A scaling property of Farey fractions. Part II: convergence at rational points

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Abstract The Farey sequence of order n consists of all reduced fractions a/b between 0 and 1 with positive denominator b less or equal to n . The sums of the inverse denominators $1/b$ of the Farey fractions in prescribed intervals with rational bounds have a simple main term, but the deviations are determined by an interesting sequence of polygonal functions f_n . In a former paper we obtained a limit function for $n \rightarrow \infty$, which describes a scaling behaviour of the functions f_n in the vicinity of any fixed rational number a/b and which is independent of a/b . In this paper we prove that $f_n(a/b)$ tends to zero for $n \rightarrow \infty$ by using elementary representation formulas for $f_n(a/b)$ as well as a variant of the prime number theorem. An application of this result immediately gives a global version of the scaling behaviour of the functions f_n around the rational numbers.

Keywords Farey sequences · Riemann zeta function · Prime number theorem

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1 Introduction

The Farey sequence $\mathcal{F}_n$ of order n consists of all reduced fractions a/b with natural denominator $b \leq n$ in the unit interval $[0, 1]$, arranged in order of increasing size. This sequence is finite and consists of $N + 1$ fractions, where $N = \sum_{k=1}^n \varphi(k)$ with Euler's totient function φ . We denote the fractions in $\mathcal{F}_n$ by

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$$\frac{0}{1} = \frac{a_0}{b_0} < \frac{a_1}{b_1} < \cdots < \frac{a_N}{b_N} = \frac{1}{1}. \quad (1)$$

Dropping just the restriction $0 \leq a/b \leq 1$ in this definition gives the infinite extended Farey sequence $\mathcal{F}_n^{\text{ext}}$ of order n , for example the extended sequence $\mathcal{F}_3^{\text{ext}}$ of order $n = 3$:

$$\cdots < \frac{-1}{1} < \frac{-2}{3} < \frac{-1}{2} < \frac{-1}{3} < \frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1} < \frac{4}{3} < \frac{3}{2} < \cdots$$

We will use the following basic property of Farey fractions:

Assume that $a, a' \in \mathbb{Z}$ and $b, b', n \in \mathbb{N}$ with $(a, b) = (a', b') = 1$. Then the reduced fractions $a/b < a'/b'$ are consecutive fractions in $\mathcal{F}_n^{\text{ext}}$ if and only if

$$a'b - ab' = 1, \quad b \leq n, \quad b' \leq n \quad \text{and} \quad b + b' > n.$$

In the former paper [1] we introduced and studied three families of 1-periodic functions

$$\Psi_n^*, \Psi_n, f_n : \mathbb{R} \rightarrow \mathbb{R}$$

which are related to the Farey sequence $\mathcal{F}_n$. We recall these functions which describe the structure of the Farey sequence and give interesting relations between the functions Ψ_n and f_n , the prime number theorem [1, Corollary 3.9] and the Riemann hypothesis [1, Corollary 3.10].

The 1-periodic functions Ψ_n^* are determined in the unit interval. We put $\xi_\alpha = a_\alpha/b_\alpha$ with the Farey fractions in (1) and for $\xi_{\alpha-1} \leq t < \xi_\alpha$, $\alpha = 1, \dots, N$, we define

$$\Psi_n^*(t) = -\frac{b_\alpha + b_{\alpha-1}}{2} \left[t - \frac{a_\alpha + a_{\alpha-1}}{b_\alpha + b_{\alpha-1}} \right]. \quad (2)$$

In addition we require $\Psi_n^*(1) = \Psi_n^*(0) = 1/2$. It easily follows from (2) that the function Ψ_n^* is linear between two subsequent Farey fractions in $\mathcal{F}_n$ and has a jump of height $1/b$ at each Farey fraction $a/b \in \mathcal{F}_n$. The plot of Ψ_{50}^* in Fig. 1 shows that the jump discontinuities are centered symmetrically around zero. This holds in general and leads to the following simple but important conclusion: For relatively prime integer numbers $a \in \mathbb{Z}$, $b \in \mathbb{N}$ we have

$$2b\Psi_n^*\left(\frac{a}{b}\right) = 1 \quad \text{for all } n \geq b. \quad (3)$$

In order to define the functions Ψ_n we put $\beta(t) = t - [t] - 1/2$ and define for $k \in \mathbb{N}$ and $t \in \mathbb{R}$ the 1-periodic function

$$q_k(t) = -\sum_{d|k} \mu(d) \beta\left(\frac{kt}{d}\right). \quad (4)$$

Fig. 1 The jumps of Ψ_{50}^* are centered around zero

Then $q_k(t) + t\varphi(k)$ is the number of reduced fractions j/k in the interval $(0, t]$, provided that $t > 0$ and $k > 1$. For $n \in \mathbb{N}$ we define the 1-periodic function

$$\Psi_n = \sum_{k=1}^n \frac{1}{k} q_k. \quad (5)$$

Like Ψ_n^* , the function Ψ_n is linear between two subsequent Farey fractions in $\mathcal{F}_n^{\text{ext}}$, and also has a jump of height $1/b$ at each Farey fraction $a/b \in \mathcal{F}_n^{\text{ext}}$. A plot of Ψ_{50} is presented in Fig. 2. Note that in general these jumps are not symmetrically centered around zero.

Thus it is natural to define $f_n: \mathbb{R} \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$ by

$$f_n(t) = \Psi_n^*(t) - \Psi_n(t). \quad (6)$$

The *polygonal functions* f_n have the following elementary properties:

- f_n is 1-periodic, continuous and odd.
- $f_n(0) = f_n(1/2) = f_n(1) = 0$.
- The function f_n is linear between two subsequent Farey fractions.
- We use (1), put $\xi_\alpha = a_\alpha/b_\alpha$, $s_n = \sum_{k=1}^n \varphi(k)/k$ and obtain

$$f_n(\xi_\alpha) = s_n \xi_\alpha + \frac{1}{2} \left(1 + \frac{1}{b_\alpha} \right) - \sum_{\beta=0}^{\alpha} \frac{1}{b_\beta} \quad \text{for all } \alpha = 0, \dots, N.$$

Fig. 2 The jumps of Ψ_{50} are not centered around zero

Fig. 3 The polygonal function f_{50}

In the former paper [1] we have shown that

$$\lim_{n \rightarrow \infty} \|\Psi_n\|_2 = \lim_{n \rightarrow \infty} \|\Psi_n^*\|_2 = \lim_{n \rightarrow \infty} \|f_n\|_2 = 0,$$

where $\|h\|_2$ is the $L^2(0, 1)$ -norm of any square integrable function h . A plot of f_{50} is given in Fig. 3.

We will show in Theorem 3.1 the main result of the paper on hand, namely

$$\lim_{n \rightarrow \infty} f_n\left(\frac{a}{b}\right) = 0$$

for each rational number a/b . The basic ingredients in the proof of Theorem 3.1 are Lemma 2.2 and Theorem 2.4(b), which enable the application of (9), a variant of the prime number theorem.

Combining Theorem 3.1 with [1, Theorem 4.6] immediately gives the following global version of a scaling behaviour of the functions f_n around the rational numbers, namely:

Assume that $a/b \in \mathcal{F}_n^{\text{ext}}$ and put

$$\tilde{g}_{a,b}(n, x) = b f_n\left(\frac{a}{b} + \frac{x}{bn}\right), \quad x \in \mathbb{R}.$$

Then for $n \rightarrow \infty$ the sequence of functions $\tilde{g}_{a,b}(n, \cdot)$ converges *uniformly* on each interval $[-x_*, x_*]$, $x_* > 0$ fixed, towards a continuous, piecewise smooth and odd limit function $g: \mathbb{R} \rightarrow \mathbb{R}$, which is *independent* of the choice of the fraction a/b .

Finally we will present a further application of (9), namely Theorem 3.3, which states that the families of functions Ψ_n^* , Ψ_n and f_n are uniformly bounded.

2 Representation formulas for Ψ_n and f_n

In this section we will derive representation formulas for Ψ_n and f_n . Especially a representation formula for f_n enables the application of the prime number theorem in the next section in order to prove that $\lim_{n \rightarrow \infty} f_n(a/b) = 0$ at each rational point a/b .

Lemma 2.1 *For relatively prime integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we have*

$$\sum_{m=1}^b \left\lfloor \frac{am}{b} \right\rfloor = a \frac{b+1}{2} - \frac{b-1}{2}.$$

Proof The set of integers $m = 1, \dots, b$ is a complete residue system modulo b , hence also the set of residues

$$r_m = am - b \left\lfloor \frac{am}{b} \right\rfloor \in \{0, \dots, b-1\}$$

modulo b corresponding to the integers $a \cdot m$, because a and b are relatively prime. Now the statement of the lemma follows from

$$\begin{aligned} \frac{b(b-1)}{2} &= \sum_{r=0}^{b-1} r = \sum_{m=1}^b r_m = \sum_{m=1}^b \left(am - b \left\lfloor \frac{am}{b} \right\rfloor \right) \\ &= a \frac{b(b+1)}{2} - b \sum_{m=1}^b \left\lfloor \frac{am}{b} \right\rfloor. \end{aligned} \quad \square$$

Lemma 2.2 For relatively prime integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ we put $\beta(t) = t - \lfloor t \rfloor - 1/2$ and $t_{a/b}(m) = 1 + 2b\beta(am/b)$. Then the series

$$\theta_{a/b} = \sum_{m=1}^{\infty} \frac{t_{a/b}(m)}{m}, \quad \theta_{a/b}(j) = \sum_{m>j} \frac{t_{a/b}(m)}{m} \quad \text{with } j \in \mathbb{N}$$

are convergent, and we have the estimates

$$|\theta_{a/b}| \leq 2b^2, \quad |\theta_{a/b}(j)| \leq \frac{4b^2}{j} \quad \text{for all } j \in \mathbb{N}.$$

Proof The expression

$$t_{a/b}(m) = 2am - 2b \left\lfloor \frac{am}{b} \right\rfloor - b + 1$$

defines a b -periodic sequence with respect to m . Due to Lemma 2.1, this sequence has mean value zero over one period, i.e.

$$\sum_{m \leq b} t_{a/b}(m) = 0.$$

We regard that $|\beta(t)| \leq 1/2$ for $t \in \mathbb{R}$, such that

$$\begin{aligned} \max_{k \in \mathbb{N}} \left| \sum_{m \leq k} t_{a/b}(m) \right| &= \max_{1 \leq k \leq b} \left| \sum_{m \leq k} \left(2b\beta\left(\frac{am}{b}\right) + 1 \right) \right| \\ &\leq \max_{1 \leq k \leq b} \sum_{m \leq k} (b+1) = b(b+1) \leq 2b^2. \end{aligned} \quad (7)$$

Abelian summation gives

$$\sum_{m \leq j} \frac{t_{a/b}(m)}{m} = \frac{1}{j} \sum_{m \leq j} t_{a/b}(m) + \sum_{k \leq j-1} \frac{1}{k(k+1)} \cdot \sum_{m \leq k} t_{a/b}(m). \quad (8)$$

We conclude from (7) and (8) that the series $\theta_{a/b}$ and $\theta_{a/b}(j)$ are convergent for all $j \in \mathbb{N}$. Moreover, (8) gives the representation

$$\theta_{a/b} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \cdot \sum_{m \leq k} t_{a/b}(m),$$

and we obtain with (7) that

$$|\theta_{a/b}| \leq \sum_{k=1}^{\infty} \frac{2b^2}{k(k+1)} = 2b^2.$$

We get in the same way

$$\begin{aligned} \theta_{a/b}(j) &= -\frac{1}{j} \sum_{m \leq j} t_{a/b}(m) + \sum_{k=j}^{\infty} \frac{1}{k(k+1)} \cdot \sum_{m \leq k} t_{a/b}(m), \\ |\theta_{a/b}(j)| &\leq \frac{2b^2}{j} + \sum_{k=j}^{\infty} \frac{2b^2}{k(k+1)} = \frac{4b^2}{j}. \end{aligned} \quad \square$$

Lemma 2.3

$$\sum_{j \leq n} \frac{1}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} = 1 \quad \text{for all } n \in \mathbb{N}.$$

Proof

$$\sum_{j \leq n} \frac{1}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} = \sum_{jk \leq n} \frac{1}{jk} \mu(k) = \sum_{m \leq n} \frac{1}{m} \sum_{d|m} \mu(d) = 1. \quad \square$$

Theorem 2.4 (representation formulas for Ψ_n and f_n)

(a) For all $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we regard $\beta(t) = t - \lfloor t \rfloor - 1/2$ and have

$$\Psi_n(t) = - \sum_{j \leq n} \frac{\beta(jt)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k}.$$

(b) With relatively prime integers $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ the following two representations are valid for all integer numbers $n \geq b$:

$$\begin{aligned} 2bf_n\left(\frac{a}{b}\right) &= \sum_{j \leq n} \frac{t_{a/b}(j)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} \\ &= \theta_{a/b} \sum_{k \leq n} \frac{\mu(k)}{k} - \sum_{j \leq n} \theta_{a/b}(j) \sum_{n/(j+1) < k \leq n/j} \frac{\mu(k)}{k}. \end{aligned}$$

Proof Part (a) follows from (4) and (5). In order to prove the first representation for $2bf_n(a/b)$ in part (b), we regard $n \geq b$ and obtain from (6), (3) and part (a) that

$$2bf_n\left(\frac{a}{b}\right) = 2b\Psi_n^*\left(\frac{a}{b}\right) - 2b\Psi_n\left(\frac{a}{b}\right) = 1 + 2b \sum_{j \leq n} \frac{\beta(aj/b)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k}.$$

Now Lemma 2.3 gives

$$2bf_n\left(\frac{a}{b}\right) = \sum_{j \leq n} \frac{1 + 2b\beta(aj/b)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} = \sum_{j \leq n} \frac{t_{a/b}(j)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k}.$$

From this the second representation for $2bf_n(a/b)$ in part (b) follows for all $n \geq b$ via Abelian summation with a zero boundary term,

$$\begin{aligned} 2bf_n\left(\frac{a}{b}\right) &= \sum_{j \leq n} \frac{t_{a/b}(j)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} \\ &= - \sum_{j \leq n} \sum_{m \leq j} \frac{t_{a/b}(m)}{m} \cdot \left(\sum_{k \leq n/(j+1)} \frac{\mu(k)}{k} - \sum_{k \leq n/j} \frac{\mu(k)}{k} \right) \\ &= \theta_{a/b} \sum_{k \leq n} \frac{\mu(k)}{k} - \sum_{j \leq n} \theta_{a/b}(j) \sum_{n/(j+1) < k \leq n/j} \frac{\mu(k)}{k}. \quad \square \end{aligned}$$

3 Convergence results and uniform boundedness for the functions f_n and Ψ_n

Now we are able to derive the main result, namely

Theorem 3.1 *There are absolute constants $c_1, c_2 > 0$ such that for any two relatively prime integers $a \in \mathbb{Z}$, $b \in \mathbb{N}$ there holds the inequality*

$$\left| f_n\left(\frac{a}{b}\right) \right| \leq c_1 b e^{-c_2 \sqrt{\log n}} \quad \text{for all } n \geq b.$$

Especially we have $\lim_{n \rightarrow \infty} f_n(a/b) = 0$.

Proof From Prachar [2, Kapitel III, Section 5] we obtain for all $x \geq 1$

$$\left| \sum_{k \leq x} \frac{\mu(k)}{k} \right| \leq \alpha_1 e^{-2\alpha_2 \sqrt{\log x}} + \alpha_3 \int_x^\infty e^{-2\alpha_2 \sqrt{\log \xi}} \frac{d\xi}{\xi}$$

with absolute constants $\alpha_1, \alpha_2, \alpha_3 > 0$. It follows for all $x \geq 1$ that

$$\left| \sum_{k \leq x} \frac{\mu(k)}{k} \right| \leq \alpha_4 e^{-\alpha_2 \sqrt{\log x}} \quad (9)$$

with the new constant

$$\alpha_4 = \alpha_1 + \alpha_3 \int_1^\infty e^{-\alpha_2 \sqrt{\log \xi}} \frac{d\xi}{\xi} > 0.$$

Regarding the second representation for $2bf_n(a/b)$ in Theorem 2.4(b) we employ Lemma 2.2, (9), and obtain for $n \geq b$ the following two estimates:

$$S_1 = \frac{1}{2b} \left| \theta_{a/b} \sum_{k \leq n} \frac{\mu(k)}{k} \right| \leq b\alpha_4 e^{-\alpha_2 \sqrt{\log n}}, \quad (10)$$

$$\begin{aligned} S_2 &= \frac{1}{2b} \left| \sum_{j \leq \sqrt{n}-1} \theta_{a/b}(j) \sum_{n/(j+1) < k \leq n/j} \frac{\mu(k)}{k} \right| \\ &\leq 2b \sum_{j \leq \sqrt{n}-1} \frac{2\alpha_4}{j} e^{-\alpha_2 \sqrt{\log \sqrt{n}}} \leq b\alpha_5 e^{-\alpha_6 \sqrt{\log n}} \end{aligned} \quad (11)$$

with absolute constants $\alpha_5, \alpha_6 > 0$ compensating the factor

$$\sum_{j \leq \sqrt{n}-1} \frac{2\alpha_4}{j} = \mathcal{O}(\log n)$$

in front of $e^{-\alpha_2 \sqrt{(\log n)/2}}$. For the first inequality in (11) we regard $n/j > n/(j+1) \geq \sqrt{n}$ and

$$\begin{aligned} \left| \sum_{n/(j+1) < k \leq n/j} \frac{\mu(k)}{k} \right| &= \left| \sum_{k \leq n/j} \frac{\mu(k)}{k} - \sum_{k \leq n/(j+1)} \frac{\mu(k)}{k} \right| \\ &\leq \alpha_4 e^{-\alpha_2 \sqrt{\log(n/j)}} + \alpha_4 e^{-\alpha_2 \sqrt{\log(n/(j+1))}} \\ &\leq 2\alpha_4 e^{-\alpha_2 \sqrt{\log \sqrt{n}}}. \end{aligned}$$

Next we obtain from Lemma 2.2 with an absolute constant $\alpha > 0$ that

$$\begin{aligned} S_3 &= \frac{1}{2b} \left| \sum_{\sqrt{n}-1 < j \leq n} \theta_{a/b}(j) \sum_{n/(j+1) < k \leq n/j} \frac{\mu(k)}{k} \right| \\ &\leq 2b \sum_{\sqrt{n}-1 < j \leq n} \frac{1}{j} \sum_{n/(j+1) < k \leq n/j} \frac{1}{k} \\ &= 2b \sum_{k \leq n/\lfloor \sqrt{n} \rfloor} \frac{1}{k \lfloor n/k \rfloor} \leq \frac{b\alpha}{\sqrt{n}}. \end{aligned} \quad (12)$$

We conclude from (10), (11), (12) and Theorem 2.4 (b) that there are absolute constants $c_1, c_2 > 0$ such that

$$\left| f_n\left(\frac{a}{b}\right) \right| \leq S_1 + S_2 + S_3 \leq c_1 b e^{-c_2 \sqrt{\log n}}$$

for all $n \geq b$. $\square$

Combining Theorem 3.1 with [1, Theorem 4.6] immediately gives a global version of the scaling behaviour of the functions f_n around the rational numbers, namely

Theorem 3.2 (scaling property of f_n near Farey fractions) *Assume that $a/b \in \mathcal{F}_n^{\text{ext}}$ and put*

$$\tilde{g}_{a,b}(n, x) = b f_n\left(\frac{a}{b} + \frac{x}{bn}\right), \quad x \in \mathbb{R}.$$

Then for $n \rightarrow \infty$ the sequence of functions $\tilde{g}_{a,b}(n, \cdot)$ converges uniformly on each interval $[-x_*, x_*]$, $x_* > 0$ fixed, towards the limit function

$$g(x) = \begin{cases} \left(\frac{6}{\pi^2} - \frac{1}{2}\right)x & \text{for } 0 \leq x \leq 1, \\ \frac{6}{\pi^2} - \frac{1}{2} + \int_1^x \left(\frac{6}{\pi^2} - \frac{1}{t} \sum_{k \leq t} \frac{\varphi(k)}{k}\right) dt & \text{for } x > 1, \\ -g(-x) & \text{for } x < 0. \end{cases}$$

Theorem 3.3 *The families of functions Ψ_n^* , Ψ_n and f_n are uniformly bounded.*

Proof Since this is true for the functions Ψ_n^* and since $f_n = \Psi_n^* - \Psi_n$, it is sufficient to show this for the functions Ψ_n .

Due to Theorem 2.4 (a) and (9) it is sufficient to show that

$$\sum_{j=1}^n \frac{1}{j} e^{-\alpha_2 \sqrt{\log(n/j)}}$$

has an upper bound independent from n . But for large values of n the order of magnitude of this sum is given by the uniformly bounded integral

$$\int_1^n e^{-\alpha_2 \sqrt{\log(n/x)}} \frac{dx}{x} = \frac{2}{\alpha_2^2} \left(1 - \frac{1 + \alpha_2 \sqrt{\log n}}{e^{\alpha_2 \sqrt{\log n}}}\right) < \frac{2}{\alpha_2^2}. \quad \square$$

4 Conclusions and outlook

The 1-periodic polygonal functions f_n from (6) carry important information about Farey sequences, see especially Theorem 3.2, which results from the combination of a former result with Theorem 3.1. The basic ingredients in the proof of Theorem 3.1 are

the second representation formula for $2bf_n(a/b)$ in Theorem 2.4(b) and estimates with the b -periodic sequence $(t_{a/b}(m))_{m \in \mathbb{N}}$ in Lemma 2.2, which enable the application of (9), a variant of the prime number theorem. In view of the present study it is natural to pose the question whether

$$\lim_{n \rightarrow \infty} f_n(t) = 0$$

also holds for all irrational numbers t .

Assume that $t \in \mathbb{R}$ is irrational and let $a'(n, t)/b'(n, t) < t < a''(n, t)/b''(n, t)$ with reduced neighbouring fractions $a'(n, t)/b'(n, t) < a''(n, t)/b''(n, t)$ in $\mathcal{F}_n^{\text{ext}}$. Then

$$\lim_{n \rightarrow \infty} b'(n, t) = \lim_{n \rightarrow \infty} b''(n, t) = \infty$$

since t is not rational, and it follows from (2) that $\lim_{n \rightarrow \infty} \Psi_n^*(t) = 0$. Thus it remains the question whether

$$\lim_{n \rightarrow \infty} \Psi_n(t) = 0$$

holds for irrational $t \in \mathbb{R}$, which will be studied in a forthcoming paper.

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