Local derivations on measurable operators and commutativity

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Abstract We prove that a von Neumann algebra M is abelian if and only if the square of every derivation on the algebra $S(M)$ of measurable operators affiliated with M is a local derivation. We also show that for general associative unital algebras this is not true.

Keywords von Neumann algebra · Measurable operators · Derivation · Local derivation

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1 Introduction

Let $\mathcal{A}$ be an algebra over the field of complex numbers. A linear operator $D: \mathcal{A} \rightarrow \mathcal{A}$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (Leibniz rule). Each element $a \in \mathcal{A}$ defines a derivation D_a on $\mathcal{A}$ given by $D_a(x) = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be *inner*. If the element a implementing the derivation D_a on $\mathcal{A}$, belongs to a larger algebra $\mathcal{B}$, containing $\mathcal{A}$ (as a proper ideal as usual) then D_a is called a *spatial derivation*.

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One of the main problems considered in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations, or to show the existence of non-inner and discontinuous derivations on various topological algebras. In particular, it is a general algebraic problem to find algebras which admit only inner derivations.

Such problems in the framework of algebras of measurable operators affiliated with von Neumann algebras have been considered in several papers (e.g. [1, 3–6]).

A linear operator Δ on an algebra $\mathcal{A}$ is called a *local derivation* if, given any $x \in \mathcal{A}$, there exists a derivation D (depending on x) such that $\Delta(x) = D(x)$. The main problem concerning these operators is to find conditions under which local derivations become derivations and to present examples of algebras which admit local derivations that are not derivations (see e.g. [12, 14]). In particular, Kadison in [12] proves that every continuous local derivation from a von Neumann algebra M into a dual M -bimodule is a derivation. This theorem gave rise to studies and several results on local derivations on C^* -algebras, culminating with a definitive contribution due to Johnson [11], who proved that every (not necessary continuous) local derivation of a C^* -algebra is a derivation. A comprehensive survey of recent results concerning local derivations on C^* - and von Neumann algebras is presented in [8].

In [2], we have studied local derivations on the algebra $S(M, \tau)$ of all τ -measurable operators affiliated with a von Neumann algebra M and a faithful normal semi-finite trace τ . One of our main results in [2] (Theorem 2.1) presents an unbounded version of Kadison's Theorem A from [12], and it asserts that every local derivation on $S(M, \tau)$ which is continuous in the measure topology automatically becomes a derivation. In particular, for type I von Neumann algebras M all such local derivations on $S(M, \tau)$ are inner derivations. We also proved that for type I finite von Neumann algebras without abelian direct summands, as well as for von Neumann algebras with the atomic lattice of projections, the continuity condition on local derivations in the above theorem is redundant. For survey of results concerning local derivations on algebras of measurable operators we refer to [7].

The most interesting effects appear when we consider the problem of existence of local derivations which are not derivations. The consideration of such examples on various finite- and infinite dimensional algebras was initiated by Kadison, Kaplansky and Jensen (see [12]). We have considered this problem on a class of commutative regular algebras, which includes the algebras of measurable operators affiliated with abelian von Neumann algebras. Unlike C^* - and von Neumann algebras cases, when all derivations, and hence all local derivations, are trivial, the abelian algebras $S(M)$ may admit non-zero derivations (see [9]). In [2, Theorem 3.2], the authors obtained necessary and sufficient conditions for the algebras of measurable and τ -measurable operators affiliated with a commutative von Neumann algebra to admit local derivations which are not derivations. In the latter paper, we have proved some properties of derivations and local derivations which seem to be specific for the abelian case as it was noted by Professor Efim Zelmanov (UCSD). In the present paper we confirm this conjecture and give some criteria for a von Neumann algebra M to be abelian in terms of properties of derivations and local derivations on the algebra $S(M)$ of measurable operators affiliated with M .

2 The main result

The following theorem is the main result of the paper.

Theorem 2.1 *Let $S(M)$ be the algebra of measurable operators affiliated with a von Neumann algebra M . The following conditions are equivalent:*

- (a) M is abelian.
- (b) For every derivation D on $S(M)$ its square D^2 is a local derivation.
- (c) A linear map $\Delta: S(M) \rightarrow S(M)$ is a local derivation if and only if $\Delta(p) = 0$ for every projection p in M , and $s(\Delta(x)) \leq s(x)$ for all x in $S(M)$, where $s(x)$ is the support projection of the element x .

Let us first present some necessary definitions and facts from the theory of measurable operators, which will be used in the proof of this theorem.

Let H be a Hilbert space over the field $\mathbb{C}$ of complex numbers and let $B(H)$ be the algebra of all bounded linear operators on H . Denote by $\mathbf{1}$ the identity operator on H and let $\mathcal{P}(H) = \{p \in B(H) : p^2 = p = p^*\}$ be the lattice of projections in $B(H)$. Consider a von Neumann algebra M on H , i.e. a weakly closed $*$ -subalgebra in $B(H)$ containing the operator $\mathbf{1}$ and denote by $\|\cdot\|_M$ the operator norm on M . The set $\mathcal{P}(M) = \mathcal{P}(H) \cap M$ is a complete orthomodular lattice with respect to the natural partial order on $M_h = \{x \in M : x^* = x\}$, generated by the cone M_+ of positive operators from M .

Two projections $e, f \in \mathcal{P}(M)$ are said to be *equivalent* (denoted by $e \sim f$) if there exists a partial isometry $v \in M$ with the initial projection e and final projection f , i.e. $v^*v = e, vv^* = f$. The relation “ $\sim$ ” is an equivalence relation on the lattice $\mathcal{P}(M)$.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u in the commutant

$$M' = \{y \in B(H) : xy = yx \text{ for all } x \in M\}$$

of the von Neumann algebra M in $B(H)$.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary u in M' .

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M , if

- $\mathcal{D}\eta M$;
- there exists a sequence of projections $\{p_n\}_{n=1}^\infty$ in $\mathcal{P}(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H . Denote by $S(M)$ the set of all measurable operators with respect to M .

It is well-known (see e.g. [15]) that the set $S(M)$ is a unital $*$ -algebra when equipped with the algebraic operations of the strong addition and multiplication and taking the adjoint of an operator.

For every $x \in S(M)$ we set $s(x) = l(x) \vee r(x)$, where $l(x)$ and $r(x)$ are left and right supports of x respectively (see [15]).

Let $\Delta: S(M) \rightarrow S(M)$ be a local derivation. Then

$$p\Delta(p)p = 0 \quad (1)$$

for every idempotent $p \in M$. Indeed, for any idempotent $p \in M$, we have

$$\Delta(p) = D_p(p) = D(p^2) = D_p(p)p + pD_p(p) = \Delta(p)p + p\Delta(p).$$

Multiplying both sides of this equality by p we obtain (1).

Let M be an abelian von Neumann algebra and let D be a derivation on $S(M)$. By [9, Proposition 2.3] or [3, Theorem], we have that $s(D(x)) \leq s(x)$ for any $x \in S(M)$ and also $D(p) = 0$ for any $p \in \mathcal{P}(M)$. Therefore by the definition, each local derivation Δ on $S(M)$ satisfies the following two conditions:

$$s(\Delta(x)) \leq s(x) \quad \text{for all } x \in S(M) \quad (2)$$

and

$$\Delta|_{\mathcal{P}(M)} = 0. \quad (3)$$

This means that (2) and (3) are necessary conditions for a linear operator Δ to be a local derivation on the algebra $S(M)$. We are going to show that these two conditions are in fact also sufficient. The following lemma in a more general setting of regular commutative algebras was obtained in [2, Lemma 3.2]. For the sake of completeness below we give an alternative and straightforward proof of this result.

Lemma 2.2 *Let M be an abelian von Neumann algebra. Then each linear operator Δ on the algebra $S(M)$ satisfying conditions (2) and (3) is a local derivation on $S(M)$.*

Proof Let us first assume that M is an abelian von Neumann algebra with a faithful normal finite trace τ . In this case the algebra $S(M)$ is a complete metrizable regular algebra with respect to the metric $\rho(x, y) = \tau(s(x - y))$, $x, y \in S(M)$ (see [9, Example 2.6]).

It is well-known that the abelian von Neumann algebra M can be identified with the algebra $C(\Omega)$ of all complex valued continuous functions on a hyperstonean compact space Ω .

Let $a \in S(M)$ be a fixed element. In order to proof that the linear map Δ is a local derivation we must find a derivation D on $S(M)$ such that $\Delta(a) = D(a)$. Let $P[t]$ be the algebra of all polynomials at variable t over $\mathbb{C}$. Consider the following subalgebra of $S(M)$:

$$\mathcal{A}(a) = \{f(a) : f \in P[t], f(0) = 0\}.$$

Consider the linear operator D from $\mathcal{A}(a)$ into $S(M)$ defined as

$$D(f(a)) = f'(a)\Delta(a), \quad (4)$$

where f' is the usual derivative of the polynomial f . It is clear that D is a derivation. Let us show that

$$s(D(f(a))) \leq s(f(a)) \quad (5)$$

for all $f \in P[t]$ with $f(0) = 0$.

Case 1. Suppose first that a is a bounded element, i.e. $a \in M \equiv C(\Omega)$. Take an arbitrary $f(t) = \sum_{k=1}^n \lambda_k t^k \in P[t]$. Denote $e = \mathbf{1} - s(f(a))$. There exists a closed subset S of Ω such that e equals to the characteristic function χ_S of the set S . Since $ef(a) = 0$, it follows that $\sum_{k=1}^n \lambda_k a(t)^k = 0$ for all $t \in S$, that is the complex numbers $a(t)$ are roots of f for all $t \in S$. Since the polynomial f may have at most n roots the set $\{a(t) : t \in S\}$ is finite. This means that ea is a simple (step) function, i.e. it is a linear combination of projections. Since by (3), $\Delta(p) = 0$ for all $p \in \mathcal{P}(M)$, it follows that $\Delta(ea) = 0$. Further from the linearity of Δ and properties of the support we have

$$\begin{aligned} s(\Delta(a)) &= s(\Delta((\mathbf{1} - e)a + ea)) = s(\Delta((\mathbf{1} - e)a) + \Delta(ea)) \\ &= s(\Delta((\mathbf{1} - e)a)) \leq s(\mathbf{1} - e) = s(f(a)). \end{aligned}$$

Therefore

$$s(D(f(a))) = s(f'(a)\Delta(a)) \leq s(\Delta(a)) \leq s(f(a)).$$

Case 2. Let $a \in S(M)$ be an arbitrary element. There exists an increasing sequence of projections $\{e_n\}$ in M such that $e_n a \in M$ for all $n \in \mathbb{N}$ and $\bigvee_{n \in \mathbb{N}} e_n = \mathbf{1}$. Taking into account the equalities $e_n D(x) = D(e_n x)$, $e_n f(a) = f(e_n a)$ (the second equality follows from $f(0) = 0$) and Case 1, we obtain

$$\begin{aligned} e_n s(D(f(a))) &= s(e_n D(f(a))) = s(D(e_n f(a))) \\ &= s(D(f(e_n a))) \leq s(f(e_n a)) = s(e_n f(a)) = e_n s(f(a)) \end{aligned}$$

for all n . Thus $s(D(f(a))) \leq s(f(a))$. Thus, the derivation D defined by (4) satisfies condition (5). Since $S(M)$ is a complete metrizable regular algebra, by [9, Theorem 3.1], the derivation D from $\mathcal{A}(a)$ into $S(M)$ can be extended to a derivation on the whole algebra $S(M)$, which we also denote by D . Taking the polynomial $p(t) = t$ in the definition (4) of D , we obtain that $\Delta(a) = D(a)$. This means that Δ is a local derivation.

Now let M be an arbitrary abelian von Neumann algebra. It is well-known that it has a faithful normal semifinite trace τ . There exists a family of mutually orthogonal projections $\{z_i\}_{i \in I}$ in M such that $\bigvee_{i \in I} z_i = \mathbf{1}$ and $\tau(z_i) < +\infty$ for all $i \in I$.

Take $x = z_i x \in z_i S(M) \equiv S(z_i M)$. Condition (2) implies that

$$s(\Delta(x)) = s(\Delta(z_i x)) \leq s(z_i x) \leq z_i.$$

Thus $\Delta(x) = z_i \Delta(x) \in S(z_i M)$. This means that Δ maps $S(z_i M)$ into itself for all $i \in I$. Therefore, since $z_i M$ is an abelian von Neumann algebra with a faithful normal

finite trace, from the above it follows that the restriction $\Delta|_{S(z_i M)}$ of Δ onto $S(z_i M)$ is a local derivation. Thus Δ is also a local derivation. $\square$

Proof of Theorem 2.1 (a) $\Rightarrow$ (c). Let M be abelian. Before Lemma 2.2 we have already mentioned that any local derivation on $S(M)$ satisfies conditions (2) and (3).

Conversely, suppose that Δ is a linear operator on $S(M)$ such that $s(\Delta(x)) \leq s(x)$ for any $x \in S(M)$ and $\Delta(p) = 0$ for all $p \in \mathcal{P}(M)$. By Lemma 2.2, it follows that Δ is a local derivation.

(c) $\Rightarrow$ (b). Let D be a derivation on $S(M)$. Then D is a local derivation. By assumption (c), we have

$$s(D^2(x)) = s(D(D(x))) \leq s(D(x)) \leq s(x)$$

for all x in $S(M)$ and

$$D^2(p) = D(D(p)) = D(0) = 0$$

for all projections p in M . Applying (c) to the linear operator D^2 , we conclude that it is a local derivation.

(b) $\Rightarrow$ (a). Suppose M is a non-commutative von Neumann algebra and $a \in M$ is a non-central element. By the assumption of theorem, a linear operator $\Delta: S(M) \rightarrow S(M)$ defined by

$$\Delta(x) = [a, [a, x]], \quad x \in S(M),$$

is a local derivation. By (1), we obtain that $p[a, [a, p]]p = 0$ for every projection $p \in M$. Thus $p(a^2p + pa^2 - 2apa)p = 0$, i.e.,

$$pa^2p = papap. \quad (6)$$

Since M is non-commutative, there exists a non-central projection $e \in M$. Then $z(e)z(1-e) \neq 0$, where $z(x)$ denotes the central support of the element x . By [13, Proposition 6.1.8], we can find mutually orthogonal, equivalent non-zero projections $p, q \in M$ such that $p \leq e$ and $q \leq 1-e$. Take a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u = q$. Let us consider the element $a = u + u^*$. Using the equality $u = uu^*u$, we obtain

$$u^2 = uu = (uu^*u)(uu^*u) = u(u^*u)(uu^*)u = uqp u = 0.$$

Then $a^2 = uu^* + u^*u = p + q$ and therefore

$$pa^2p = p. \quad (7)$$

Further, $pap = uu^*(u + u^*)uu^* = uu^*uuu^* + uu^*u^*uu^* = 0$, and therefore

$$papap = 0. \quad (8)$$

Combining (7), (8) and (6), we obtain a contradiction. This contradiction implies that M is commutative. $\square$

Remark 2.3 In [9], the authors have proved that if M is an abelian von Neumann algebra with a non-atomic lattice of projections $\mathcal{P}(M)$, then the algebra $S(M)$ admits non-zero derivations. If D is such a derivation then from the above theorem we have that D^2 is a non-zero local derivation. Moreover D^2 is not a derivation, because from [2, Lemma 3.3] it follows that D^2 is a derivation if and only if D is identically zero.

3 Counterexample for general associative algebras

The following example shows that Theorem 2.1 and the last remark fail in the general case of unital associative algebras.

Example 3.1 Let $T_2(\mathbb{C})$ be the algebra of all upper triangular 2×2 matrices over $\mathbb{C}$, i.e.,

$$T_2(\mathbb{C}) = \left\{ \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbb{C}, 1 \leq i \leq j \leq 2 \right\}.$$

Let e_{11}, e_{12}, e_{22} be the matrix units in $T_2(\mathbb{C})$. It is known [10] that any derivation on the algebra $T_2(\mathbb{C})$ is inner. Direct computations show that the inner derivation generated by an element $a = \sum_{1 \leq i \leq j} a_{ij} e_{ij} \in T_2(\mathbb{C})$ acts on $T_2(\mathbb{C})$ as

$$D\left(\sum_{1 \leq i \leq j}^2 \lambda_{ij} e_{ij}\right) = [(a_{11} - a_{22})\lambda_{12} - a_{12}(\lambda_{11} - \lambda_{22})]e_{12}.$$

Thus

$$D^2\left(\sum_{1 \leq i \leq j}^2 \lambda_{ij} e_{ij}\right) = [(a_{11} - a_{22})^2 \lambda_{12} - (a_{11} - a_{22})a_{12}(\lambda_{11} - \lambda_{22})]e_{12}$$

and D^2 is an inner derivation generated by the matrix

$$\begin{pmatrix} a_{11}^2 + a_{22}^2 & (a_{11} - a_{22})a_{12} \\ 0 & 2a_{11}a_{22} \end{pmatrix}.$$

So, the square of every derivation on the algebra $T_2(\mathbb{C})$ is a derivation (and hence a local derivation), but $T_2(\mathbb{C})$ is non-commutative.

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