Canonical and log canonical thresholds of Fano complete intersections

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Abstract It is proved that the global log canonical threshold of a Zariski general Fano complete intersection of index 1 and codimension k in $\mathbb{P}^{M+k}$ is equal to one, if $M \geq 2k + 3$ and the maximum of the degrees of defining equations is at least 8. This is an essential improvement of the previous results about log canonical thresholds of Fano complete intersections. As a corollary we obtain the existence of Kähler–Einstein metrics on generic Fano complete intersections described above.

Keywords Fano variety · Log canonical singularity · Hypertangent divisor · Kähler–Einstein metric

Mathematics Subject Classification 14E05 · 14E07 · 14J45

1 Introduction

1.1 Statement of the main result

The aim of the present paper is to show that the global (log) canonical threshold of a general Fano complete intersection of index 1 is at least (respectively, equal to) one, except for a sufficiently narrow class of Fano complete intersections, defined by equations of low degree. More precisely, let $\underline{d} = (d_1, \dots, d_k)$ be an ordered integral vector, where $k \geq 1$ (the value k is *not* fixed), $2 \leq d_1 \leq \dots \leq d_k$, and

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$$M = d_1 + \cdots + d_k - k \geq 3.$$

To every such vector corresponds a family $\mathcal{F}(\underline{d})$ of non-singular Fano complete intersections of codimension k in the complex projective space $\mathbb{P}^{|\underline{d}|}$, where $|\underline{d}| = d_1 + \cdots + d_k = M + k$, which we will in the sequel for simplicity denote by the symbol $\mathbb{P}$:

$$\mathcal{F}(\underline{d}) \ni F = Q_1 \cap \cdots \cap Q_k \subset \mathbb{P},$$

$\deg Q_i = d_i$. Obviously, F is a non-singular Fano variety of index 1, that is, $\text{Pic } F = \mathbb{Z}K_F$, where $K_F = -H_F$ is the canonical class of the variety F , and H_F is the class of its hyperplane section in $\mathbb{P}$. The varieties of the family $\mathcal{F}(\underline{d})$ are naturally parametrized by the coefficients of the polynomials, defining the hypersurfaces Q_i .

Conjecture 1.1 *For a general (in the sense of Zariski topology) variety $F \in \mathcal{F}(\underline{d})$ and an arbitrary effective divisor $D \sim nH_F$ on F the pair $(F, (1/n)D)$ is canonical, that is, for any exceptional prime divisor E over F the inequality*

$$\text{ord}_E D \leq n \cdot a(E, F),$$

holds, where $a(E, F)$ is the discrepancy of E with respect to the model F .

The claim of Conjecture 1.1 is usually stated in the following way: the (global) canonical threshold of the variety F is at least one. Recall that the (global) canonical threshold is defined by the equality

$$\text{ct}(F) = \sup \left\{ \lambda \in \mathbb{Q}_+ : \left(F, \frac{\lambda}{n} D \right) \text{ is canonical for all } D \in |nH_F| \text{ and all } n \geq 1 \right\},$$

and the log canonical threshold, respectively, by the equality

$$\begin{aligned} \text{lct}(F) \\ = \sup \left\{ \lambda \in \mathbb{Q}_+ : \left(F, \frac{\lambda}{n} D \right) \text{ is log canonical for all } D \in |nH_F| \text{ and all } n \geq 1 \right\}, \end{aligned}$$

The importance of canonical and log canonical thresholds is connected with their applications to the complex differential geometry and birational geometry. Tian, Nadel, Demailly and Kollár showed in [7, 12, 20], that the inequality

$$\text{lct}(F) > \frac{M}{M+1}$$

implies the existence of the Kähler–Einstein metric on F (this fact was shown for arbitrary Fano varieties, not only for complete intersections in the projective space), in fact, the inequality above implies the K -stability of F , see the most recent paper [9] on this subject. Since the property of being canonical is stronger than that of being log canonical, the claim of Conjecture 1.1 implies the existence of the Kähler–Einstein on

a general Fano complete intersection of index 1. This application alone is sufficient to justify the importance of Conjecture 1.1. For the applications to birational geometry, see Sect. 1.3.

Now let us state the main result of the present paper. Let $\mathcal{D}$ be the set of ordered integral vectors $\underline{d}$, such that $2 \leq d_1 \leq \dots \leq d_k, k \geq 2$. For an integer $a \geq 2$ set

$$\mathcal{D}_{\geq a} = \{\underline{d} : d_k \geq a\}.$$

Recall that $|\underline{d}| = d_1 + \dots + d_k$.

Theorem 1.2 *Assume that $\underline{d} \in \mathcal{D}_{\geq 8}$ and $|\underline{d}| \geq 3k + 3$. Then for a Zariski general variety $F \in \mathcal{F}(\underline{d})$ the inequality $\text{ct}(F) \geq 1$ holds.*

Corollary 1.3 *In the assumptions of Theorem 1.2 the equality $\text{lct}(F) = 1$ holds, so that on the variety F there is a Kähler–Einstein metric.*

Note that the inequality $\text{ct}(F) \geq 1$ was shown for a general variety $F \in \mathcal{F}(\underline{d})$, $\underline{d} \in \mathcal{D}_{\geq 8}$, under the assumption that $M \geq 4k + 1$ (that is, $|\underline{d}| \geq 5k + 1$), in [16], and under the assumption that $M \geq 3k + 4$ (that is, $|\underline{d}| \geq 4k + 4$), in [8]. For more details about the history of this problem, see Sect. 1.5. One should keep in mind that the smaller are the degrees d_i of the equations defining F (respectively, the higher is the degree $d = d_1 \dots d_k = \deg F$ with the dimension $M = \dim F$ fixed), the harder is to prove the inequality $\text{ct}(F) \geq 1$. The case is similar with proving the birational superrigidity of Fano complete intersections of index 1 [19]: the birational superrigidity remains an open problem in arbitrary dimension only for three types of complete intersections,

$$\underline{d} \in \{(2, \dots, 2, 2), (2, \dots, 2, 3), (2, \dots, 2, 4)\}.$$

Note also that the canonicity of the pair $(F, (1/n)D)$ for any divisor $D \sim -nK_F$ is a much stronger fact, than the canonicity of this pair for a general divisor D of an arbitrary mobile linear system $\Sigma \subset |-nK_F|$, and for that reason it is harder to prove the inequality $\text{ct}(F) \geq 1$, than the birational rigidity.

1.2 Regular complete intersections

We understand the condition that the variety $F \in \mathcal{F}(\underline{d})$ is Zariski general in the sense that at every point $o \in F$ the regularity condition (R), which we will now state, is satisfied. This condition was used in [8, 16].

Let $F = Q_1 \cap \dots \cap Q_k \in \mathcal{F}(\underline{d})$ and $o \in F$ be an arbitrary point. Fix a system of affine coordinates $z_* = (z_1, \dots, z_{M+k})$ on $\mathbb{A}^{M+k} \subset \mathbb{P}^{M+k+1}$ with the origin at the point $o \in \mathbb{A}^{M+k}$. Let $f_i(z_*)$ be the (non-homogeneous) polynomial defining the hypersurface Q_i in the affine chart $\mathbb{A}^{M+k}$, $\deg F_i = d_i$. Write down

$$f_i = q_{i,1} + q_{i,2} + \dots + q_{i,d_i},$$

where $q_{i,j}(z_*)$ are homogeneous polynomials of degree j . On the set $\{q_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d_i\}$ we introduce the *standard order*, setting:

- $q_{i,j}$ precedes $q_{a,l}$, if $j < l$,
- $q_{i,j}$ precedes $q_{a,j}$, if $i < a$.

Thus placing the polynomials $q_{i,j}$ in the standard order, we get a sequence of $d_1 + \dots + d_k = M + k$ homogeneous polynomials

$$q_{1,1}, q_{2,1}, \dots, q_{k,d_k} \quad (1)$$

in $M + k$ variables z_* .

Definition 1.4 (i) The complete intersection F is *regular at the point o* , if the linear forms $q_{1,1}, \dots, q_{k,1}$ are linearly independent and for any linear form

$$h(z_*) \notin \langle q_{1,1}, \dots, q_{k,1} \rangle$$

the sequence of homogeneous polynomials, which is obtained from (1) by removing the last two polynomials and adding the form h , is regular in $\mathcal{O}_{o,\mathbb{P}}$.

(ii) The complete intersection F satisfies the *condition (R)*, if it is regular at every point $o \in F$.

In other words, the regularity at the point o means that, removing from the sequence (1) the last two polynomials and adding the form h , we obtain $M + k - 1$ homogeneous polynomials in $M + k$ variables, the set of common zeros of which is a finite set of lines in $\mathbb{A}^{M+k}$, passing through the point $o = (0, \dots, 0)$.

Theorem 1.5 For every tuple $\underline{d} \in \mathcal{D}$ there exists a non-empty Zariski open set $\mathcal{F}_{\text{reg}}(\underline{d}) \subset \mathcal{F}(\underline{d})$, such that every variety $F \in \mathcal{F}_{\text{reg}}(\underline{d})$ satisfies the condition (R).

Now for $\underline{d} \in \mathcal{D}_{\geq 8}$ the claim of Theorem 1.2 follows from Theorem 1.5 and the following claim.

Theorem 1.6 Assume that $\underline{d} \in \mathcal{D}_{\geq 8}$. Then for a variety $F \in \mathcal{F}_{\text{reg}}(\underline{d})$ the inequality $\text{ct}(F) \geq 1$ holds.

1.3 The canonical threshold and birational rigidity

Theorem 1.2 has the following application in birational geometry. For an arbitrary non-singular primitive Fano variety X (that is, $\text{Pic } X = \mathbb{Z}K_X$) of dimension $\dim X$ define the *mobile canonical threshold* $\text{mct}(X)$ as the supremum of such $\lambda \in \mathbb{Q}_+$, that the pair $(X, (\lambda/n)D)$ is canonical for a general divisor D of an arbitrary mobile linear system $\Sigma \subset |-nK_X|$. The inequality $\text{mct}(X) \geq 1$ is “almost equivalent” to birational superrigidity of the variety X (for the definition of birational rigidity and superrigidity see [18, Chapter 2]).

In [15] the following general fact was shown.

Theorem 1.7 Assume that primitive Fano varieties $F_1, \dots, F_K$, $K \geq 2$, satisfy the conditions $\text{lct}(F_i) = 1$ and $\text{mct}(F_i) \geq 1$. Then their direct product

$$V = F_1 \times \cdots \times F_K$$

is a birationally superrigid variety. In particular,

- (i) Every structure of a rationally connected fiber space on the variety V is given by a projection onto a direct factor. More precisely, let $\beta: V^\# \rightarrow S^\#$ be a rationally connected fiber space and $\chi: V \dashrightarrow V^\#$ a birational map. Then there exists a subset of indices $I = \{i_1, \dots, i_k\} \subset \{1, \dots, K\}$ and a birational map

$$\alpha: F_I = \prod_{i \in I} F_i \dashrightarrow S^\#,$$

such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\chi} & V^\# \\ \pi_I \downarrow & & \downarrow \beta \\ F_I & \xrightarrow{\alpha} & S^\# \end{array}$$

commutes, that is, $\beta \circ \chi = \alpha \circ \pi_I$, where

$$\pi_I: \prod_{i=1}^K F_i \rightarrow \prod_{i \in I} F_i$$

is the natural projection onto a direct factor.

- (ii) Let $V^\#$ be a variety with $\mathbb{Q}$ -factorial terminal singularities, satisfying the condition

$$\dim_{\mathbb{Q}}(\text{Pic } V^\# \otimes \mathbb{Q}) \leq K,$$

and $\chi: V \dashrightarrow V^\#$ a birational map. Then χ is a (biregular) isomorphism.

- (iii) The groups of birational and biregular self-maps of the variety V coincide:

$$\text{Bir } V = \text{Aut } V.$$

In particular, the group $\text{Bir } V$ is finite.

- (iv) The variety V admits no structures of a fibration into rationally connected varieties of dimension strictly smaller than $\min\{\dim F_i\}$. In particular, V admits no structures of a conic bundle or a fibration into rational surfaces.
- (v) The variety V is non-rational.

Since the inequality $\text{ct}(F) \geq 1$ implies that $\text{mct}(F_i) \geq 1$ and $\text{lct}(F_i) = 1$, Theorem 1.2 implies that generic complete intersections $F \in \mathcal{F}(\underline{d})$ with $\underline{d} \in \mathcal{D}_{\geq 8}$ for $|\underline{d}| \geq 3k + 3$ satisfy the assumptions of Theorem 1.7.

1.4 The structure of the paper

In Sects. 2–3 we prove Theorem 1.6. We reproduce the proof sketched in [16, Section 3.1] in full detail, somewhat modifying the argument given in [16], adjusting it to a wider class of Fano complete intersections. In principle, the new argument is potentially applicable to proving the inequality $\text{ct}(F) \geq 1$ for complete intersections $F \in \mathcal{F}(d)$ with $d \notin \mathcal{D}_{\geq 8}$.

Our main tool is the technique of hypertangent linear systems. This is a procedure (described in Sect. 3), the “input” of which is an effective divisor $D \sim nH_F$, such that the pair $(F, (1/n)D)$ is not canonical (under the assumption that such pairs exist), and the “output” of which is an effective 1-cycle C that has a high multiplicity at some point $o \in F$. More precisely, if $d \in \mathcal{D}_{\geq 8}$, then $\text{mult}_o C > \deg C$, which is impossible. This contradiction proves Theorem 1.6.

In Sect. 4, we prove Theorem 1.5.

1.5 Historical remarks

As we pointed out above, the connection between the existence of Kähler–Einstein metrics and the global log canonical thresholds was established in [7, 12, 20]. The special importance of those papers is in that they connected some concepts of complex differential geometry with some objects of higher-dimensional birational geometry, which makes it possible to use the results of birational geometry to prove the existence of Kähler–Einstein metrics. That work was started in [1] and continued in [2–6, 8, 11, 16, 17]. Every time, a computation or estimate for the global log canonical threshold, obtained by the methods of birational geometry (the connectedness principle, inversion of adjunction, the technique of hypertangent divisors) yielded a proof of existence of Kähler–Einstein metrics for new classes of varieties. Such results are important by themselves, not speaking of their applications to birational geometry (Theorem 1.7), that is, of new classes of birationally rigid varieties.

2 Tangent divisors

In this section we start the proof of Theorem 1.6. We begin (Sect. 2.1) with some preparatory work: assuming that the pair $(F, (1/n)D)$ is not canonical, we show the existence of a hyperplane section Δ of the variety F , such that the multiplicity of the restriction of the divisor D onto Δ at the point o is strictly higher than $2n$. After that (Sect. 2.2) using the regularity condition (R), we construct a subvariety $Y \subset \Delta$ of codimension $k + 1$ with a high multiplicity at the point o .

2.1 Inversion of adjunction

Assume that there exists an effective divisor $D \sim nH_F$ such that the pair $(F, (1/n)D)$ is not canonical, that is, there is an exceptional divisor E over F , satisfying the Noether–Fano inequality

$$\mathrm{ord}_E D > na(E, F).$$

By linearity of this inequality in the divisor D (the integer $n \in \mathbb{Z}_+$ depends linearly on D), we may assume that D is a prime divisor. Let $B \subset F$ be the centre of the exceptional divisor E . It is well known that the estimate

$$\mathrm{mult}_B D > n$$

holds, whence by for example [16, Proposition 3.6], we immediately conclude that $\dim B \leq k - 1$. Consider a point $o \in B$ of general position. Let $\sigma: F^+ \rightarrow F$ be its blow up, $E^+ = \sigma^{-1}(o) \cong \mathbb{P}^{M-1}$ the exceptional divisor. For some hyperplane $\Theta \subset E^+$ the inequality

$$\mathrm{mult}_o D + \mathrm{mult}_\Theta D^+ > 2n$$

holds, where D^+ is the strict transform of the divisor D on F^+ (see [16, Proposition 2.5] or [18, Chapter 7, Proposition 2.3]).

Now let us consider a general hyperplane section Δ of the complete intersection F , containing the point o and cutting out the hyperplane Θ on E^+ in the sense that $\Delta^+ \cap E^+ = \Theta$. It is easy to see that the restriction $D_\Delta = D|_\Delta = (D \circ \Delta)$ of the divisor D on Δ satisfies the inequality

$$\mathrm{mult}_o D_\Delta > 2n. \quad (2)$$

The hyperplane section Δ can be viewed as a complete intersection of the type $\underline{d}$ in $\mathbb{P}^{M+k-1}$.

2.2 Intersection with tangent hyperplanes

Now assume that F satisfies the condition (R). In the notations of Sect. 1.2 the system of linear equations

$$q_{1,1} = \cdots = q_{k,1} = 0$$

defines the (embedded) tangent space $T_o F \subset T_o \mathbb{P}$. Obviously, $E^+ = \mathbb{P}(T_o F)$. Let $h(z_*)$ be the linear form, defining the hyperplane that cuts out Δ . In particular,

$$\{h|_{E^+} = 0\} = \Theta$$

and $h \notin \langle q_{1,1}, \dots, q_{k,1} \rangle$. Let

$$T_i = \{q_{i,1}|_\Delta = 0\},$$

$i = 1, \dots, k$, be the tangent hyperplane sections of the variety Δ . By the condition (R), the inequality $\dim \Delta \geq 2k + 2$ and the Lefschetz theorem (taking into account

that the singularities of the variety Δ are at most zero-dimensional and $o \in \Delta$ is a non-singular point), we may conclude that for any $j = 1, \dots, k$,

$$T_1 \cap \dots \cap T_j = (T_1 \circ \dots \circ T_j)$$

is an irreducible subvariety of codimension j in Δ , which has multiplicity precisely 2^j at the point o . We will show that the effective divisor $D_\Delta \sim nH_\Delta$ (where H_Δ is the class of a hyperplane section of the complete intersection $\Delta \subset \mathbb{P}^{M+k-1}$), satisfying inequality (2), cannot exist. Again by the linearity of inequality (2) (we will need no other information about the divisor D_Δ), we assume that D_Δ is a prime divisor. In particular, inequality (2) implies that $D_\Delta \neq T_1$ (since $\text{mult}_o T_1 = 2$), so that the effective cycle $(D_\Delta \circ T_1) = Y_1^*$ of the scheme-theoretic intersection of these divisors is well defined and satisfies the inequality

$$\text{mult}_o Y_1^* > 4n,$$

and moreover, $Y_1^* \sim nH_\Delta^2$; in particular,

$$\frac{\text{mult}_o Y_1^*}{\deg Y_1^*} > \frac{4}{d},$$

where $d = \deg F = d_1, \dots, d_k$. In the sequel for simplicity of notations we write

$$\frac{\text{mult}_o}{\deg}$$

for the ratio of multiplicity at the point o to the degree. Let Y_1 be an irreducible component of the cycle Y_1^* with the maximal value of $\text{mult}_o/\deg$; in particular,

$$\frac{\text{mult}_o}{\deg} Y_1 > \frac{4}{d}.$$

Since by construction $Y_1 \subset T_1$ and

$$\frac{\text{mult}_o}{\deg} (T_1 \cap T_2) = \frac{4}{d},$$

we conclude that $Y_1 \not\subset T_2$ and the effective cycle $(Y_1 \circ T_2) = Y_2^*$ is well defined and satisfies the inequality

$$\frac{\text{mult}_o}{\deg} Y_2^* > \frac{8}{d}.$$

Let Y_2 be an irreducible component of the cycle Y_2^* with the maximal value of $\text{mult}_o/\deg$.

Continuing in the same way, we construct a sequence of irreducible subvarieties

$$D_{\Delta} = Y_0 \supset Y_1 \supset \cdots \supset Y_k$$

of codimension $\text{codim}(Y_j \subset \Delta) = j + 1$, satisfying the inequality

$$\frac{\text{mult}_o}{\deg} Y_j > \frac{2^{j+1}}{d}.$$

The inequality $M \geq 2k + 3$ is needed to justify the last step in this construction: by the Lefschetz theorem, $T_1 \cap \cdots \cap T_k = (T_1 \circ \cdots \circ T_k)$ is an irreducible subvariety of Δ of codimension k , with the multiplicity 2^k at the point o and degree d , which makes it possible to form the effective cycle $Y_k^* = (Y_{k-1} \circ T_k)$ of codimension $k + 1$.

We have shown the following claim.

Proposition 2.1 *Assume that the pair $(F, (1/n)D)$ is not canonical. Then for some point $o \in F$ and a hyperplane section $\Delta \ni o$, non-singular at the point o , there exists an irreducible subvariety $Y \subset \Delta$ of codimension $k + 1$ in Δ , satisfying the inequality*

$$\frac{\text{mult}_o}{\deg} Y > \frac{2^{k+1}}{d}. \quad (3)$$

In order to complete the proof of Theorem 1.6, we now need the technique of hypertangent divisors. It is considered in the next section.

3 Hypertangent divisors

In this section we complete the proof of Theorem 1.6. First (Sect. 3.1) we construct hypertangent linear systems on the variety Δ and study their properties. After that (Sect. 3.2) we select a sequence of general divisors from the hypertangent systems. Finally, intersecting the subvariety Y with the hypertangent divisors, we complete the proof of Theorem 1.6 (Sect. 3.3).

3.1 Hypertangent linear systems

For $j \in \{1, \dots, d_i\}$ let

$$f_{i,j} = q_{i,1} + \cdots + q_{i,j}$$

be the truncated equation of the hypersurface Q_i . By the symbol $\mathcal{P}_{a,M+k}$ we denote the linear space of homogeneous polynomials of degree a in the coordinates $z_1, \dots, z_{M+k}$. We use this symbol for $a < 0$ as well, setting in that case $\mathcal{P}_{a,M+k} = \{0\}$.

Definition 3.1 The linear system of divisors

$$\Lambda_a = \left\{ \left(\sum_{i=1}^k \sum_{j=1}^{d_i-1} f_{i,j} s_{a-j} \right) \Big|_{\Delta} : s_l \in \mathcal{P}_{l, M+k} \right\}$$

is the a -th hypertangent linear system on Δ at the point o .

Note that by our convention about the negative degrees only the polynomials $f_{i,j}$ of degree $j \leq a$ are really used in the construction of the system Λ_a .

Set $\delta = d_k$ and for $a \geq 2$ set

$$k_a = \# \{i : 1 \leq i \leq k, d_i = a\} \in \mathbb{Z}_+.$$

Obviously, $k_a = 0$ for $a \geq \delta + 1$. The equality $d_1 + \dots + d_k = M + k$ implies that $\delta \leq M$. Obviously,

$$k = k_2 + \dots + k_{\delta}.$$

We say that we are in

$$\begin{array}{ll} \text{CASE I,} & \text{if } k_{\delta} \geq 3, \\ \text{CASE IIA,} & \text{if } k_{\delta} = 2, \\ \text{CASE IIB,} & \text{if } k_{\delta} = 1 \text{ and } k_{\delta-1} \geq 1, \\ \text{CASE III,} & \text{if } k_{\delta} = 1 \text{ and } k_{\delta-1} = 0. \end{array}$$

Obviously, one of these cases takes place: we simply listed all options.

For $a \geq 2$ set

$$m_a = \sum_{i \geq a} k_i.$$

It is easy to see that m_a is the number of polynomials of degree a in the sequence (1). In the next proposition we sum up the properties of hypertangent systems that we will need. The symbol codim_o stands for the codimension in a neighborhood of the point o with respect to Δ .

Proposition 3.2 (i) *The following inclusion holds: $\Lambda_a \subset |aH_{\Delta}|$, where H_{Δ} is the class of a hyperplane section of Δ .*

(ii) *The following equality holds: $\text{mult}_o \Lambda_a = a + 1$.*

(iii) *In Cases I and IIA for $a = 1, \dots, \delta - 2$, and in Cases IIB and III for $a = 1, \dots, \delta - 3$ the following equality holds:*

$$\text{codim}_o \text{Bs } \Lambda_a = \sum_{i=2}^{a+1} m_i.$$

(iv) In Case I for $a = \delta - 1$, in Cases IIA and IIB for $a = \delta - 2$, and in Case III for $a = \delta - 3$ the following equality holds: $\dim \text{Bs} \Lambda_a = 1$.

Note that (iii) in Case IIA for $a = \delta - 2$ and in Case III for $a = \delta - 3$ coincides with (iv) for these cases.

Proof These are the standard facts of the technique of hypertangent divisors, following immediately from the regularity condition (Definition 1.4), see [18, Chapter 3]. Claim (i) is obvious, claim (ii) follows from the equality

$$f_{i,j}|_{\Delta} = -q_{i,j+1}|_{\Delta} + \cdots, \quad (4)$$

where the dots stand for the components of degree $j + 2$ and higher, and from the regularity condition. Claims (iii) and (iv) follow from equality (4) and the counting of polynomials of degree j in the sequence (1). For the details, see [18, Chapter 3]. $\square$

3.2 Hypertangent divisors

The next step is constructing a sequence of hypertangent divisors $D_{i,j} \in \Lambda_i$. From each hypertangent linear system Λ_i we select l_i divisors, where the integer l_i is defined in the following way: $l_2 = m_3 - 1$, $l_i = m_{i+1}$ for $i = 3, \dots, \delta - 3$, finally,

- in Case I, $l_{\delta-2} = m_{\delta-1}$, $l_{\delta-1} = m_{\delta} - 2$,
- in Case IIA, $l_{\delta-2} = m_{\delta-1}$,
- in Case IIB, $l_{\delta-2} = m_{\delta-1} - 1$.

For all other values of i set $l_i = 0$.

Furthermore, for $l_i \neq 0$ we set

$$\mathcal{L}_i = \Lambda_i^{\times l_i}$$

and a tuple of divisors $(D_{i,1}, \dots, D_{i,l_i}) \in \mathcal{L}_i$ is denoted by the symbol $D_{i,*}$. Finally, set

$$\mathcal{L} = \prod_{i \geq 2} \mathcal{L}_i,$$

where the direct product is taken over all i such that $l_i \neq 0$, see the definition of the integers l_i above. It is easy to see that $\mathcal{L}$ is the direct product of

$$\sum_{i \geq 2} l_i = M - k - 3$$

factors (precisely the number of polynomials in the sequence (1), from which all linear and quadratic forms are removed, together with one cubic polynomial and the last two polynomials). The elements of the space $\mathcal{L}$, that is, the tuples of divisors

$$(D_{2,*}, D_{3,*}, \dots)$$

are denoted by the symbol $D_{*,*}$.

For an arbitrary equidimensional effective cycle W on Δ , $\dim W \geq 2$, and a divisor $D_{i,j} \in \Lambda_i$, such that none of the components of W is contained in its support $|D_{i,j}|$, we denote by the symbol

$$(W \circ D_{i,j})_o$$

the effective cycle of dimension $\dim W - 1$, which is obtained from the cycle $(W \circ D_{i,j})$ of the scheme-theoretic intersection of W and $D_{i,j}$ (see [10, Chapter 2]) by removing all irreducible components, not containing the point o .

3.3 Proof of Theorem 1.6

Now everything is ready to apply the technique of hypertangent systems to the subvariety $Y \subset \Delta$, constructed in Sect. 2. The tuple $D_{*,*} \in \mathcal{L}$ is understood as a tuple of divisors

$$(D_{2,1}, D_{2,2}, \dots, D_{3,1}, \dots),$$

which makes it possible to apply the construction of the scheme-theoretic intersection at the point o , described above, many times.

Proposition 3.3 *For a general tuple $D_{*,*} \in \mathcal{L}$ the effective 1-cycle*

$$C = (Y \circ D_{*,*})_o = (\cdots ((Y \circ D_{2,1}) \circ D_{2,2})_o \cdots)_o$$

is well defined and satisfies the inequalities

$$\deg C \leq \deg Y \cdot \prod_{i \geq 2} i^{l_i}$$

and

$$\text{mult}_o C \geq \text{mult}_o Y \cdot \prod_{i \geq 2} (i+1)^{l_i}.$$

Proof The procedure of constructing the cycle C is justified by Proposition 3.2 (iii)–(iv), and the inequalities for the degree and multiplicity follow from claims (i) and (ii). $\square$

Let us prove Theorem 1.6. Assume that $\delta = d_k \geq 8$. Combining inequality (3) with inequalities of Proposition 3.3, we obtain the estimate

$$\frac{\text{mult}_o}{\deg} C > \frac{2^{k+1}}{d} \cdot \prod_{i \geq 2} \frac{(i+1)^{l_i}}{i^{l_i}},$$

and after cancellations we see that the inequality $\text{mult}_o C > \deg C$ holds. (For the details, see [16, Section 3].) This contradiction completes the proof of Theorem 1.6.

4 Regular complete intersections

In this section we prove Theorem 1.5. First (Sect. 4.1), we reduce the problem to a local problem about violation of the regularity condition at a fixed point. After that (Sect. 4.2), we estimate the codimension of the set of tuples of polynomials, vanishing simultaneously on some line. Finally (Sect. 4.3), we estimate the codimension of the set of tuples of polynomials, the set of common zeros of which has an “incorrect” dimension, but is not a line. This completes the proof of Theorem 1.5.

4.1 Reduction to the local problem

Following the standard scheme of proving the regularity conditions (see [18, Chapter 3] or any paper that makes use of the technique of hypertangent divisors, for example, [14] or [8]), we have to show that a violation of the local regularity condition at a fixed point o (that is, Definition 1.4(i)) imposes at least $M + 1$ independent conditions on the coefficients of the polynomials (1). The complete intersection F is non-singular, so let us fix the linear forms $q_{1,1}, \dots, q_{k,1}$ and so the linear space

$$T_o F = \{q_{1,1} = \dots = q_{k,1} = 0\}.$$

The last two polynomials in the sequence (1) are not used in the regularity condition. Let us re-label the polynomials of the sequence (1), from which all linear forms and the last two polynomials are removed, by the symbols

$$p_1, \dots, p_{M-2}.$$

Now the local regularity condition can be stated as follows: for any hyperplane $S \subset T_o F$ (in the notations of Definition 1.4, $S = \{h = 0\} \cap T_o F$) the sequence

$$p_1|_S, \dots, p_{M-2}|_S$$

is regular at the origin. If $\mathbb{S} = \mathbb{P}(S) \cong \mathbb{P}^{M-2}$, then this means that the closed subset

$$\{p_1|_{\mathbb{S}} = \dots = p_{M-2}|_{\mathbb{S}} = 0\}$$

is zero-dimensional. Fix an isomorphism $T_o F \cong \mathbb{C}^M$. Set $\delta(i) = \deg p_i$ and $\mathbb{T} = \mathbb{P}(T_o F) \cong \mathbb{P}^{M-1}$. Let $\mathcal{P}_{a,M}$ be the space of homogeneous polynomials of degree a on $\mathbb{C}^M$ (or $\mathbb{P}^{M-1}$) and

$$\mathcal{P}_{\mathbb{T}} = \prod_{i=1}^{M-2} \mathcal{P}_{\delta(i),M}.$$

If all polynomials p_i vanish on a line $L \subset \mathbb{T}$, then, obviously, the local regularity condition is violated: it is sufficient to take any hyperplane $\mathbb{S} \supset L$. For that reason

the case when the set $\{p_1 = \cdots = p_{M-2} = 0\}$ contains a line will be considered separately.

4.2 The case of a line

Let $\mathcal{B}^{\text{line}} \subset \mathcal{P}_{\mathbb{T}}$ be a closed subset of tuples $(p_1, \dots, p_{M-2})$, such that for some line $L \subset \mathbb{T}$,

$$p_1|_L \equiv \cdots \equiv p_{M-2}|_L \equiv 0.$$

Proposition 4.1 *The following inequality holds: $\text{codim}(\mathcal{B}^{\text{line}} \subset \mathcal{P}_{\mathbb{T}}) \geq M + 1$.*

Proof The proof is obtained by elementary but not quite trivial computations.

Lemma 4.2 *The following inequality holds:*

$$\text{codim}(\mathcal{B}^{\text{line}} \subset \mathcal{P}_{\mathbb{T}}) = \sum_{i=1}^{M-2} (\delta(i) + 1) - 2(M - 2).$$

Proof The first component in the right hand side is the codimension of the set of tuples of polynomials vanishing on a fixed line $L \subset \mathbb{T}$. Subtracting the dimension of the Grassmanian of lines, we complete the proof. $\blacksquare$

Considering the polynomials $q_{i,j}$ for each $i = 1, \dots, k$ separately, we conclude that

$$\sum_{i=1}^{M-2} \delta(i) = \sum_{i=1}^k \left(\frac{a_i(a_i + 1)}{2} - 1 \right) = \sum_{i=1}^k \frac{a_i(a_i + 1)}{2} - k,$$

where $a_i = d_i$ for $i = 1, \dots, k - 2$, $a_{k-1} = a_k = d_k - 1$ in Cases I and IIA and $a_{k-1} = d_{k-1}$, $a_k = d_k - 2$ in Cases IIB and III. In any case $a_i \geq 2$ and

$$a_1 + \cdots + a_k = M + k - 2.$$

Lemma 4.3 *The minimum of the quadratic function*

$$\xi(a_1, \dots, a_k) = \sum_{i=1}^k a_i(a_i + 1)$$

on the set of integral vectors $(a_1, \dots, a_k)$ such that all $a_i \geq 2$ and $a_1 + \cdots + a_k = A$, where $A = ka + l$, $a \in \mathbb{Z}$ and $l \in \{0, 1, \dots, k - 1\}$, is equal to

$$ka^2 + (k + 2l)a + 2l.$$

Proof Without loss of generality we assume that the set $(a_1, \dots, a_k)$ is ordered: $a_i \leq a_{i+1}$. It is easy to check that if two positive integers u, v satisfy the inequality $u \leq v-2$, then

$$u(u+1) + v(v+1) > (u+1)(u+2) + (v-1)v.$$

Therefore, if $a_i \leq a_{i+1} - 2$, then, replacing the vector $\underline{a} = (a_1, \dots, a_k)$ by the vector $\underline{a}' = (a'_1, \dots, a'_k)$, where $a'_j = a_j$ for $j \neq i, i+1$, $a'_i = a_i + 1$ and $a'_{i+1} = a_{i+1} - 1$, we decrease the value of the function ξ . Similarly, if

$$a_i + 1 = a_{i+1} = \dots = a_{i+\alpha} = a_{i+\alpha+1} - 1,$$

then, replacing the vector $\underline{a}$ by the vector $\underline{a}'$ with

$$a'_i = a'_{i+1} = \dots = a'_{i+\alpha+1} = a_{i+1},$$

we decrease the value of the function ξ . In both cases the vector $\underline{a}'$ remains ordered and satisfies the restrictions $a'_1 + \dots + a'_k = A$, $a'_i \geq 2$. Since the set of such vectors is finite, applying finitely many modifications of the two types described above, we obtain a vector with

$$a_1 = \dots = a_{k-l} = a \quad \text{and} \quad a_{k-l+1} = \dots = a_k = a + 1,$$

which realizes the minimum of the function ξ . Simple computations complete the proof of the lemma. $\blacksquare$

Now writing $M - 2 = ka + l$ with $l \in \{0, \dots, k-1\}$ and applying Lemmas 4.2 and 4.3, after obvious simplifications we obtain the inequality

$$\text{codim}(\mathcal{B}^{\text{line}} \subset \mathcal{P}) \geq \frac{1}{2} (k(a+1)^2 + k(a-1) + 2l(a+1)) - (M-2).$$

Now the inequality of Proposition 4.1 follows from the estimate

$$k(a^2 - a) + 2l(a-1) \geq 6,$$

which is easy to check (recall that by assumption $M \geq 2k+3$, so that $ak+l \geq 2k+1$ and, in particular, $a \geq 2$). $\square$

Starting from this moment, we assume that the polynomials $p_1, \dots, p_{M-2}$ do not vanish simultaneously on a line $L \subset \mathbb{T}$.

4.3 End of the proof of Theorem 1.5

Fix a hyperplane $\mathbb{S} \subset \mathbb{T}$ and its isomorphism $\mathbb{S} \cong \mathbb{P}^{M-2}$. Set

$$\mathcal{P} = \prod_{i=1}^{M-2} \mathcal{P}_{\delta(i), M-1}.$$

Since the hyperplane $\mathbb{S}$ varies in an $(M-1)$ -dimensional family, it is sufficient to show that the codimension of the set of tuples $(p_1, \dots, p_{M-2}) \in \mathcal{P}$ such that the closed set

$$\{p_1 = \dots = p_{M-2} = 0\}$$

has a component of positive dimension, *which is not a line*, is of codimension at least $(M+1) + (M-1) = 2M$ in $\mathcal{P}$. Let us check this fact. The check is not difficult, arguments of that type were published many times in full detail, so we will just sketch the main steps.

Let $\mathcal{B}_i \subset \mathcal{P}$ be the set of such tuples that the closed set

$$\{p_1 = \dots = p_{i-1} = 0\} \subset \mathbb{P}^{M-2} \quad (5)$$

(if $i = 0$, then this set is assumed to be equal to $\mathbb{P}^{M-2}$) is of codimension $i-1$ in $\mathbb{P}^{M-2}$, but for some irreducible component B of this set we have $p_i|_B \equiv 0$, and moreover if $i = M-2$, then B is a curve of degree at least two.

Obviously, Theorem 1.5 is implied by the following fact.

Proposition 4.4 *The following inequality holds:*

$$\text{codim}(\mathcal{B}_i \subset \mathcal{P}) \geq 2M.$$

Proof Using the method that was applied in [13] (see also [18, Chapter 3, Subsection 1.3]), for $i = 1, \dots, k$ we obtain the estimate

$$\text{codim}(\mathcal{B}_i \subset \mathcal{P}) \geq \binom{M+1-i}{2}$$

(recall that $\delta(i) = 2$ for $i = 1, \dots, k$). The minimum of the right hand side is attained at $i = k$ and it is easy to check that

$$\binom{M+1-k}{2} - 2M = \frac{1}{2} [(M - (2k+3))M + k^2 - k] > 0.$$

Therefore, we may assume that $i \geq k+1$, so that $\delta(i) \geq 3$. Now we use the technique that was developed in [14] (see also [18, Chapter 3, Section 3]). Let $\mathcal{B}_{i,b} \subset \mathcal{P}$ be the set of tuples such that the closed set (5) is of codimension $i-1$, and moreover, there is an irreducible component B of this set, such that

$$\text{codim}(\langle B \rangle \subset \mathbb{P}^{M-2}) = b,$$

$b \in \{0, 1, \dots, i-1\}$, $b \neq M-3$, and $p_i|_B \equiv 0$. Since

$$\mathcal{B}_i = \bigcup_{b=0}^{i-1} \mathcal{B}_{i,b}$$

(the condition $b \neq M - 3$ for $i = M - 2$ is meant, but not shown, in order for the formula not to be ugly), it is sufficient to show the inequality

$$\operatorname{codim}(\mathcal{B}_{i,b} \subset \mathcal{P}) \geq 2M$$

for $i \geq k + 1$, $b \in \{0, \dots, i - 1\}$, $b \neq M - 3$.

Now the technique of good sequences and associated subvarieties, which we do not give here, see [14] or [18, Chapter 3, Section 3], gives the estimate (taking into account the dimension of the Grassmanian of linear subspaces of codimension b in $\mathbb{P}^{M-2}$)

$$\operatorname{codim}(\mathcal{B}_{i,b} \subset \mathcal{P}) \geq M(2b + 3) - 2b^2 - 6b - 5.$$

The right hand side of this inequality, considered as a function on the set $\{0, \dots, i - 1\}$, can decrease or increase or first increase and then decrease. In any case the minimum of the right hand side is attained either at $b = 0$ (and equals $3M - 5 \geq 2M$), or at $b = i - 1$ (if $i \leq M - 3$) or $b = M - 4$ (if $i = M - 2$), when it is also not smaller than $2M$. $\square$

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