Quasi-one-fibered and projectively full ideals in 2-dimensional Muhly rational singularities

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Abstract Let (R, m) be a 2-dimensional rational singularity. Assume that R is a Muhly domain: (R, m) is an integrally closed Noetherian local domain with algebraically closed residue field R/m and the associated graded ring $\text{gr}_m R$ is an integrally closed domain. Let I be a complete m -primary ideal of R and let $v_1, \dots, v_n$ be the Rees valuations of I . For every $v_i \neq \text{ord}_R$, there is a complete m -primary ideal I_i of R such that I_i is quasi-one-fibered (i.e. I_i has exactly one Rees valuation different from ord_R) and the degree coefficient of I_i with respect to v_i is $d(I_i, v_i) = 1$. We show the following formula: $\text{Im}^s = I_1^{u_1} \dots I_n^{u_n}$ for some $s, u_1, \dots, u_n \geq 0$. We then prove that if ord_R is not a Rees valuation of I and if $\gcd(u_1, \dots, u_n) = 1$, then I is projectively full.

Keywords Degree function · Rees valuation · Complete ideal · 2-dimensional rational singularity · Quasi-one-fibered ideal · Projectively full ideal

Mathematics Subject Classification 13B22 · 13H10

1 Introduction

Let (R, m) be a Noetherian local ring and I an m -primary ideal of R . An element $x \in R$ is *integral* over I if there exist $a_1, \dots, a_n \in R$ such that

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

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and $a^i \in I^i$ for all $i = 1, \dots, n$. The set of all elements of R that are integral over I is an ideal of R and is denoted by $\bar{I}$. $\bar{I}$ is called the *integral closure* of I . If $I = \bar{I}$, the ideal I is *complete*. The integral closure of I and of the powers of I can be described using the Rees valuation rings of I . Let $P_1, \dots, P_n$ be the minimal primes of $(t^{-1})\overline{R[It, t^{-1}]}$. Then for $i = 1, \dots, n$

$$V_i = (\overline{R[It, t^{-1}]})_{P_i} \cap K$$

is a discrete valuation ring. Here K denotes the fraction field of R . The rings $V_1, \dots, V_n$ are called the *Rees valuation rings* of I . The *Rees valuations* of I are the corresponding discrete valuations $v_1, \dots, v_n$. Let $T(I)$ denote the set of Rees valuations of I : $T(I) = \{v_1, \dots, v_n\}$. One has for any $k \geq 1$

$$\bar{I}^k = \bigcap_{i=1}^n I^k V_i \cap R.$$

It is natural to ask the following question: when do two ideals have the same integral closure? This question has been answered in a number of ways, one of which is by using degree functions. This was done in the work of Rees and Sharp in [12]. The *degree function* d_I of I is defined by

$$d_I(x) = e\left(\frac{I + xR}{xR}\right), \quad x \in m \setminus \{0\},$$

where $e((I + xR)/xR)$ stands for the multiplicity of the ideal $(I + xR)/xR$. Let $P(R)$ denote the set of all prime divisors of R . In [10, Theorem 3.2], Rees showed that for every prime divisor v of R , there is a natural number $d(I, v)$ such that

$$d_I(x) = \sum_{v \in P(R)} d(I, v) v(x), \quad x \in m \setminus \{0\},$$

and $d(I, v) \neq 0$ if and only if v is a Rees valuation of I

$$d(I, v) \neq 0 \iff v \in T(I).$$

The coefficients $d(I, v)$ are unique and are called the *degree coefficients* of I . For I and J , m -primary ideals of R , Rees and Sharp showed in [12, Lemma 5.1] that

$$d(IJ, v) = d(I, v) + d(J, v), \quad \forall v \in P(R).$$

If R is analytically unramified and normal, the previous result implies that

$$T(IJ) = T(I) \cup T(J).$$

Rees and Sharp also showed the following formula for the multiplicity of an m -primary ideal I :

$$e(I) = \sum_{v \in P(R)} d(I, v) v(I). \quad (1)$$

For I and J , m -primary ideals in a 2-dimensional Cohen–Macaulay local domain (R, m) , one has the following equivalence [12, Corollary 5.3]:

$$\bar{I} = \bar{J} \iff d(I, v) = d(J, v), \quad \forall v \in P(R).$$

As $d(I, v) \neq 0$ if and only if $v \in T(I)$, this implies

$$\bar{I} = \bar{J} \iff T(I) = T(J) \quad \text{and} \quad d(I, v) = d(J, v), \quad \forall v \in T(I). \quad (2)$$

So, two ideals have the same closure if and only if they have the same degree coefficients. For more information about degree functions, we refer to [10, 12].

Complete ideals have been studied extensively in the case where (R, m) is a 2-dimensional regular local ring. The study of complete ideals in 2-dimensional regular local rings started with Zariski's 1938 paper [16] and was continued in [17, Appendix 5]. Assume from now on till specified otherwise, that (R, m) is a 2-dimensional regular local ring. Then the product of two complete ideals of R is complete again. This result is called Zariski's Product Theorem. Every complete ideal I of (R, m) can be factored into a product of simple complete ideals in a unique way: $I = I_1^{e_1} \dots I_n^{e_n}$. This is Zariski's Unique Factorization Theorem. An ideal is simple if it cannot be factored into a product of proper ideals of R . A simple m -primary ideal is one-fibered, i.e. it has only one Rees valuation. There is a $(1, 1)$ -correspondence between the prime divisors of R and the simple complete m -primary ideals of R (Zariski's $(1, 1)$ -correspondence): for every prime divisor v of R there is a unique simple complete m -primary ideal K of R such that $T(K) = \{v\}$. Let I be a complete m -primary ideal of R and $T(I) = \{v_1, \dots, v_n\}$. For $i = 1, \dots, n$, let I_i be the simple complete ideal corresponding to v_i . Then the ideals $I_1, \dots, I_n$ are precisely the ideals that occur in the unique factorization of I

$$I = I_1^{e_1} \dots I_n^{e_n}.$$

One has that $d(I_i, v_i) = 1$ and $e_i = d(I, v_i)$.

The work on complete ideals in 2-dimensional regular local rings has been extended and generalized in a.o. [3, 9, 11].

In this paper we focus on 2-dimensional Muhly rational singularities. These form a class of rings that contains the regular local rings of dimension 2. We will now recall the definition of a 2-dimensional rational singularity. Let (R, m) be a 2-dimensional analytically normal local ring. For an m -primary ideal I of R , one has that $\ell(R/I^n)$ is given by a polynomial $\bar{P}_I(n)$ for all $n \gg 0$. This polynomial $\bar{P}_I(n)$ is called the *normal Hilbert polynomial* of I , and one has the following:

$$\bar{P}_I(n) = \bar{e}_0(I) \binom{n+1}{2} - \bar{e}_1(I) \binom{n}{1} + \bar{e}_2(I).$$

Then (R, m) is a rational singularity if and only if $\bar{e}_2(I) = 0$ for every complete m -primary ideal I of R . In a 2-dimensional rational singularity, the product of complete ideals is complete again. From now on, we will assume not only that (R, m) is a 2-dimensional rational singularity, but also a Muhly local domain. According to [4, p. 1140], a Muhly local domain is an integrally closed Noetherian local domain with algebraically closed residue field R/m and for which the associated graded ring $\text{gr}_m R$ is an integrally closed domain. This implies that m has ord_R as unique Rees valuation and that $B\ell_m R$ is a desingularization of R [6, p. 403], [8, p. 259]. Here $B\ell_m R$ denotes the scheme $\text{Proj}(\bigoplus_{n \geq 0} m^n)$ obtained by blowing up m . Following Debremaeker in [5], an m -primary ideal with exactly one Rees valuation $v \neq \text{ord}_R$ is called *quasi-one-fibered*. In Sect. 2, we associate with every prime divisor $v \neq \text{ord}_R$ of R , a unique quasi-one-fibered complete ideal I such that $d(I, v) = 1$. In the case of a 2-dimensional regular local ring R , the quasi-one-fibered ideal associated to a prime divisor v of R is just the simple complete m -primary ideal with v as unique Rees valuation.

There is no notion of simple ideal in a 2-dimensional rational singularity. However, a 2-dimensional rational singularity satisfies the condition (N): for every prime divisor v of R , there is a unique complete m -primary ideal A_v of R such that $T(A_v) = \{v\}$ and the only complete m -primary ideals that have v as unique Rees valuation are the powers of A_v [6, Corollary 3.11]. In a 2-dimensional rational singularity one no longer has unique factorization into simple complete ideals. However, there exists a positive integer k such that for every complete m -primary ideal I of R , one has a unique decomposition

$$I^k = A_{v_1}^{e_1} \dots A_{v_n}^{e_n},$$

where $T(I) = \{v_1, \dots, v_n\}$ and the ideals A_{v_i} are as described above. Lipman showed in [8, Theorem 20.1] that one can take $k = 1$ if and only if the completion of R is a UFD. If (R, m) is a 2-dimensional regular local ring, then $k = 1$, and for a 2-dimensional Muhly rational singularity the converse is true as well. We can prove this using degree functions.

Theorem 1.1 *Let (R, m) be a 2-dimensional Muhly rational singularity. Then R is regular if and only if for every complete m -primary ideal I of R there exist positive integers e_i such that $I = A_{v_1}^{e_1} \dots A_{v_n}^{e_n}$ (where the ideals A_{v_i} are the one-fibered ideals associated to the Rees valuations of I).*

Proof One implication is Zariski's Unique Factorization Theorem. For the other implication, assume that (R, m) is a 2-dimensional Muhly rational singularity in which factorization holds. Let I be a first-neighborhood ideal of R , i.e. I is the inverse transform of the maximal ideal in an immediate quadratic transform R_1 of R . The fact that R is a Muhly local domain implies that R_1 is a regular local ring. It now follows from [14, Corollary 3.4] that I is a quasi-one-fibered ideal: $T(I) = \{v\}$ or $\{v, \text{ord}_R\}$. Here v denotes the ord_{R_1} -valuation. Let A_v be the unique one-fibered ideal of R with Rees valuation v . According to the assumption, I can be written as follows:

$$I = A_v^e m^f$$

for some $e > 0$ and $f \geq 0$ (since $\{v\} \subseteq T(I) \subseteq \{v, \text{ord}_R\}$). So $I \subseteq A_v \subseteq m$. As I is a first-neighborhood ideal, it follows that I is adjacent to m . Since $A_v \neq m$ (as $v \neq \text{ord}_R$), it follows that $I = A_v$. So $T(I) = \{v\}$. It then follows from [15, Proposition 3.2] that R is regular. $\square$

In this paper, we prove the following formula for a complete m -primary ideal I of R :

$$\text{Im}^s = I_1^{u_1} \cdots I_n^{u_n},$$

where $T(I) = \{v_1, \dots, v_n\}$ and I_i is the quasi-one-fibered ideal associated to v_i . We then use this result to find a class of projectively full ideals. More precisely, we show that if $\text{ord}_R \notin T(I)$ and $\gcd(u_1, \dots, u_n) = 1$, then I is projectively full.

2 Quasi-one-fibered ideals

In this section we will show that Im^s can be factorized into a product of quasi-one-fibered ideals. We will make use of the following result of Rees and Sharp. If I and J are m -primary ideals in a 2-dimensional Cohen–Macaulay local domain (R, m) , Rees and Sharp defined in [12]

$$d_I(J) = \min\{d_I(x) : 0 \neq x \in J\}.$$

We recall that the degree function associated with the ideal I is given by $d_I(x) = e((I+xR)/xR)$ for all $x \in m \setminus \{0\}$. The following theorem can be found in [12, Theorem 5.2].

Theorem 2.1 (Rees and Sharp) *Let (R, m) be a 2-dimensional Cohen–Macaulay local domain and let I, J be m -primary ideals of (R, m) . Then*

$$d_I(J) = \sum_{v \in P(R)} d(I, v)v(J)$$

$$\text{and } d_I(J) = d_J(I).$$

The following lemma now follows immediately.

Lemma 2.2 *Let (R, m) be a 2-dimensional Muhly rational singularity. Let $v \neq \text{ord}_R$ be a prime divisor of R and let I be a quasi-one-fibered ideal with $\{v\} \subseteq T(I) \subseteq \{v, \text{ord}_R\}$ and $d(I, v) = 1$. Then*

$$d(I, \text{ord}_R) = e(m)\text{ord}_R(I) - v(m).$$

Proof Since $\{v\} \subseteq T(I) \subseteq \{v, \text{ord}_R\}$, it follows from Theorem 2.1 that $d_I(m) = d_m(I)$ and that

$$d(I, v)v(m) + d(I, \text{ord}_R)\text{ord}_R(m) = d(m, \text{ord}_R)\text{ord}_R(I).$$

Since $T(m) = \{\text{ord}_R\}$, it follows from (1) that $d(m, \text{ord}_R) = e(m)$. So $d(I, \text{ord}_R) = e(m)\text{ord}_R(I) - v(m)$. $\square$

Let (R, m) be a 2-dimensional Muhly rational singularity. For $x \in m \setminus m^2$ and N a maximal ideal of $R[m/x]$ lying over m (i.e. $N \cap R = m$), the ring $R_1 = R[m/x]_N$ is called an *immediate quadratic transform* of R . As noted in the introduction, the fact that (R, m) is a Muhly local domain implies that R_1 is a 2-dimensional regular local ring. For an m -primary ideal I of R , put $r = \text{ord}_R(I)$. Then one has that $IR_1 = x^r I_1$ for some ideal I_1 of R_1 . The ideal I_1 is called the *transform* of I in R_1 . If $I_1 \neq R_1$, then R_1 is called an *immediate base point* of I .

Let v be a prime divisor of R with $v \neq \text{ord}_R$. In [14, Theorem 3.3, Corollary 3.4] it was shown that there exists an m -primary complete ideal I of R such that $T(I) = \{v\}$ or $\{v, \text{ord}_R\}$, and $d(I, v) = 1$. This ideal I is defined as follows. Let A_v be the complete one-fibered ideal of R with Rees valuation v and such that any complete one-fibered ideal of R with Rees valuation v is a power of A_v . A_v has a unique immediate base point (R_1, m_1) . Let I_1 be the simple complete m_1 -primary ideal of R_1 such that $T(I_1) = \{v\}$. Let I be the inverse transform of I_1 in R , i.e. $I = x^a I_1 \cap R$ where a is the smallest positive integer such that $x^a I_1 = JR_1$ for some ideal J of R . Then I is a complete m -primary ideal of R with $T(I) = \{v\}$ or $\{v, \text{ord}_R\}$ and $d(I, v) = 1$ [14, Corollary 3.4]. We will call I the *quasi-one-fibered ideal associated to v* . It is obvious that any ideal of the form $m^n I$ (with $n \geq 0$) is also quasi-one-fibered with $d(I, v) = 1$. In the following lemma we show that these are the only complete quasi-one-fibered ideals with $d(I, v) = 1$.

Lemma 2.3 *Let (R, m) be a 2-dimensional Muhly rational singularity. For $v \neq \text{ord}_R$ a prime divisor of R , let I be the quasi-one-fibered ideal associated to v . If J is any quasi-one-fibered ideal with v as Rees valuation and with $d(J, v) = 1$, then $J = m^n I$ for some $n \geq 0$.*

Proof Let J be quasi-one-fibered with $T(J) = \{v\}$ or $\{v, \text{ord}_R\}$ and $d(J, v) = 1$. If $T(J) = \{v\}$, then $J = A_v$, and according to [15, Proposition 3.1], this implies that $J = I$. So assume that $T(J) = \{v, \text{ord}_R\}$. First consider the case where $T(I) = \{v\}$, so $I = A_v$. Then there exists some $k > 0$ such that

$$J^k = m^f I^e.$$

Since $kd(J, v) = ed(I, v)$, it follows that $k = e$. Also, $k\text{ord}_R(J) = f + k\text{ord}_R(I)$. So it follows that

$$J = m^{\text{ord}_R(J) - \text{ord}_R(I)} I.$$

Next, assume that $T(I) = \{v, \text{ord}_R\}$. Put $r = \text{ord}_R(I)$ and $s = \text{ord}_R(J)$. Since $x^s I_1 = JR_1$, it follows that $r \leq s$. We have $d(J, v) = d(m^{s-r} I, v)$. From Lemma 2.2, it follows that

$$d(I, \text{ord}_R) = e(m)r - v(m), \quad d(J, \text{ord}_R) = e(m)s - v(m).$$

So $d(J, \text{ord}_R) = d(I, \text{ord}_R) + (s - r)e(m)$. This implies that

$$d(J, \text{ord}_R) = d(m^{s-r}I, \text{ord}_R).$$

Since $T(J) = T(m^{s-r}I)$, it follows from (2) that $J = m^{s-r}I$. $\square$

We are now ready to prove our main result. To simplify the formulation of the following theorem, we define m to be the quasi-one-fibered ideal associated to ord_R . Note that $d(m, \text{ord}_R) \neq 1$ if R is not regular.

Theorem 2.4 *Let (R, m) be a 2-dimensional Muhly rational singularity and let I be a complete m -primary ideal of R . Let $T(I) = \{v_1, \dots, v_n\}$ and, for $i = 1, \dots, n$, let I_i be the quasi-one-fibered ideal associated to v_i . Then there exist natural numbers $s, u_1, \dots, u_n$ such that*

$$\text{Im}^s = I_1^{u_1} \dots I_n^{u_n}.$$

Proof First we consider the case where $\text{ord}_R \notin T(I)$: $v_i \neq \text{ord}_R$ for all i . For every $i = 1, \dots, n$, put $u_i = d(I, v_i)$ and define

$$s = \frac{1}{e(m)} \sum_{i=1}^n d(I, v_i) d(I_i, \text{ord}_R).$$

We first show that $s \in \mathbb{Z}$. From Lemma 2.2, it follows that $d(I_i, \text{ord}_R) = e(m)\text{ord}_R(I_i) - v_i(m)$. So

$$se(m) = \sum_{i=1}^n (d(I, v_i) e(m) \text{ord}_R(I_i) - d(I, v_i) v_i(m)).$$

Using Theorem 2.1, we have

$$\begin{aligned} se(m) &= e(m) \sum_{i=1}^n d(I, v_i) \text{ord}_R(I_i) - d_I(m) \\ &= e(m) \sum_{i=1}^n d(I, v_i) \text{ord}_R(I_i) - d_m(I) \\ &= e(m) \sum_{i=1}^n d(I, v_i) \text{ord}_R(I_i) - e(m) \text{ord}_R(I). \end{aligned}$$

So

$$s = \sum_{i=1}^n d(I, v_i) \text{ord}_R(I_i) - \text{ord}_R(I)$$

and $s \in \mathbb{Z}$. It follows from the definition of s that $s \geq 0$.

According to (2), we need to show the following to prove the theorem in this case:

$$T(\mathrm{Im}^s) = T(I_1^{u_1} \dots I_n^{u_n})$$

and

$$d(\mathrm{Im}^s, v) = d(I_1^{u_1} \dots I_n^{u_n}, v) \quad \text{for all } v \in T(\mathrm{Im}^s).$$

Note that $s = 0$ if and only if $\mathrm{ord}_R \notin T(I_i)$ for all i , i.e. the ideals I_i are all one-fibered. This implies that $T(\mathrm{Im}^s) = T(I_1^{u_1} \dots I_n^{u_n})$.

Since we are working in the case where $\mathrm{ord}_R \neq v_i$ for all i , we have for every $i = 1, \dots, n$

$$d(\mathrm{Im}^s, v_i) = d(I, v_i) + sd(m, v_i) = d(I, v_i).$$

On the other hand,

$$d(I_1^{u_1} \dots I_n^{u_n}, v_i) = u_i d(I_i, v_i)$$

and since I_i is the quasi-one-fibered ideal associated to v_i , we have $d(I_i, v_i) = 1$. So $d(\mathrm{Im}^s, v_i) = d(I_1^{u_1} \dots I_n^{u_n}, v_i)$.

Now we compare the degree coefficients with respect to ord_R . As $\mathrm{ord}_R \notin T(I)$, we have

$$d(\mathrm{Im}^s, \mathrm{ord}_R) = sd(m, \mathrm{ord}_R) = se(m).$$

Since

$$d(I_1^{u_1} \dots I_n^{u_n}, \mathrm{ord}_R) = \sum_{i=1}^n u_i d(I_i, \mathrm{ord}_R) = \sum_{i=1}^n d(I, v_i) d(I_i, \mathrm{ord}_R),$$

we find that these degree coefficients are equal. So we can conclude that $\mathrm{Im}^s = I_1^{u_1} \dots I_n^{u_n}$.

Next, we turn to the case where $\mathrm{ord}_R \in T(I)$, say $v_1 = \mathrm{ord}_R$. We put $u_1 = 0$, $u_i = d(I, v_i)$ for all $i = 2, \dots, n$, and define

$$s = \frac{1}{e(m)} \left(\sum_{i=2}^n d(I, v_i) d(I_i, \mathrm{ord}_R) - d(I, \mathrm{ord}_R) \right).$$

To see that $s \in \mathbb{Z}$, we use again Lemma 2.2 and Theorem 2.1,

$$se(m) = \sum_{i=2}^n d(I, v_i) d(I_i, \mathrm{ord}_R) - d(I, \mathrm{ord}_R)$$

$$\begin{aligned}
&= \sum_{i=2}^n d(I, v_i) (e(m) \operatorname{ord}_R(I_i) - v_i(m)) - d(I, \operatorname{ord}_R) \\
&= e(m) \sum_{i=2}^n d(I, v_i) \operatorname{ord}_R(I_i) - \sum_{i=2}^n d(I, v_i) v_i(m) - d(I, \operatorname{ord}_R) \operatorname{ord}_R(m) \\
&= e(m) \sum_{i=2}^n d(I, v_i) \operatorname{ord}_R(I_i) - d_I(m) \\
&= e(m) \sum_{i=2}^n d(I, v_i) \operatorname{ord}_R(I_i) - e(m) \operatorname{ord}_R(I).
\end{aligned}$$

So $s \in \mathbb{Z}$.

Just like in the first part of the proof, we have for all $i = 2, \dots, n$ that $T(\operatorname{Im}^s) = T(I_1^{u_1} \dots I_n^{u_n})$ and $d(\operatorname{Im}^s, v_i) = d(I_1^{u_1} \dots I_n^{u_n}, v_i)$. From the definition of s , and since $u_1 = 0$, we find

$$\begin{aligned}
d(\operatorname{Im}^s, \operatorname{ord}_R) &= d(I, \operatorname{ord}_R) + se(m) = \sum_{i=2}^n d(I, v_i) d(I_i, \operatorname{ord}_R) \\
&= d(I_1^{u_1} \dots I_n^{u_n}, \operatorname{ord}_R).
\end{aligned}$$

By replacing s by $s + |s|$ and u_1 by $u_1 + |s|$, we can assume that $s \geq 0$. So again we find $\operatorname{Im}^s = I_1^{u_1} \dots I_n^{u_n}$. $\square$

Remark 2.5 There are two cases where $s = 0$: (a) when $\operatorname{ord}_R \notin T(I)$ and all ideals $I_1, \dots, I_n$ are one-fibered, and (b) when $\operatorname{ord}_R \in T(I)$ and all ideals I_i for $i = 2, \dots, n$ are one-fibered. In these two cases we get $I = I_1^{u_1} \dots I_n^{u_n}$. From the proof and the properties of degree coefficients, it follows that the exponents $s, u_1, \dots, u_n$ are unique when $\operatorname{ord}_R \notin T(I)$.

3 Projectively full ideals

In this section we show that if $\operatorname{Im}^s = I_1^{u_1} \dots I_n^{u_n}$, where I_i are the quasi-one-fibered ideals associated to the Rees valuations of I , $\operatorname{ord}_R \notin T(I)$ and $\gcd(u_1, \dots, u_n) = 1$, then I is projectively full.

Let I and J be two ideals of a Noetherian ring R . If $\overline{I^i} = \overline{J^j}$ for some positive integers i and j , then I and J are called *projectively equivalent*. The notion of projectively equivalent ideals was introduced by Samuel in [13]. A regular ideal I of R is called *projectively full* if the following holds: if J is a complete ideal that is projectively equivalent to I , then $J = \overline{I^i}$ for some $i \in \mathbb{Z}^+$. Projectively full ideals were introduced by Ciuperca et al. in [2]. The work on projectively full ideals was continued in a.o. [1, 7]. In [1, Example 3.2] the authors note that a complete ideal $I = I_1^{e_1} \dots I_n^{e_n}$ of a 2-dimensional regular local ring (where $I_1^{e_1} \dots I_n^{e_n}$ is the unique factorization of I into simple complete ideals) is projectively full if and only if $\gcd(e_1, \dots, e_n) = 1$. We can generalize this as follows.

Theorem 3.1 *Let (R, m) be a 2-dimensional Muhly rational singularity and let I be a complete m -primary ideal of R with $\text{ord}_R I \notin T(I)$. Suppose that $T(I) = \{v_1, \dots, v_n\}$ and that $\text{Im}^s = I_1^{u_1} \dots I_n^{u_n}$ with $s \in \mathbb{N}$ and $u_i \in \mathbb{Z}^+$ and I_i the quasi-one-fibered ideal associated to v_i . If $\gcd(u_1, \dots, u_n) = 1$, then I is projectively full.*

Proof Suppose that for a complete m -primary ideal I of R , $\text{Im}^s = I_1^{u_1} \dots I_n^{u_n}$, $\text{ord}_R I \notin T(I)$ and $\gcd(u_1, \dots, u_n) = 1$. Let J be a complete m -primary ideal of R that is projectively equivalent to I : $J^j = I^i$ for some i, j . This implies that $T(I) = T(J)$. According to Theorem 2.4, we have

$$Jm^k = I_1^{t_1} \dots I_n^{t_n}$$

for some $k \in \mathbb{N}$ and $t_i \in \mathbb{Z}^+$. Then from $J^j m^{kj} m^{si} = I^i m^{si} m^{kj}$, it follows that

$$I_1^{jt_1} \dots I_n^{jt_n} m^{si} = I_1^{iu_1} \dots I_n^{iu_n} m^{kj}.$$

So

$$jt_r d(I_r, v_r) = iu_r d(I_r, v_r)$$

and $jt_r = iu_r$ for all $i = 1, \dots, n$. Since $\gcd(u_1, \dots, u_n) = 1$, it follows that $j|i$. As $d(J, v_r) = t_r = iu_r/j = id(I, v_r)/j$ for all $r = 1, \dots, n$, and as $T(J) = T(I)$, it follows from (2) that $J = I^{i/j}$. So I is projectively full. $\square$

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