*
(c) *If $q = 2$ and A is simple, then $T_{A,2} + T_{A,4} \geq 2$. This is strict if $g \geq 4$.*

Proof Parts (a) and (b) are apparent from Lemma 2.1. To check (c), we check for $g \leq 6$ using LMFDB;[1] see Table 2 for the detailed results. For $g > 6$, Lemma 2.2 and (2.3) yield $T_{A,2} + T_{A,4} = 2g + t_1 + t_2 \geq 2g - 1 - 2 \geq 2$, as desired. □

3 Constant extensions

In this section, we prove Theorem 1.1. We recall a point from the introduction: for any abelian variety A over $\mathbb{F}_q$ and any positive integer d, the Weil restriction of A from $\mathbb{F}_{q^d}$ to $\mathbb{F}_q$ is isogenous to the product of A with the "Prym variety" A'.

Lemma 3.1 *Let A be an abelian variety over $\mathbb{F}_q$ such that $\#A(\mathbb{F}_q) = \#A(\mathbb{F}_{q^d})$ for some prime $d > 2$. Then $q = 2$, $d = 3$, and the Weil polynomial of every simple isogeny factor of A belongs to $\{T^2 + T + 2, T^2 + 2T + 2\}$.*

Proof Since $[A(\mathbb{F}_{q^d}) : A(\mathbb{F}_q)] = \#A'(\mathbb{F}_q)$ is an integer, the hypothesis that $\#A(\mathbb{F}_q) = \#A(\mathbb{F}_{q^d})$ implies the same for the isogeny factors of A; we may thus assume that A is simple. Let $P(T)$ be the Weil polynomial of A. Then the Weil polynomial of A' is $\prod_{i=1}^{d-1} P(\zeta_d^i T)$, and hence has roots $\alpha_1, \dots, \alpha_{d-1}$ such that

$$\alpha_1 \zeta_d = \cdots = \alpha_{d-1} \zeta_d^{d-1};$$

by Lemma 2.1, this is impossible if $q > 2$. If $q = 2$, then by (2.2) there must exist roots of unity $\eta_1, \dots, \eta_{d-1}$ with

$$\alpha_i^2 + (\eta_i - 1)\alpha_i - 2\eta_i = 0 \qquad (i = 1, \dots, d-1).$$

For $1 \leq i < j \leq d-1$, applying [34, Lemma 5.2, Lemma 7.2] to the equation $\alpha_i = \alpha_j \zeta_d^{j-i}$ shows that $(\eta_i, \eta_j, \zeta_d^{j-i})$ either appears in one of the parametric solutions in [34, (7.2.1)] or is a sporadic solution fitting a pattern listed in [34, Table 2].

If only parametric solutions occur, then from [34, (7.2.1)] we have $\eta_1 = \cdots = \eta_{d-1}$, leaving only two distinct values for $\alpha_1, \dots, \alpha_{d-1}$. Hence $d = 3$; from [34, (7.2.1)] again, $\eta_1 = \eta_2 = -\zeta_3$ has order 6. This yields the Weil polynomial $T^2 + T + 2$.

If we get a sporadic solution for some i, j, then [34, Table 2] indicates that ζ_d^{j-i} has order dividing 21, 24, or 30; this forces $d \leq 7$. For $d \in \{5, 7\}$, the η_i must all have order 30 or 7, respectively; however, if α satisfies (2.2) for some root of unity η of this order, then at most two of the quantities $\{\alpha \zeta_d^i : i = 1, \dots, d-1\}$ do likewise, and this leaves no options for A'. Hence $d = 3$; from [34, Table 2] (taking $\eta_3 = \zeta_3$), $\eta_1 = \eta_2$ has order 4. This yields the Weil polynomial $T^2 + 2T + 2$. □

Lemma 3.2 *Let C be an algebraic curve of genus $g > 0$ over $\mathbb{F}_q$ such that $\#J(C)(\mathbb{F}_q) = \#J(C)(\mathbb{F}_{q^d})$ for some integer $d > 1$. Then*

$$(q, d, g) \in \{(2, 2, 1), (2, 2, 2), (2, 2, 3), (2, 3, 1), (3, 2, 1), (4, 2, 1)\}.$$

[1] On an LMDFB page, the entry "Point counts of the curve" lists $q^i + 1 - T_{A,q^i}$ for $i = 1, \dots, 10$.

Table 1 Upper bounds on $\#C(\mathbb{F}_q)$ for a genus-g curve C from [35]

g	1	2	3	4	5	6	7	8	9	10
$q = 2$	5	6	7	8	9	10	10	11	12	13
$q = 2^2$	9	10	14	15	17	20	21	23	26	27
$q = 2^3$	25	33	38	45	53	65	69	75	81	86
$q = 3$	7	8	10	12	13	14	16	18	19	21
$q = 3^2$	16	20	28	30	35	38	43	46	50	54

Proof It suffices to prove the claim when d is prime, as the result will then rule out composite values of d. By Lemma 2.1, $q \leq 4$. By Lemma 3.1 applied with $A = J(C)$, if $d > 2$ then $(q, d, g) = (2, 3, 1)$.

Assume now that $d = 2$. Then the Prym variety A' is the quadratic twist of $J(C)$, so $T_{A',q^i} = (-1)^i T_{C,q^i}$. If $q \in \{3, 4\}$, then by Lemma 2.3,

$$0 \leq \#C(\mathbb{F}_{q^2}) - \#C(\mathbb{F}_q) = (q^2 + 1 - T_{A',q}) - (q + 1 + T_{A',q})$$
$$= q^2 - q - T_{A',3} - T_{A',9} = q^2 - q - q(q-1)g$$

and so $g \leq 1$. If $q = 2$, then

$$0 \leq \#C(\mathbb{F}_4) - \#C(\mathbb{F}_2) = (2^2 + 1 - T_4) - (2 + 1 + T_2) = 2 - T_2 - T_4 \leq 0$$

with the last inequality strict unless A' is simple of dimension at most 3. □

Lemma 3.3 *Let C be a curve over $\mathbb{F}_q$ such that $\#J(C)(\mathbb{F}_q) = \#J(C)(\mathbb{F}_{q^d})$ for some $d > 1$. Then C appears in Theorem 1.1.*

Proof As this property only depends on the isogeny class of $J(C)$, it suffices to search over the isogeny classes in LMFDB permitted by Lemma 3.2. □

4 Bounds on rational points on curves

We next compile some explicit upper bounds for the number of rational points on a curve over $\mathbb{F}_q$. For $g \leq 10$, we reproduce in Table 1 some data from [35] (see therein for underlying references). For larger g, we use the "linear programming" method of Oesterlé. (All decimal expansions herein refer to *exact* rational numbers.)

Lemma 4.1 *Let C be a curve of genus g over $\mathbb{F}_q$ with $q \in \{2, 3, 4\}$. Then*

$$\#C(\mathbb{F}_q) \leq \begin{cases} 0.6272g + 9.562 & (q = 2) \\ 1.153g + 11.67 & (q = 3) \\ 1.435g + 21.75 & (q = 4). \end{cases}$$

Proof For $q = 2$, this is the "third choice" bound of [18, (7.1.4)]. For $q = 3, 4$, we adapt the proof of the "first choice" bound of [18, (7.1.1)]). For $x_1, x_2, \ldots \geq 0$, define $c = 1 + 2x_1^2 + 2x_2^2 + \cdots$ and consider the function

$$f(\theta) = \frac{1}{c}(1 + 2x_1\cos(\theta) + 2x_2\cos(2\theta) + \cdots)^2 = 1 + 2\sum_{n \geq 1} c_n \cos(n\theta).$$

By construction, $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$ and $c_n \geq 0$ for all n (that is, f is *doubly positive* in the sense of Serre). Define $\psi(t) = \sum_{n=1}^{\infty} c_n t^n$; then by [18, Theorem 5.3.3].

$$\#C(\mathbb{F}_q)\psi(q^{-1/2}) \leq g + \psi(q^{-1/2}) + \psi(q^{1/2}),$$

or in other words

$$\#C(\mathbb{F}_q) \leq \frac{1}{\psi(q^{-1/2})} g + 1 + \frac{\psi(q^{1/2})}{\psi(q^{-1/2})}.$$

For $x_1 = 1, x_2 = 0.7, x_3 = 0.2, x_4 = \cdots = 0$, this yields the indicated results. □

The bounds produced by linear programming also include some correction terms counting points over extension fields. We make one such bound explicit for $q = 2$.

Lemma 4.2 *Let C be a curve of genus g over $\mathbb{F}_2$. For $d = 1, 2, \dots$, let a_d be the number of closed points of degree d on C. Then*

$$a_1 + 2a_2(0.3366) + 3a_3(0.1382) + 4a_4(0.0537) \leq 0.8042g + 5.619. \tag{4.1}$$

Proof With notation as in the proof of Lemma 4.1, define $\psi_d(t) = \sum_{n=1}^{\infty} c_{dn} t^{dn}$. Then by [18, Theorem 5.3.3] again,

$$\sum_{d=1}^{\infty} d a_d \psi_d(q^{-1/2}) \leq g + \psi(q^{-1/2}) + \psi(q^{1/2}),$$

or in other words

$$a_1 + \sum_{d=2}^{\infty} d a_d \frac{\psi_d(q^{-1/2})}{\psi(q^{-1/2})} \leq \frac{1}{\psi(q^{-1/2})} g + 1 + \frac{\psi(q^{1/2})}{\psi(q^{-1/2})}. \tag{4.2}$$

We apply this with $x_1 = 1, x_2 = 0.85, x_3 = 0.25, x_4 = \cdots = 0$. This yields (4.1) by discarding the terms $d \geq 5$ in (4.2). □

5 Numerical estimates

We next apply the bounds on rational points to bound the genera of function fields occurring in a purely geometric extension with relative class number 1. We will later take a closer account of the degree of the extension; see §7.

For the remainder of the paper, let F'/F be a purely geometric extension of degree d such that $g_{F'} > g_F$ and $h_{F'/F} = 1$. For brevity, we write q, g, g' in place of $q_F, g_F, g_{F'}$. Let A be the Prym variety of $C' \to C$; then A has order 1, so Lemma 2.1 implies $q \leq 4$. By Riemann–Hurwitz,

$$\dim(A) = g' - g = (d-1)(g-1) + \delta \qquad \text{with} \qquad \delta \geq 0, \tag{5.1}$$

with equality if and only if $C' \to C$ is étale. Since $T_{C',q^i} = T_{C,q^i} + T_{A,q^i}$, we have

$$0 \leq \#C'(\mathbb{F}_{q^i}) = q^i + 1 - T_{C',q^i} = q^i + 1 - T_{C,q^i} - T_{A,i} = \#C(\mathbb{F}_{q^i}) - T_{A,q^i} \tag{5.2}$$

for each positive integer i, and hence

$$T_{A,q^i} \leq \#C(\mathbb{F}_{q^i}) \qquad (i = 1, 2, \dots). \tag{5.3}$$

Lemma 5.1 *If $q > 2$, then $g \leq 6$.*

Proof By combining Lemma 2.3, Lemma 4.1, (5.1), and (5.3), we obtain

$$q(g-1) \leq q(g'-g) \leq \#C(\mathbb{F}_q) \leq \begin{cases} 1.153g + 11.67 & (q=3) \\ 1.435g + 21.75 & (q=4). \end{cases} \tag{5.4}$$

Comparing the ends of this equation yields

$$g \leq \begin{cases} (11.67+3)/(3-1.153) \leq 7.95 & (q=3) \\ (21.75+4)/(4-1.435) \leq 10.04 & (q=4); \end{cases}$$

hence $g \leq 7$ if $q = 3$ and $g \leq 10$ if $q = 4$. Replacing the right-hand side of (5.4) with the explicit bounds given in Table 1, we may eliminate the case $g = 7$. □

For $q = 2$, it is not enough to control $\#C(\mathbb{F}_2)$ because there exists a simple abelian variety of order 1 with trace 0 (namely 2.2.a_ae). Instead, we use a bound modeled on Lemma 4.2. For A an abelian variety over $\mathbb{F}_2$, define its *excess* as

$$1.3366T_{A,2} + 0.3366T_{A,4} + 0.1137(T_{A,8} - T_{A,2}) + 0.0537(T_{A,16} - T_{A,4}) - 1.5612g.$$

Lemma 5.2 *For A an abelian variety of order 1 and dimension g over $\mathbb{F}_2$, the excess of A is nonnegative.*

Proof We may assume that A is simple; define n as in Lemma 2.1. We again treat the case $g \leq 6$ using LMFDB; see Table 2. For $g \geq 7$, we have $g = \phi(n)$; per Lemma 2.2 we can write the excess as

$$0.112g - 0.1012t_1 - 0.1656t_2 + 0.1011t_3 + 0.0537t_4.$$

For $g \in \{7, 8\}$, we have $n \in \{15, 16, 20, 24, 30\}$; we compute the excess using (2.3) to obtain a lower bound of 0.4807. For $g \geq 9$, we apply (2.3) to deduce that $|t_d| \leq d$ and then obtain a lower bound of $0.112g - 0.9505 \geq 0.112 \cdot 9 - 0.9505 \geq 0.0575$. □

Lemma 5.3 *For $q = 2$, we have*

$$g' \leq 0.4313\#C(\mathbb{F}_2) + 1.5152g + 3.6.$$

Proof We combine Lemma 4.2, (5.3), and Lemma 5.2 to obtain

$$\begin{aligned} 1.5612(g'-g) &\leq 1.3366T_{A,q} + 0.3366T_{A,q^2} + 0.1137(T_{A,q^3} - T_{A,q}) \\ &\quad + 0.0537(T_{A,q^4} - T_{A,q^2}) \\ &= (1.3366 - 0.1137)T_{A,q} + (0.3366 - 0.0537)T_{A,q^2} \\ &\quad + 0.1137T_{A,q^3} + 0.0537T_{A,q^4} \\ &\leq (1.3366 - 0.1137)\#C(\mathbb{F}_q) + (0.3366 - 0.0537)\#C(\mathbb{F}_{q^2}) \\ &\quad + 0.1137\#C(\mathbb{F}_{q^3}) + 0.0537\#C(\mathbb{F}_{q^4}) \\ &= 1.3366a_1 + 0.3366(a_1 + 2a_2) + 0.1137(3a_3) + 0.0537(4a_4) \\ &= 1.6732a_1 + 0.3366(2a_2) + 0.1137(3a_3) + 0.0537(4a_4) \\ &\leq 0.6732\#C(\mathbb{F}_2) + 0.8042g + 5.619, \end{aligned}$$

which yields the claimed inequality. □

Table 2 Simple abelian varieties over $\mathbb{F}_2$ of order 1 and dimension at most 6, from LMFDB. For the definitions of n and the excess, see Lemma 2.1 and Lemma 5.2

A	n	$T_{A,2}$	$T_{A,4}$	$T_{A,8}$	$T_{A,16}$	$T_{A,2}+T_{A,4}$	Excess
1.2.ac	2	2	0	−4	−8	2	0.0002
2.2.a_ae	1	0	8	0	16	8	0.0000
2.2.ad_f	3	3	−1	0	7	2	0.6393
2.2.ac_c	4	2	0	8	8	2	0.6626
2.2.ab_ab	6	1	3	10	−1	4	0.0325
3.2.ad_c_b	7	3	5	6	−11	8	0.4911
3.2.ae_j_ap	7	4	−2	1	10	2	0.2929
4.2.af_m_au_bd	5	5	1	5	−3	6	0.5600
4.2.ae_g_ae_c	8	4	4	4	0	8	0.2332
4.2.ad_c_a_b	10	3	5	9	13	8	0.5598
4.2.ae_f_c_al	12	4	6	−2	−2	10	0.0094
4.2.ae_e_h_av	30	4	8	−5	4	12	0.5563
4.2.af_n_az_bn	30	5	−1	5	7	4	0.5312
6.2.ag_p_av_y_abn_cn	9	6	6	9	−6	12	0.3687
6.2.af_j_ah_d_ab_ab	14	5	7	11	15	12	0.7838
6.2.ag_p_at_g_bb_acj	18	6	6	3	18	12	0.9753

Corollary 5.4 *For $q = 2$, we have $g \leq 40$. Moreover, if $d \geq 3$ then $g \leq 6$; if $d \geq 4$ then $g \leq 4$; if $d \geq 5$ then $g \leq 3$; and if $d \geq 6$ then $g \leq 2$.*

Proof By (5.1) and Lemma 5.3,

$$(d - 1.5152)g \leq 0.4313\#C(\mathbb{F}_2) + (d + 2.6).$$

Taking $d = 2$ and using the bound on $\#C(\mathbb{F}_2)$ from Lemma 4.1 yields $g \leq 40$. For $d \geq 3$ we obtain $g \leq 8$; we then use Table 1 to obtain the remaining bounds. □

6 Exhaustion over Weil polynomials

We next describe an exhaustive search over Weil polynomials which rules out some additional pairs (g, g'); compare [18, Theorem 7.2.1] for an example in the context of bounding rational points on curves. This will yield Theorem 1.2; for $g > 1$, we will do better with constraints depending on d (see §7).

We first make a list of candidate Weil polynomials for A. For $q > 2$ this consists of the single polynomial $(T^2 - qT + q)^{g'-g}$. For $q = 2$, we identify isogeny classes of simple abelian varieties A of order 1 such that for $i = 1, 2$, $T_{A,2^i}$ and $T_{A,4}$ is at most the value listed in Table 1 for the pair $(g, 2^i)$, and moreover

$$\begin{aligned} &T_{A,2} + 0.3366(T_{A,4} - T_{A,2}) + 0.1137(T_{A,8} - T_{A,2}) + 0.0537(T_{A,16} - T_{A,4}) \\ &\quad \leq 0.8042g + 5.619; \end{aligned}$$

these are all necessary conditions by Lemma 4.2 and (5.3).

We next identify candidate Weil polynomials for C for which the resulting values of $\#C(\mathbb{F}_q)$ (and $\#C(\mathbb{F}_{q^2})$ for $q = 2$) are consistent with at least one choice of A, and eliminate those that are ruled out by any of the following.

- Bounds on point counts from Table 1.
- The *positivity condition*: the number of degree-i places on C must be nonnegative for all $i \geq 1$.
- Data from LMFDB (genus ≤ 3), [23] (genus 4), and [24] (genus 5) indicating which curves have a particular Weil polynomial.
- The *resultant*-1 *and resultant*-2 *criteria* of Serre [18, Theorem 2.4.1] as extended by Howe–Lauter [36, Proposition 2.8]. The resultant-2 criterion forces C to occur as a double cover of another curve, whose Weil polynomial can sometimes be ruled out. (Compare Corollary 9.2.)

Finally, we exhaust over pairs of candidate Weil polynomials for C and A to confirm that the resulting Weil polynomial for C' is not ruled out. This yields the following.

Lemma 6.1 *For $q = 2$, for $g = 0, \dots, 6$ we have $g' \leq 4, 6, 8, 10, 12, 14, 16$, respectively. Hence by (5.1), if $d \geq 4$ then $g \leq 3$; and if $d \geq 5$ then $g \leq 2$.*

Proof From Lemma 5.3, for $g = 0, \dots, 6$ we obtain $g' \leq 4, 7, 9, 11, 13, 15, 17$, respectively. We rule out the pairs $(g, g') \in \{(1, 7), (2, 9), (3, 11), (4, 13), (5, 15), (6, 17)\}$ by exhausting over Weil polynomials as described above. □

We can now prove Theorem 1.2 as follows. By (5.4), Lemma 6.1, and Table 1, for $(q, g) = (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1)$ we have respectively $g' \leq 4, 6, 1, 3, 1, 3$. We may settle all cases by table lookups except $(q, g, g') = (2, 1, 6)$, which we settle as follows.

- The isogeny class 6.2.ad_c_a_a_m_abg is ruled out by [1, Proposition 5.2], whose proof we summarize. By the resultant-2 criterion (compare Remark 10.3), C' is a double cover of a curve C_0 with real Weil polynomial $T^2 - 2T - 2$; this is inconsistent with $\#C_0(\mathbb{F}_2) = 1, \#C'(\mathbb{F}_4) = 0$.
- The isogeny class 6.2.ad_c_a_f_am_q occurs for a cyclic étale quintic cover of a genus-2 curve listed in Table 5 (see also Remark 6.2).

Remark 6.2 Table 3 includes a column counting Jacobians in the isogeny class of $J(C')$. This can be obtained by table lookups except for 6.2.ad_c_a_f_am_q, for which Table 3 reports a *unique* Jacobian; this will be proved in [25, Lemma 10.2].

7 Additional constraints on Weil polynomials

We assume hereafter that $g > 1$ and introduce constraints on the Weil polynomials of C and C' based on d. Note that none of these presumes $h_{F'/F} = 1$, and so may be applicable in other cases of interest.

We start with the full form of Riemann–Hurwitz:

$$2g' - 2 = d(2g - 2) + 2\delta, \qquad 2\delta = \sum_P (e_P - 1) \tag{7.1}$$

where P runs over geometric points of C' and e_P is the ramification index at P.

Let t denote the number of geometric ramification points, i.e., the number of P for which $e_P > 1$. Then $t = 0$ iff $\delta = 0$, and $t \leq 2\delta$ in general. If q is even, then e_P can never equal 2, so $t \leq \delta$; in particular,

$$\delta = 1 \Longrightarrow t = 1 \Longrightarrow \#C'(\mathbb{F}_q) \geq 1 \tag{7.2}$$

because the unique ramification point of C' is $\mathbb{F}_q$-rational, and similarly

$$\delta = 2 \Longrightarrow \#C'(\mathbb{F}_{q^2}) \geq 1, \qquad t = 2 \Longrightarrow \#C'(\mathbb{F}_{q^2}) \geq 2. \tag{7.3}$$

If $C' \to C$ is cyclic of prime degree $d = p \mid q$, then the Deuring–Shafarevich formula holds (e.g., see [37]): for $\gamma_C, \gamma_{C'}$ the p-ranks of C, C',

$$\gamma_{C'} - 1 = d(\gamma_C - 1) + t \tag{7.4}$$

If $\delta = 0$ and $C' \to C$ is cyclic (e.g., if $d = 2$), then by class field theory,

$$\#J(C)(\mathbb{F}_q) \equiv 0 \pmod{d}. \tag{7.5}$$

For small d, we have the following additional constraints (building on [38, Lemma 8]).

- When $d = 2$, every $\mathbb{F}_{q^i}$-rational point of C lifts to either an $\mathbb{F}_{q^i}$-rational ramification point or two $\mathbb{F}_{q^{2i}}$-rational points of C'. Hence

$$\#C'(\mathbb{F}_{q^{2i}}) \geq 2\#C(\mathbb{F}_{q^i}) - t; \tag{7.6}$$

by (5.2) and (5.3), this yields

$$2T_{A,q^i} + T_{A,q^{2i}} - t \leq 2\#C(\mathbb{F}_{q^i}) + T_{A,q^{2i}} - t \leq \#C(\mathbb{F}_{q^{2i}}). \tag{7.7}$$

For $i = 2j - 1$ odd, every degree i-place of C' projects to a degree-i place of C. If $t \leq 2$, then for $i > 1$ these points occur in pairs in fibers, and so

$$t \leq 2 \Longrightarrow \#C'(\mathbb{F}_{q^{2j-1}}) \equiv \#C'(\mathbb{F}_q) \pmod{2} \qquad (j > 0). \tag{7.8}$$

- When $d = 3$, every $\mathbb{F}_{q^i}$-rational point of C lifts to either at least one $\mathbb{F}_{q^i}$-rational point or three $\mathbb{F}_{q^{3i}}$-rational points of C'. Hence $\#C'(\mathbb{F}_{q^{3i}}) - \#C'(\mathbb{F}_{q^i}) \geq 3(\#C(\mathbb{F}_{q^i}) - \#C'(\mathbb{F}_{q^i}))$; by (5.3), this yields

$$\#C(\mathbb{F}_{q^i}) + 2T_{A,q^i} + T_{A,q^{3i}} \leq \#C(\mathbb{F}_{q^{3i}}). \tag{7.9}$$

- When $d = 4$, every $\mathbb{F}_{q^i}$-rational point of C lifts to at least one $\mathbb{F}_{q^i}$-rational point, two $\mathbb{F}_{q^{2i}}$-rational ramification points, or four $\mathbb{F}_{q^{4i}}$-rational points of C'. Hence $\#C'(\mathbb{F}_{q^{4i}}) \geq 4(\#C(\mathbb{F}_{q^i}) - \#C'(\mathbb{F}_{q^i})) - 2t$; by (5.3), this yields

$$4T_{A,q^i} + T_{A,q^{4i}} - 2\delta \leq 4T_{A,q^i} + T_{A,q^{4i}} - 2t \leq \#C(\mathbb{F}_{q^{4i}}). \tag{7.10}$$

Remark 7.1 For $d = 2$, the compositum $F' \cdot \mathbb{F}_{q^2}$ contains another purely geometric quadratic extension F''/F. We call the corresponding cover $C'' \to C$ the *relative quadratic twist* of $C' \to C$; it also obeys the conditions listed in §6.

8 Purely geometric extensions: q>2

We settle Theorem 1.3(a) as follows. For $q > 2$, Lemma 5.1 implies $g \leq 6$. If $d = 2$, then by Lemma 2.3 plus (7.7),

$$\begin{aligned} \#C(\mathbb{F}_{q^2}) \geq 2T_{A,q} + T_{A,q^2} - c(g' - 2g + 1) &= q^2(g' - g) - c(g' - 2g + 1) \\ &\geq q^2(g - 1). \end{aligned} \tag{8.1}$$

Combining (8.1) with Table 1, we deduce that

$$(q, g, g') \in \{(3, 2, 3), (3, 2, 4), (3, 3, 5), (3, 3, 6), (3, 4, 7), (4, 2, 3), (4, 2, 4), (4, 3, 5)\}.$$

If $d > 2$, then by upgrading (5.4) using Table 1, we deduce that $(d, g, g') = (3, 2, 4)$. We also have the following.

Lemma 8.1 *If $q > 2$, $g = 2$, and $d = 3$, then $C' \to C$ is cyclic.*

Proof Suppose first that $C' \to C$ is a non-Galois cover which becomes Galois after a quadratic constant field extension. By Lemma 2.3, the quadratic twist $\tilde{C}$ of C admits a cyclic cubic étale cover $\tilde{C}'$ whose Prym has Weil polynomial $(T^2+qT+q)^2$. Since each $\mathbb{F}_q$-point of $\tilde{C}$ lifts to at most three $\mathbb{F}_q$-points of $\tilde{C}'$, we have $\#\tilde{C}(\mathbb{F}_q)+2q = \tilde{C}'(\mathbb{F}_q) \leq 3\#\tilde{C}(\mathbb{F}_q)$ and so $\tilde{C}(\mathbb{F}_q) \geq q$. However, $\#C(\mathbb{F}_q) \geq 2q$ by (5.3), yielding the impossibility

$$2q + 2 = \#C(\mathbb{F}_q) + \#\tilde{C}(\mathbb{F}_q) \geq 3q.$$

Suppose next that $C' \to C$ is geometrically non-Galois. In this case, the Galois closure F'' of F'/F is itself the function field of a curve C'' with $q_{F''} = q_F$. The abelian variety $J(C'')$ is isogenous to $J(C) \times A^2 \times E$ for some elliptic curve E, so

$$\#C''(\mathbb{F}_q) = \#C(\mathbb{F}_q) - 2T_{A,q} - T_{E,q} = \#C(\mathbb{F}_q) - 4q - T_{E,q} \leq \#C(\mathbb{F}_q) - 3q;$$

this yields $\#C(\mathbb{F}_q) \geq 3q$, which is inconsistent with Table 1. □

We now know that in all cases $C' \to C$ is cyclic, so we may proceed as follows.

- We again exhaust over Weil polynomials for C and A, but this time accounting for (7.4), (7.5), (7.7), (7.9), (8.1). At this point the cases $(q, d, g, g') = (3, 2, 3, 6), (4, 2, 2, 4)$ drop out.
- For each candidate Weil polynomial for C, we consult LMFDB to find all candidates for C. At this point the case $(q, d, g, g') = (3, 2, 4, 7)$ drops out: the only isogeny class for $J(C)$ is 4.3.f_v_ca_eg, which contains no Jacobian.
- We then use Magma to compute all cyclic extensions of F with the desired degree and ramification behavior and check the resulting Weil polynomial for A. At this point the case $(q, d, g, g') = (4, 2, 3, 5)$ drops out.

This yields Theorem 1.3(a).

9 A refined resultant criterion

In preparation for the case $q = 2$, we next introduce a refinement of the resultant criteria, modeled on [36, Proposition 2.8] (applicable over any finite base field).

Lemma 9.1 *Let $f : C' \to C$ be a finite flat morphism of degree d between smooth projective curves over an arbitrary field k. Let $f^* : J(C) \to J(C')$ denote the pullback map and let $f_* : J(C') \to J(C)$ denote the pushforward map. Let A be the Prym variety of f, defined as the reduced closed subscheme of the identity component of $\ker(f_*)$. Then there is an exact sequence*

$$0 \to \Delta \to J(C) \times_k A \to J(C') \to 0 \tag{9.1}$$

where the map $J(C) \to J(C')$ is f^* and Δ is a finite flat group scheme killed by d.

Proof The composition $J(C) \xrightarrow{f^*} J(C') \xrightarrow{f_*} J(C)$ equals the isogeny $[d]$; consequently, f_* is surjective (as a morphism of group schemes) and $J(C) \xrightarrow{f^*} J(C') \to J(C')/\ker(f_*)$ is surjective. The latter implies that $J(C) \times_k \ker(f_*) \to J(C')$ is surjective, as then is $\ker(f_*) \to \operatorname{coker}(f^*)$; since the target is connected and reduced, $A \to \operatorname{coker}(f^*)$ is surjective, as then is $J(C) \times_k A \to J(C')$.

Let S be an arbitrary k-scheme and suppose $x \in (J(C) \times_k A)(S)$ maps to zero to $J(C')$. Write $x = (x_1, x_2)$ with $x_1 \in J(C)(S)$ and $x_2 \in A(S)$. By definition, x_1 and $-x_2$ have the same image in $J(C')(S)$; that is, $f^*(x_1) = -x_2$. Applying f_*, we deduce that $f_* f^*(x_1) = 0$, and so $[d](x_1) = 0$; it follows that $[d](x) = ([d](x_1), [d](x_2)) = (0, [d](x_2))$ maps to zero in $J(C')$, and hence $[d](x_2) = 0$. □

Corollary 9.2 *In Lemma 9.1, let h_1 and h_2 be the radicals of the real Weil polynomials associated to $J(C)$ and A. Let $\widetilde{\mathrm{res}}(h_1, h_2)$ be the modified reduced resultant of h_1 and h_2 in the sense of* [36, Proposition 2.8]. *Then*

$$\gcd(d, \widetilde{\mathrm{res}}(h_1, h_2)) > 1. \tag{9.2}$$

Proof In (9.1), Δ cannot be trivial: otherwise, $J(C')$ would be decomposable as a principally polarized abelian variety, violating Torelli [39, Theorem 12.1]. The exponent of Δ divides d by Lemma 9.1 and $\widetilde{\mathrm{res}}(h_1, h_2)$ by [36, Proposition 2.8]. □

10 Purely geometric extensions: $q = 2$

To conclude, we establish parts (b) and (c) of Theorem 1.3.

Lemma 10.1 *If $q = 2$ and $d = 2$, then $g \leq 9$. Moreover, for $g = 2, \ldots, 9$ we have respectively $g' \leq 7, 9, 10, 11, 13, 14, 15, 17$.*

Proof Combining (5.1), Lemma 5.2,and (7.7) yields

$$\begin{aligned}
1.5612(g-1) &\leq 1.5612(g'-g) - 0.3366\delta \\
&\leq T_{A,q} + 0.3366(T_{A,q} + T_{A,q^2} - \delta) \\
&\quad + 0.1137(T_{A,q^3} - T_{A,q}) + 0.0537(T_{A,q^4} - T_{A,q^2}) \\
&\leq (1 - 0.3366 - 0.1137 + 0.0537)T_{A,q} \\
&\quad + (0.3366 - 0.0537)(T_{A,q} + T_{A,q^2} - t) + 0.1137 T_{A,q^3} + 0.0537 T_{A,q^4} \\
&\leq (1 - 0.3366 - 0.1137 + 0.0537)\#C(\mathbb{F}_q) \\
&\quad + (0.3366 - 0.0537)(\#C(\mathbb{F}_{q^2}) - \#C(\mathbb{F}_q)) \\
&\quad + 0.1137\#C(\mathbb{F}_{q^3}) + 0.0537\#C(\mathbb{F}_{q^4}) \\
&= a_1 + 0.3366(2a_2) + 0.1137(3a_3) + 0.0537(4a_4) \leq 0.8042g + 5.619.
\end{aligned}$$

This yields the claimed results. □

Lemma 10.2 *Suppose that $q = 2$ and $g > 1$.*

(a) *If $d = 2$, then*

$$(g, g') \in \{(2, 3), (2, 4), (2, 5), (3, 5), (3, 6), (4, 7), (4, 8), (5, 9), (6, 11), (7, 13)\}.$$

(b) *If $d = 3$, then $(g, g') \in \{(2, 4), (2, 6), (3, 7), (4, 10)\}$.*
(c) *If $d = 4$, then $(g, g') \in \{(2, 5), (2, 6), (3, 9)\}$.*
(d) *If $d > 4$, then $g = 2$ and $(d, g') \in \{(5, 6), (6, 7), (7, 8)\}$.*

Proof We run an exhaustive search over Weil polynomials as in §6, but also accounting for (7.2), (7.3), (7.5) (for $d = 2$), (7.6) (taking $i = 1, 2, 3$), (7.9) (taking $i = 1, 2$), (7.10) (taking $i = 1$), and (9.2). This rules out

$$(d, g, g') \in \{(2,2,6),(2,2,7),(2,3,7),(2,3,8),(2,3,9),(2,4,9),(2,4,10),(2,4,11),\\ (2,5,10),(2,5,11),(2,6,12),(2,6,13),(2,7,14),(2,8,15),(2,9,17),\\ (3,2,5),(3,2,7),(3,2,8),(3,3,8),(3,3,9),(3,3,10),(3,4,11),(3,4,12),\\ (3,5,13),(3,5,14),(3,6,16),(4,2,7),(4,2,8),(4,3,10),(5,2,7),(5,2,8),(6,2,8)\}.$$

(The runtime is dominated by the cases $(d, g, g') = (2, 8, 15), (2, 9, 17)$.) We may thus deduce (a) from (5.1) and Lemma 10.1, (b) from Lemma 5.4 and Lemma 6.1, and (c) and (d) from Lemma 6.1. □

We obtain Theorem 1.3(b) by a similar calculation which also accounts for (7.8) (taking $j = 2$), Remark 7.1, and the following Remark 10.3.

Remark 10.3 If $C' \to C$ is étale and geometrically cyclic (i.e., cyclic after base extension from $\mathbb{F}_2$ to an algebraic closure), we can upgrade Lemma 9.1 to say that Δ has exponent exactly d (because $\ker(f^*)$ is étale and cyclic of order d; compare (7.5)), and Corollary 9.2 to say that $\widetilde{\mathrm{res}}(h_1, h_2)$ must be divisible by d.

If we drop these conditions on $C' \to C$, we can still say something when $\gcd(d, \widetilde{\mathrm{res}}(h_1, h_2)) = 2$: as in [36, Theorem 2.2] there must be a degree-2 map from C' to another curve D whose Jacobian is isogenous to $J(C)$ or A. By (5.1), the second option cannot occur if $g' > 2g + 1$; in characteristic 2, (7.4) also applies.

In the context of Theorem 1.3(b), the condition that $\gcd(d, \widetilde{\mathrm{res}}(h_1, h_2)) = 2$ rules out some cases with $(d, g, g') \in \{(4, 2, 6), (4, 3, 9), (6, 2, 7)\}$: there would have to be a double cover $C' \to D$ with $J(D)$ isogenous to $J(C)$, but this is forbidden by Lemma 10.2(a). Similarly, if $(d, g, g') = (4, 2, 5)$, then $J(D)$ cannot be isogenous to A: otherwise D would admit an étale double cover while $\#J(D)(\mathbb{F}_2) = 1$. Hence $J(C), J(C')$ must occur in Theorem 1.3(c) with $(d, g, g') = (2, 2, 5)$.

Remark 10.4 When $d = 2$ and $\delta \leq 1$, A admits a principal polarization; over $\mathbb{C}$ this is classical [40, Theorem 12.3.3], and a characteristic-free argument will appear in [41]. Our formulation of Theorem 1.3(b) does not account for this constraint; it would rule out a further 16 pairs, which are marked with stars in Table 7.

To obtain Theorem 1.3(c), we use table lookups to find candidates for C with a given Weil polynomial (see §6), then use Magma to enumerate cyclic extensions. As a consistency check, for each triple (d, g, g') listed in Lemma 10.2 with $g \leq 5$, we enumerated cyclic extensions for *all* curves C of genus g; this took about 14 h and yielded no new results.

Data Availibility Statement All data generated during this project is fully reproducible using the code available from [33].

Declarations

conflict of interests
The author asserts that there are no conflicts of interest.

Appendix A: Extensions of relative class number 1

Table 3 Purely geometric extensions with $g_F \leq 1, g_{F'} > g_F$. The column $\#C'$ counts Jacobians in the isogeny class. The star indicates a conjectural value; see Remark 6.2

q_F	g_F	$g_{F'}$	$J(C)$	$J(C')$	$\#C'$
2	0	1	0	1.2.ac	1
2	0	2	0	2.2.ad_f	1
2	0	2	0	2.2.ac_c	1
2	0	3	0	3.2.ad_c_b	1
2	0	3	0	3.2.ad_d_ac	1
2	0	4	0	4.2.ad_c_a_b	1
2	1	2	1.2.a	2.2.ac_e	1
2	1	2	1.2.b	2.2.ab_c	1
2	1	2	1.2.c	2.2.a_a	1
2	1	3	1.2.ac	3.2.ad_d_ac	1
2	1	3	1.2.ab	3.2.ad_g_ak	1
2	1	3	1.2.ab	3.2.ac_c_ad	1
2	1	3	1.2.b	3.2.ad_g_ai	1
2	1	3	1.2.b	3.2.ac_e_ah	1
2	1	4	1.2.a	4.2.ad_e_af_i	1
2	1	4	1.2.a	4.2.ad_f_ai_m	2
2	1	4	1.2.c	4.2.ad_f_ag_i	1
2	1	4	1.2.c	4.2.ac_c_ae_i	2
2	1	5	1.2.b	5.2.ad_c_d_ag_h	3
2	1	5	1.2.b	5.2.ad_c_e_ai_i	3
2	1	5	1.2.b	5.2.ad_e_ag_k_ao	3
2	1	6	1.2.c	6.2.ad_c_a_f_am_q	1*
3	0	1	0	1.3.ab	1
3	1	2	1.3.ab	2.3.ae_j	1
3	1	2	1.3.a	2.3.ad_g	1
3	1	2	1.3.b	2.3.ac_d	2
3	1	2	1.3.c	2.3.ab_a	1
3	1	3	1.3.c	3.3.ae_g_ag	1
3	1	3	1.3.d	3.3.ad_a_j	2
4	0	1	0	1.4.ae	1
4	1	2	1.4.a	2.4.ae_i	1

Table 4 Purely geometric extensions with $q_F > 2$ and $g_F > 1$

q_F	d	g_F	$g_{F'}$	$J(C)$	F
3	2	2	3	2.3.ab_c	$y^2 + x^5 + 2x^2 + x$
3	2	2	3	2.3.ab_e	$y^2 + x^6 + x^4 + 2x^3 + x^2 + 2x$
3	2	2	3	2.3.b_c	$y^2 + 2x^5 + x^2 + 2x$
3	2	2	3	2.3.b_e	$y^2 + 2x^6 + 2x^4 + x^3 + 2x^2 + x$
3	2	2	4	2.3.c_h	$y^2 + 2x^6 + x^4 + 2x^3 + x^2 + 2$
3	2	3	5	3.3.c_g_i	$y^2 + 2x^8 + x^7 + x^5 + x^3 + 2x^2 + 2x$
3	2	3	5	3.3.c_g_m	$y^2 + x^7 + 2x^5 + x^4 + x^3 + x^2 + 2$
3	3	2	4	2.3.c_d	$y^2 + 2x^6 + 2x^4 + x^3 + x + 2$
3	3	2	4	2.3.c_g	$y^2 + 2x^6 + 2x^5 + x^4 + x^3 + x^2 + 2x + 2$
4	2	2	3	2.4.ab_e	$y^2 + xy + x^5 + x$
4	2	2	3	2.4.b_e	$y^2 + xy + x^5 + ax^2 + x$
4	3	2	4	2.4.d_h	$y^2 + (x^3 + x + 1)y + ax^5 + ax^4 + ax^3 + ax$

Table 5 Cyclic purely geometric extensions with $q_F = 2, g_F > 1, d > 2$. Conjecture 1.5 asserts that no noncyclic extensions occur

d	g_F	$g_{F'}$	$J(C)$	F
3	2	4	2.2.ac_e	$y^2 + y + x^5 + x^4 + 1$
3	2	4	2.2.b_b	$y^2 + (x^3 + x + 1)y + x^6 + x^3 + x^2 + x$
3	2	6	2.2.a_c	$y^2 + y + x^5 + x^4 + x^3$
3	2	6	2.2.b_c	$y^2 + xy + x^5 + x^3 + x^2 + x$
3	2	6	2.2.b_d	$y^2 + (x^3 + x + 1)y + x^6 + x^5 + x^4 + x^2$
3	3	7	3.2.a_b_a	$y^4 + (x^3 + 1)y + x^4$
3	3	7	3.2.a_b_d	$y^3 + x^2y^2 + x^3y + x^4 + x^3 + x$
3	3	7	3.2.b_b_b	$y^3 + xy^2 + (x^3 + 1)y + x^4$
3	3	7	3.2.b_b_e	$y^3 + (x^2 + x)y^2 + y + x^3$
3	3	7	3.2.b_c_b	$xy^3 + xy^2 + y + x^3$
3	3	7	3.2.b_c_e	$y^4 + xy^2 + y + x^4$
3	3	7	3.2.b_e_e	$y^3 + x^2y^2 + xy + x^4 + x$
3	4	10	4.2.d_f_k_s	$x^2y^3 + (x^4 + x^2 + 1)y + x^4 + x^2 + x + 1$
3	4	10	4.2.e_j_q_z	$xy^3 + (x^2 + x + 1)y^2 + (x^4 + x)y + x^5 + x^4$
4	2	5	2.2.ab_c	$y^2 + xy + x^5 + x^3 + x$
5	2	6	2.2.a_a	$y^2 + y + x^5$
5	2	6	2.2.b_c	$y^2 + xy + x^5 + x^3 + x^2 + x$
5	2	6	2.2.c_e	$y^2 + y + x^5 + x^4$
7	2	8	2.2.c_d	$y^2 + (x^2 + x + 1)y + x^5 + x^4 + x^2 + x$

Table 6 Purely geometric extensions with $q_F = 2, g_F > 1, d = 2$. Completeness of the list is confirmed above the double line and conjectural below it (Conjecture 1.5). For $g_F = 7$, $J(C)$ does not appear in LMFDB, so we list $J(C)(\mathbb{F}_{2^i})$ for $i = 1, \ldots, 7$

g_F	$g_{F'}$	$J(C)$	F
2	3	2.2.ab_c	$y^2 + xy + x^5 + x^3 + x$
2	3	2.2.b_c	$y^2 + xy + x^5 + x^3 + x^2 + x$
2	4	2.2.a_a	$y^2 + y + x^5$
2	4	2.2.a_c	$y^2 + y + x^5 + x^4 + x^3$
2	4	2.2.b_b	$y^2 + (x^3 + x + 1)y + x^6 + x^3 + x^2 + x$
2	5	2.2.b_d	$y^2 + (x^3 + x + 1)y + x^6 + x^5 + x^4 + x^2$
2	5	2.2.c_e	$y^2 + y + x^5 + x^4$
3	5	3.2.ad_g_ai	$y^2 + (x^4 + x^2 + 1)y + x^8 + x + 1$
3	5	3.2.ab_a_c	$xy^3 + (x^2 + x)y^2 + y + x^4$
3	5	3.2.ab_a_c	$xy^3 + x^2y^2 + (x^2 + 1)y + x^4$
3	5	3.2.ab_c_ac	$y^2 + xy + x^7 + x^5 + x$
3	5	3.2.a_a_f	$xy^3 + y + x^3$
3	5	3.2.a_c_ab	$y^2 + (x^4 + x^2 + x + 1)y + x^6 + x^5 + x^2 + 1$
3	5	3.2.a_c_b	$y^2 + (x^4 + x^2 + x + 1)y + x^8 + x^6 + x^5 + x^4$
3	5	3.2.b_c_c	$y^2 + xy + x^7 + x^5 + x^2 + x$
3	5	3.2.b_c_e	$y^2 + (x^4 + x^2)y + x^2 + x$
3	6	3.2.b_d_c	$y^3 + x^2y^2 + x^2y + x^4 + x^3 + x^2 + x$
3	6	3.2.b_d_e	$xy^3 + (x + 1)y^2 + x^4 + x^3 + x$
3	6	3.2.b_e_d	$y^3 + x^2y^2 + (x^3 + x^2)y + x^4 + x$
3	6	3.2.c_d_d	$(x + 1)y^3 + y + x^3$
4	7	4.2.a_c_ab_c	$x^3y^3 + (x^3 + x^2)y + x^6 + x^3 + 1$
4	7	4.2.a_c_ab_g	$(x^2 + 1)y^4 + (x^3 + x^2 + x + 1)y^3 + (x^5 + x^4)y + x^6 + x^3 + x^2$
4	7	4.2.a_c_b_c	$x^2y^4 + (x^3 + 1)y^2 + (x^3 + x^2 + x + 1)y + x^6 + x^5 + x^3 + x^2$
4	7	4.2.a_c_d_c	$(x^2+x+1)y^4+(x^3+x^2)y^3+(x^4+x^3+1)y^2+(x^4+x^3+x^2)y+x^5+x^4+x^3+x$
4	7	4.2.a_d_b_f	$x^3y^3 + (x^3 + x^2)y + x^6 + x^5 + 1$

Table 6 continued

g_F	$g_{F'}$	$J(C)$	F
4	7	4.2.a_d_b_h	$(x^2+x+1)y^4+x^3y^3+(x^4+x^2+1)y^2+x^5+x^3+x$
4	7	4.2.b_b_c_f	$(x+1)y^3+(x^2+x)y^2+(x^3+x)y+x^5$
4	7	4.2.b_c_a_a	$y^2+x^2y+x^9+x^7+x+1$
4	7	4.2.c_e_h_k	$y^2+(x^3+x+1)y+x^9+x^7$
4	8	4.2.d_i_o_x	$(x+1)y^3+(x^3+x^2+1)y^2+xy+x^4$
5	9	5.2.ab_d_b_b_j	$(x^4+x^3+x^2)y^4+(x^5+x^3+x)y^3+(x^3+1)y^2+(x^7+x+1)y+x^7+x^4+x+1$
5	9	5.2.b_c_e_i_i	$y^2+x^3y+x^{11}+x^9+x^5+x^3+x^2+x$
5	9	5.2.b_c_e_i_i	$y^4+(x^4+x^2)y^2+(x^4+x^2+1)y+x^8+x^6+x^4+x$
5	9	5.2.b_f_f_p_l	$(x^4+x^3+x^2)y^4+(x^5+x^3+x^2)y^3+(x^6+x^3+x^2+x+1)y^2+(x^7+x^5+x^4+x^3+1)y+x^5+x^4+x^3+x^2+x+1$
5	9	5.2.b_f_f_p_p	$y^4+x^2y^3+(x^4+x^3+x)y^2+(x^5+1)y+x^3+x^2+x+1$
5	9	5.2.c_e_f_k_o	$y^2+(x^3+x+1)y+x^{12}+x^{11}+x^{10}+x^7+x^5+x^3$
5	9	5.2.c_f_i_n_r	$y^4+(x^2+x)y^3+(x^4+x^3+x^2+1)y^2+(x^6+x^5+x^4+1)y+x^7+x^6+x+1$
5	9	5.2.c_f_i_p_t	$(x^4+x^2+x)y^4+(x^4+x^3+x+1)y^3+(x^6+x^2)y^2+(x^6+x^3+x^2+x)y+x^6+x^5+1$
5	9	5.2.c_f_i_p_v	$(x^2+x+1)y^6+xy^5+(x^4+x)y^4+(x^5+x^4+x^3+x^2+x+1)y^3+(x^5+x^3+1)y^2+(x^6+x^4+x^2)y+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x$
5	9	5.2.c_g_j_q_u	$y^4+y^3+(x^4+x^3+x^2)y^2+(x^3+x^2+1)y+x^6+1$
5	9	5.2.d_h_n_z_bl	$y^2+(x^6+x^5+x^4+x^3+x^2+x+1)y+x^{10}+x^6+x^4+x^3$
5	9	5.2.d_i_q_bc_bs	$y^4+(x^4+x^2)y^2+(x^4+x^2+1)y+x^6+x^5$
6	11	6.2.c_h_k_z_bd_cg	$x^2y^5+(x^3+x)y^4+x^4y^3+(x^5+x^4+x^3+x^2+x+1)y^2+(x^6+x^3+x^2)y+x^7+x^3+x+1$
6	11	6.2.d_j_t_bn_cl_du	$(x^2+x+1)y^4+(x^3+x+1)y^3+(x^4+x^2+1)y^2+(x^5+x^4+1)y+x^5+x^4+x^3+x$
7	13	(6, 18, 12, 18, 6, 60, 174)	$y^4+(x^6+x^4+x^3+x^2+1)y^2+(x^6+x^4+x^3+x^2)y+x^{10}+x^9+x^7+x^6$

Table 7 Candidates for A and $J(C)$ in Theorem 1.3(b). ⋆ means A is not principally polarizable (Remark 10.4). For $g=7$, $J(C)$ does not appear in LMFDB, so we list $J(C)(\mathbb{F}_{2^i})$ for $i=1,\dots,7$

(d,g,g')	A	$J(C)$
(2, 2, 3)	1.2.ac	2.2.ab_c, 2.2.b_c
(2, 2, 4)	2.2.ab_ab⋆	2.2.ab_d
(2, 2, 4)	2.2.ac_c	2.2.a_a, 2.2.a_c
(2, 2, 4)	2.2.ad_f	2.2.b_b
(2, 2, 5)	3.2.ad_d_ac	2.2.b_d
(2, 2, 5)	3.2.ae_i_am	2.2.c_e
(2, 3, 5)	2.2.a_ae	3.2.ad_g_ai
(2, 3, 5)	2.2.ac_c	3.2.ab_a_c, 3.2.ab_c_ac, 3.2.b_c_c
(2, 3, 5)	2.2.ad_f	3.2.a_a_f, 3.2.a_c_ab, 3.2.a_c_b, 3.2.c_e_f
(2, 3, 5)	2.2.ae_i	3.2.b_c_e
(2, 3, 6)	3.2.ad_c_b	3.2.b_e_d
(2, 3, 6)	3.2.ad_d_ac	3.2.b_d_c, 3.2.b_d_c, 3.2.b_d_e
(2, 3, 6)	3.2.ae_i_am	3.2.c_e_e, 3.2.c_e_g
(2, 3, 6)	3.2.ae_j_ap	3.2.c_d_d

Table 7 continued

(d, g, g′)	A	J(C)
(2, 4, 7)	3.2.ad_c_b	4.2.a_d_ab_f, 4.2.a_d_ab_h, 4.2.a_d_b_f, 4.2.a_d_b_h, 4.2.a_d_d_f
(2, 4, 7)	3.2.ad_d_ac	4.2.a_c_ab_c, 4.2.a_c_ab_e, 4.2.a_c_ab_g, 4.2.a_c_b_c, 4.2.a_c_b_e
(2, 4, 7)	3.2.ad_d_ac	4.2.a_c_b_g, 4.2.a_c_d_a, 4.2.a_c_d_c, 4.2.a_e_b_k, 4.2.c_g_j_q
(2, 4, 7)	3.2.ae_i_am	4.2.b_c_a_a, 4.2.b_c_a_c, 4.2.b_c_a_e, 4.2.b_c_c_c
(2, 4, 7)	3.2.ae_i_am	4.2.b_c_c_e, 4.2.b_c_c_g, 4.2.b_c_e_e, 4.2.b_e_c_i, 4.2.b_e_e_k
(2, 4, 7)	3.2.ae_j_ap	4.2.b_b_a_b, 4.2.b_b_a_d, 4.2.b_b_c_d, 4.2.b_b_c_f, 4.2.b_b_c_h, 4.2.b_d_c_h, 4.2.b_d_e_j
(2, 4, 7)	3.2.af_n_aw	4.2.c_e_h_k
(2, 4, 8)	4.2.ae_g_ae_c	4.2.c_g_i_q
(2, 4, 8)	4.2.af_m_au_bd	4.2.d_i_o_x
(2, 4, 8)	4.2.af_n_az_bn⋆	4.2.d_h_l_r
(2, 5, 9)	4.2.ac_ab_ac_n	5.2.ab_d_b_b_j
(2, 5, 9)	4.2.ad_c_a_b	5.2.a_d_c_j_d
(2, 5, 9)	4.2.ad_d_ag_o	5.2.a_c_d_e_g
(2, 5, 9)	4.2.ae_f_c_al	5.2.b_f_f_p_l, 5.2.b_f_f_p_n, 5.2.b_f_f_p_p
(2, 5, 9)	4.2.ae_g_ae_c	5.2.b_e_c_i_a, 5.2.b_e_c_i_c, 5.2.b_e_c_k_e, 5.2.b_e_e_k_i
(2, 5, 9)	4.2.ae_g_ae_c	5.2.b_e_e_k_k, 5.2.b_e_e_k_m, 5.2.b_e_e_m_k, 5.2.b_e_e_m_m, 5.2.b_e_g_m_q
(2, 5, 9)	4.2.ae_h_ak_p	5.2.b_d_d_h_d, 5.2.b_d_d_h_f, 5.2.b_d_d_h_h, 5.2.b_d_d_h_j
(2, 5, 9)	4.2.ae_h_ak_p	5.2.b_d_d_j_h, 5.2.b_d_d_j_j, 5.2.b_d_d_j_l, 5.2.b_d_f_j_j, 5.2.b_d_f_j_l
(2, 5, 9)	4.2.ae_i_aq_bc	5.2.b_c_e_i_
(2, 5, 9)	4.2.af_l_ao_q	5.2.c_g_j_q_u, 5.2.c_g_j_s_w, 5.2.c_g_j_u_y, 5.2.c_g_l_u_bc
(2, 5, 9)	4.2.af_m_au_bd	5.2.c_f_g_l_l, 5.2.c_f_g_n_p, 5.2.c_f_i_n_r, 5.2.c_f_i_n_t, 5.2.c_f_i_p_t, 5.2.c_f_i_p_v, 5.2.c_f_k_r_z
(2, 5, 9)	4.2.af_n_aba_bq	5.2.c_e_f_k_m, 5.2.c_e_f_k_o, 5.2.c_e_f_m_q
(2, 5, 9)	4.2.af_n_az_bn⋆	5.2.c_e_e_g_f, 5.2.c_e_e_g_h, 5.2.c_e_e_i_j, 5.2.c_e_e_k_n, 5.2.c_e_g_k_l, 5.2.c_e_g_k_n, 5.2.c_e_g_k_p
(2, 5, 9)	4.2.af_n_az_bn⋆	5.2.c_e_g_k_r, 5.2.c_e_g_m_p, 5.2.c_e_g_m_r, 5.2.c_e_g_m_t, 5.2.c_e_i_o_t, 5.2.c_e_i_o_v, 5.2.c_g_i_s_v
(2, 5, 9)	4.2.ag_s_abk_ce	5.2.d_i_q_bc_bs
(2, 5, 9)	4.2.ag_t_abp_co	5.2.d_h_o_z_bk, 5.2.d_h_o_z_bm
(2, 5, 9)	4.2.ag_t_abq_cr	5.2.d_h_n_z_bl
(2, 6, 11)	5.2.ae_e_a_l_abh	6.2.b_g_i_v_ba_bz
(2, 6, 11)	5.2.ae_f_ae_p_abi	6.2.b_f_h_p_t_bk, 6.2.b_f_h_p_v_b, 6.2.b_f_h_r_v_bq
(2, 6, 11)	5.2.af_k_ak_f_ac	6.2.c_h_k_z_bd_cg
(2, 6, 11)	5.2.af_l_as_bg_aca	6.2.c_g_l_w_bg_ca, 6.2.c_g_l_w_bg_cc, 6.2.c_g_l_w_bg_ce
(2, 6, 11)	5.2.af_l_as_bg_aca	6.2.c_g_l_w_bi_ca, 6.2.c_g_l_w_bi_cc, 6.2.c_g_l_w_bi_ce
(2, 6, 11)	5.2.af_m_aw_bk_acb	6.2.c_f_i_q_v_bh, 6.2.c_f_i_q_v_bj, 6.2.c_f_i_q_v_bl, 6.2.c_f_i_q_x_bj
(2, 6, 11)	5.2.af_m_aw_bk_acb	6.2.c_f_i_q_x_bl, 6.2.c_f_i_q_x_bn, 6.2.c_f_i_q_x_bp, 6.2.c_f_i_q_z_bn
(2, 6, 11)	5.2.af_m_aw_bk_acb	6.2.c_f_i_q_z_bp, 6.2.c_f_i_s_z_bp, 6.2.c_f_i_s_z_br
(2, 6, 11)	5.2.ag_r_abg_bx_acs	6.2.d_j_r_bh_bx_cy, 6.2.d_j_r_bh_bx_da, 6.2.d_j_r_bh_bz_dc, 6.2.d_j_r_bj_cb_d
(2, 6, 11)	5.2.ag_r_abg_bx_acs	6.2.d_j_r_bj_cd_dm, 6.2.d_j_t_bn_cl_ds, 6.2.d_j_t_bn_cl_du, 6.2.d_j_t_bn_cl_dw
(2, 6, 11)	5.2.ag_t_abt_di_afe	6.2.d_h_m_x_bi_ca, 6.2.d_h_m_x_bk_ce, 6.2.d_h_m_x_bm_c
(2, 7, 13)	6.2.ag_p_aw_bh_acu_ey	(6, 18, 12, 18, 6, 60, 174), (6, 18, 12, 18, 6, 72, 132), (6, 18, 12, 18, 6, 84, 90)
(2, 7, 13)	6.2.ah_y_ace_ea_agn_jq	(7, 15, 7, 31, 12, 69, 126), (7, 15, 7, 31, 22, 45, 112)
(2, 7, 13)	6.2.ah_y_ace_ea_agn_jq	(7, 15, 7, 31, 22, 57, 70), (7, 15, 7, 31, 22, 57, 84)

Table 7 continued

(d, g, g')	A	$J(C)$
(3, 2, 4)	2.2.ab_ab	2.2.ac_e
(3, 2, 4)	2.2.ad_f	2.2.a_ab, 2.2.a_c
(3, 2, 4)	2.2.ae_i	2.2.b_b, 2.2.c_c
(3, 3, 6)	4.2.ad_b_g_am	2.2.a_c
(3, 3, 6)	4.2.ae_e_h_av	2.2.b_d
(3, 3, 6)	4.2.ae_f_c_al	2.2.b_c
(3, 3, 6)	4.2.af_l_ao_q	2.2.c_c
(3, 3, 7)	4.2.ac_ac_e_a	3.2.ab_c_a
(3, 3, 7)	4.2.ad_b_g_am	3.2.a_b_a, 3.2.a_b_d
(3, 3, 7)	4.2.ae_e_h_av	3.2.b_c_b, 3.2.b_c_e
(3, 3, 7)	4.2.ae_e_i_ay	3.2.b_c_a
(3, 3, 7)	4.2.ae_e_i_ay	3.2.b_c_d, 3.2.b_d_e
(3, 3, 7)	4.2.ae_e_i_ay	3.2.c_e_h, 3.2.d_h_l
(3, 3, 7)	4.2.ae_f_c_al	3.2.b_b_b, 3.2.b_b_e
(3, 3, 7)	4.2.ae_f_c_al	3.2.b_c_d, 3.2.b_e_e
(3, 3, 7)	4.2.ae_f_c_al	3.2.c_d_f, 3.2.c_e_h
(3, 4, 10)	6.2.ag_p_ar_ag_cg_aei	4.2.d_f_i_n
(3, 4, 10)	6.2.ag_p_at_g_bb_acj	4.2.d_f_k_s
(3, 4, 10)	6.2.ah_v_abe_a_dk_ahc	4.2.e_j_q_z, 4.2.e_k_u_bg
(3, 4, 10)	6.2.ai_bc_abw_m_ey_alc	4.2.f_o_bc_bs
(4, 2, 5)	3.2.ac_ac_i	2.2.ab_c
(4, 2, 5)	3.2.ae_i_am	2.2.b_a
(4, 2, 5)	3.2.ae_i_am	2.2.b_c, 2.2.c_e
(4, 2, 6)	4.2.ae_e_i_ay	2.2.c_e
(4, 3, 9)	6.2.af_i_ab_ag_an_br	3.2.c_e_f
(4, 3, 9)	6.2.ag_o_am_am_bw_adc	3.2.d_g_i
(5, 2, 6)	4.2.ad_b_g_am	2.2.a_b
(5, 2, 6)	4.2.ad_c_a_b	2.2.a_a
(5, 2, 6)	4.2.ae_e_h_av	2.2.b_c
(5, 2, 6)	4.2.ae_e_i_ay	2.2.b_d
(5, 2, 6)	4.2.ae_h_ak_p	2.2.b_c
(5, 2, 6)	4.2.af_l_ao_q	2.2.c_e, 2.2.d_f
(5, 2, 6)	4.2.af_n_az_bn	2.2.c_e
(6, 2, 7)	5.2.ae_e_e_am_q	2.2.b_c, 2.2.c_e
(6, 2, 7)	5.2.af_k_ak_f_ac	2.2.c_c, 2.2.c_d
(6, 2, 7)	5.2.af_l_as_bg_aca	2.2.c_c
(6, 2, 7)	5.2.ag_q_aba_bh_abr	2.2.d_f
(6, 2, 7)	5.2.ag_r_abg_bx_acs	2.2.d_f
(7, 2, 8)	6.2.af_j_ah_d_ab_ab	2.2.c_d

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References

1. Faber, X., Grantham, J., Howe, E.W.: On the maximum gonality of a curve over a finite field. https://arxiv.org/abs/2207.14307
2. Bae, S., Kang, P.: Class numbers of cyclotomic function fields. Acta Arith. **102**, 251–259 (2002)
3. Jung, H., Ahn, J.: Divisor class number one problem for abelian extensions over rational function fields. J. Algebra **310**, 1–14 (2007)
4. Kida, M., Murabayashi, N.: Cyclotomic function fields with divisor class number one. Tokyo J. Math. **14**, 45–56 (1991)

5. Leitzel, J.R.C., Madan, M.L.: Algebraic function fields with equal class number. Acta Arith. **30**, 169–177 (1976)
6. Macrae, R.: On unique factorization in certain rings of algebraic functions. J. Algebra **17**, 243–261 (1971)
7. Stark, H.: Some effective cases of the Brauer-Siegel theorem. Invent. Math. **23**, 135–152 (1974)
8. Odlyzko, A.M.: Some analytic estimates of class numbers and discriminants. Invent. Math. **29**, 275–286 (1975)
9. Hofmann, T., Sircana, C.: Normal CM-fields with class number one. https://arxiv.org/abs/2011.12089
10. Leitzel, J.R.C., Madan, M.L., Queen, C.S.: Algebraic function fields with small class number. J. Number Theory **7**, 11–27 (1975)
11. Mercuri, P., Stirpe, C.: Classification of algebraic function fields with class number one. J. Number Theory **154**, 365–374 (2015)
12. Shen, Q., Shi, S.: Function fields of class number one. J. Number Theory **154**, 375–389 (2015)
13. Stirpe, C.: A counterexample to Algebraic function fields with small class number. J. Number Theory **143**, 402–404 (2014)
14. Booher, J., Voloch, J.F.: Recovering affine curves over finite fields from L-functions. Pac. J. Math. **314**, 1–28 (2021)
15. Sutherland, A.V., Voloch, F.: Maps between curves and arithmetic obstructions. In: Arithmetic Geometry: Computations and Applications, Contemporary Mathematics, vol. 722, 167–175. American Mathematical Society (2019)
16. Madan, M.L., Pal, S.: Abelian varieties and a conjecture of R. M. Robinson. J. Reine Angew. Math. **291**, 78–91 (1977)
17. Robinson, R.M.: Conjugate algebraic units in a special interval. Math. Z. **154**, 31–40 (1977)
18. Serre, J.-P.: Rational Points on Curves over Finite Fields, Doc. Math., vol. 18. Soc. Math. France (2020)
19. Kedlaya, K.S.: Search techniques for root-unitary polynomials. In: Computational Arithmetic Geometry. Contemporary Mathematics, vol. 463, pp. 71–82. American Mathematical Society (2008)
20. Dupuy, T., Kedlaya, K.S., Roe, D., Vincent, C.: Isogeny classes of abelian varieties over finite fields in the LMFDB. In: Arithmetic Geometry. Number Theory, and Computation, pp. 375–448. Springer, Simons Symposia (2022)
21. Lauter, K.: Zeta functions of curves over finite fields with many rational points. In: Coding Theory. Cryptography and Related Areas (Guanajuato, 1998), pp. 167–174. Springer, Berlin (2000)
22. The LMFDB Collaboration, L-Functions and Modular Forms Database. https://lmfdb.org
23. Xarles, X.: A census of all genus 4 curves over the field with 2 elements. http://arxiv.org/abs/2007.07822
24. Dragutinović, D.: Computing binary curves of genus five, arXiv https://arxiv.org/abs/2207.07809; associated repository. https://github.com/DusanDragutinovic/MT_Curves
25. Kedlaya, K.S.: The relative class number one problem for function fields, II. https://arxiv.org/abs/2206.02084
26. Kedlaya, K.S.: The relative class number one problem for function fields, III. https://arxiv.org/abs/2208.11277
27. Mukai, S.: Curves and Grassmannians. In: Algebraic Geometry and Related Topics, pp. 19–40. International Press, Cambridge, MA (1993)
28. Mukai, S.: Curves and symmetric spaces, I. Am. J. Math. **117**, 1627–1644 (1995)
29. Kedlaya, K.S.: Abelian varieties over F_2 of prescribed order. https://arxiv.org/abs/2107.12453; to appear in Publ. Math. Besançon (special issue for GTA 2021)
30. Smith, A.: Algebraic integers with conjugates in a prescribed distribution. https://arxiv.org/abs/2111.12660
31. The Sage Developers, SageMath version 9.7, 2022. https://www.sagemath.org
32. The Magma Group, Magma version 2.27-1, 2022. http://magma.maths.usyd.edu.au
33. Kedlaya, K.S.: GitHub repository https://github.com/kedlaya/same-class-number
34. D'Nelly-Warady, T., Kedlaya, K.S.: Geometric Decomposition of Abelian Varieties of Order 1. https://arxiv.org/abs/2109.03986; to appear in the Women in Numbers 5 Proceedings
35. manYPoints—Table of Curves with Many Points. https://manypoints.org. Accessed Nov 2021
36. Howe, E.W., Lauter, K.E.: New methods for bounding the number of points on curves over finite fields. In: Geometry and Arithmetic, pp. 173–212. Zürich, European Mathematical Society (2012)
37. Shiomi, D.: On the Deuring-Shafarevich formula. Tokyo Math. J. **34**, 313–318 (2011)
38. Howe, E.W., Lauter, K.E.: Improved upper bounds for the number of points on curves over finite fields. Ann. Inst. Fourier (Grenoble) **53**, 1677–1737 (2000)
39. Milne, J.S.: Jacobian varieties. In: Arithmetic Geometry. Springer, New York (1986)
40. Birkenhake, C., Lange, H.: Complex Algebraic Varieties, second edition, Grundlehren der math. Wissenschaften 302, Springer, Berlin (2004)
41. Achter, J., Casalaina-Martin, S.: Putting the p back in Prym, in preparation