A scaling property of Farey fractions

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Abstract The Farey sequence of order n consists of all reduced fractions a/b between 0 and 1 with positive denominator b less than or equal to n . The sums of the inverse denominators $1/b$ of the Farey fractions in prescribed intervals with rational bounds have a simple main term, but the deviations are determined by an interesting sequence of polygonal functions f_n . For $n \rightarrow \infty$ we also obtain a certain limit function, which describes an asymptotic scaling property of functions f_n in the vicinity of any fixed fraction a/b and which is independent of a/b . The result can be obtained by using only elementary methods. We also study this limit function and especially its decay behaviour by using the Mellin transform and analytical properties of the Riemann zeta function.

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1 Introduction

The Farey sequence $\mathcal{F}_n$ of order n consists of all reduced fractions a/b with natural denominator $b \leq n$ in the unit interval $[0, 1]$, arranged in order of increasing size. Dropping just the restriction $0 \leq a/b \leq 1$ gives the infinite extended Farey sequence $\mathcal{F}_n^{\text{ext}}$ of order n . We use these notations from Definitions 2.1 and 2.3 and present an overview of new results in each section.

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If the denominator b of a reduced fraction a/b is small compared to n , then the Farey sequence $\mathcal{F}_n^{\text{ext}}$ has a regular behaviour in a sufficiently small vicinity of a/b . To illustrate this we consider the Farey sequence $\mathcal{F}_{100}$ in the vicinity of $0/1$. In this case we have 50 subsequent fractions

$$\frac{1}{100} < \frac{1}{99} < \frac{1}{98} < \frac{1}{97} < \cdots < \frac{1}{51}. \quad (1)$$

These are followed by 16 blocks consisting of two fractions $1/(q+2)$, $2/(2q+3)$, ranging from $q = 48$ down to $q = 33$, namely

$$\frac{1}{50} < \frac{2}{99} < \frac{1}{49} < \frac{2}{97} < \cdots < \frac{1}{36} < \frac{2}{71} < \frac{1}{35} < \frac{2}{69}. \quad (2)$$

After a small “transient section”, consisting of the three fractions

$$\frac{1}{34} < \frac{2}{67} < \frac{3}{100}, \quad (3)$$

the reader will find further regular patterns with bigger blocks in $\mathcal{F}_{100}$.

In Sect. 2 we determine the general law behind these examples, which describes the local behaviour of $\mathcal{F}_n^{\text{ext}}$ for large values of n in the vicinity of a fixed reduced fraction a/b . For this purpose we derive a simple explicit representation formula for the corresponding neighbouring fractions in terms of the function $\xi_+(q)$ in Lemma 2.5. This lemma enables a simple representation and calculation of sections in $\mathcal{F}_n^{\text{ext}}$ like (1) and (2), whereas general transient sections in $\mathcal{F}_n^{\text{ext}}$ like (3) are defined and determined in Lemma 2.7. These results are summarized in Theorem 2.8, the main result of Sect. 2, which is needed for Sect. 4.

In Sect. 3 we study three families of 1-periodic functions

$$\Psi_n^*, \Psi_n, f_n: \mathbb{R} \rightarrow \mathbb{R}$$

which are related to the Farey sequence $\mathcal{F}_n$. Now we define these functions and explain how they describe the structure of the Farey sequence. In addition, we obtain interesting relations between the functions Ψ_n and f_n , the prime number theorem and the Riemann hypothesis.

The 1-periodic functions Ψ_n^* are determined in the unit interval. With the Farey fractions in Definition 2.1 and for $a_{\alpha-1}/b_{\alpha-1} \leq t < a_{\alpha}/b_{\alpha}$, $\alpha = 1, \dots, N$, we put

$$\Psi_n^*(t) = -\frac{b_{\alpha} + b_{\alpha-1}}{2} \left[t - \frac{a_{\alpha} + a_{\alpha-1}}{b_{\alpha} + b_{\alpha-1}} \right].$$

In addition we require $\Psi_n^*(1) = \Psi_n^*(0) = 1/2$. Then the function Ψ_n^* is linear between two subsequent Farey fractions in $\mathcal{F}_n$, and has a jump of height $1/b_{\alpha}$ at each Farey fraction $a_{\alpha}/b_{\alpha} \in \mathcal{F}_n$. Using only elementary properties of Farey sequences, we obtain in Theorem 3.2 the following $L^2(0, 1)$ -estimate:

Fig. 1 The jumps of Ψ_{20}^* are centered around zero

$$\|\Psi_n^*\|_2^2 = \int_0^1 \Psi_n^*(t)^2 dt = \mathcal{O}\left(\frac{\log n}{n}\right), \quad n \rightarrow \infty. \quad (4)$$

The plot of Ψ_{20}^* in Fig. 1 shows that the jump discontinuities are centered symmetrically around zero. This holds in general and enables the estimation of the L^2 -norm of Ψ_n^* .

Using $\beta(t) = t - [t] - 1/2$, the 1-periodic functions $\Psi_n: \mathbb{R} \rightarrow \mathbb{R}$ introduced in Definition 3.4 may be rewritten as

$$\Psi_n(t) = - \sum_{j \leq n} \frac{\beta(jt)}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k}, \quad t \in \mathbb{R}.$$

Like Ψ_n^* , the function Ψ_n is linear between two subsequent Farey fractions in $\mathcal{F}_n$, and also has a jump of height $1/b_\alpha$ at each Farey fraction $a_\alpha/b_\alpha \in \mathcal{F}_n$. A plot of Ψ_{20} is given in Fig. 2.

We derive the general Theorem 3.8 to estimate the L^2 -norm of arbitrary linear combinations of $\beta(j \cdot)$. Its application to Ψ_n gives absolute constants $c_1, c_2 > 0$ such that there holds for all $n \geq 3$,

$$\begin{aligned} \frac{c_1}{\log n \log \log n} \sum_{j=1}^n \frac{1}{j^2} \left(\sum_{k \leq n/j} \frac{\mu(k)}{k} \right)^2 &\leq \|\Psi_n\|_2^2 \\ &\leq c_2 \log n \log \log n \sum_{j=1}^n \frac{1}{j^2} \left(\sum_{k \leq n/j} \frac{\mu(k)}{k} \right)^2. \end{aligned} \quad (5)$$

Fig. 2 The jumps of Ψ_{20} are not centered around zero

Here $\|h\|_2$ is the $L^2(0, 1)$ -norm of any square integrable function h .

Recall that Ψ_n and Ψ_n^* both have its jumps of height $1/b$ at each Farey fraction a/b . Thus it is natural to define the following *continuous* function $f_n: \mathbb{R} \rightarrow \mathbb{R}$ for each $n \in \mathbb{N}$ by

$$f_n(t) = \Psi_n^*(t) - \Psi_n(t).$$

The elementary properties of these *polygonal* functions f_n are collected in Theorem 3.6. Using the notations from Definition 2.1 and

$$\xi_\alpha = \frac{a_\alpha}{b_\alpha} \in \mathcal{F}_n, \quad s_n = \sum_{k=1}^n \frac{\varphi(k)}{k},$$

we especially obtain that f_n gives control of the partial sums of the inverse denominators $1/b$ of the Farey fractions $a/b \in \mathcal{F}_n$, namely

$$f_n(\xi_\alpha) = s_n \xi_\alpha + \frac{1}{2} \left(1 + \frac{1}{b_\alpha} \right) - \sum_{\beta=0}^{\alpha} \frac{1}{b_\beta} \quad \text{for all } \alpha = 0, \dots, N.$$

A plot of f_{20} is given in Fig. 3.

It follows from (5), (4) and the prime number theorem in Prachar [15, Kapitel III, Satz 5.1] that the L^2 -norms of Ψ_n and f_n tend to zero for $n \rightarrow \infty$, see Corollary 3.9. It also results Corollary 3.10, a variant of the Franel–Landau theorem for the characterization of the Riemann hypothesis, see [3, Chapter 12.2], [4, 8, 10, 13].

Section 4 contains the main result, Theorem 4.6, which was obtained by using only elementary properties of Farey fractions, namely from Theorem 2.8. Theorem 4.6

Fig. 3 The polygonal function f_{20}

describes the behaviour of the Farey sequence and the function f_n with large order n in the vicinity of any fixed fraction in terms of a limit function g as follows: Assume that $a/b \in \mathcal{F}_n^{\text{ext}}$ and put

$$g_{a,b}(n, x) = b \left(f_n \left(\frac{a}{b} + \frac{x}{bn} \right) - f_n \left(\frac{a}{b} \right) \right), \quad x \in \mathbb{R}.$$

Then for $n \rightarrow \infty$ the sequence of functions $g_{a,b}(n, \cdot)$ converges *uniformly* on each interval $[-x_*, x_*]$, $x_* > 0$ fixed, towards a continuous, piecewise smooth and odd limit function $g: \mathbb{R} \rightarrow \mathbb{R}$, which is *independent* of the choice of the fraction a/b .

The finer shades in the distribution of the Farey fractions can be described in terms of the polygonal functions f_n , and especially in the vicinity of any fixed fraction a/b they are determined by the asymptotic scaling property of the functions f_n in Theorem 4.6.

In Sect. 5, Theorem 5.4, we have derived an integral representation for the limit function g by using analytical properties of the Riemann zeta function, the Mellin transform and the theory of Hardy spaces. This integral representation and Fourier analysis is used in order to justify in Corollary 5.5 that the limit function g decays to zero.

2 Farey sequences of order n

Definition 2.1 For $n \in \mathbb{N}$ and $N = \sum_{k=1}^n \varphi(k)$ the Farey sequence $\mathcal{F}_n$ of order n consists of all reduced and ordered fractions

$$\frac{0}{1} = \frac{a_0}{b_0} < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_N}{b_N} = \frac{1}{1}$$

with $1 \leq b_\alpha \leq n$ for $\alpha = 0, 1, \dots, N$.

Example 2.2 For $n = 5$ and $N = \sum_{k=1}^5 \varphi(k) = 10$ we have the following $N + 1 = 11$ fractions:

$$\frac{0}{1} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{1}{1}.$$

Definition 2.3 The Farey sequence $\mathcal{F}_n^{\text{ext}}$ is the extension of $\mathcal{F}_n$ consisting of all reduced fractions a/b with $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, $b \leq n$.

The following theorem provides two important and well-known properties of the Farey sequences $\mathcal{F}_n^{\text{ext}}$, which can be obtained by employing results from the textbook [6, Chapter III].

Theorem 2.4 (basic properties of Farey fractions)

- (a) Assume that $a, a' \in \mathbb{Z}$ and $b, b', n \in \mathbb{N}$ with $(a, b) = (a', b') = 1$. Then the reduced fractions $a/b < a'/b'$ are consecutive fractions in $\mathcal{F}_n^{\text{ext}}$ if and only if

$$a'b - ab' = 1, \quad b \leq n, \quad b' \leq n \quad \text{and} \quad b + b' > n.$$

- (b) Assume that $a/b < a^*/b^*$ are consecutive reduced fractions in the extended Farey sequence $\mathcal{F}_b^{\text{ext}}$ of order b . If $b \leq n$, then the two numbers

$$\frac{a}{b} < \frac{a^* + a \lfloor (n - b^*)/b \rfloor}{b^* + b \lfloor (n - b^*)/b \rfloor}$$

are consecutive reduced fractions in the extended Farey sequence $\mathcal{F}_n^{\text{ext}}$ of order n .

Lemma 2.5 Assume that $a/b < a^*/b^*$ are consecutive reduced fractions in the extended Farey sequence $\mathcal{F}_b^{\text{ext}}$ of order $b \leq n$, where $n \in \mathbb{N}$. We define

$$\xi_+(q) = \frac{a^* + aq}{b^* + bq}, \quad q \geq 0, \quad (6)$$

and for $\alpha \in \mathbb{N}$ the quantities

$$Q_\alpha^* = \begin{cases} \left\lfloor \frac{n - b^*}{b} \right\rfloor, & \alpha = 1, \\ \left\lfloor \frac{n - \alpha b^* + b}{\alpha b} \right\rfloor, & \alpha \geq 2, \end{cases} \quad (7)$$

as well as

$$q_\alpha^* = \left\lceil \frac{n + 1 - \alpha b^* - b}{\alpha b} \right\rceil. \quad (8)$$

Then the inequalities

$$0 \leq q_{\alpha+1}^* \leq q \leq Q_\alpha^* - 1$$

determine exactly those integer numbers $q \geq 0$ for which $\xi_+(q + q')$ gives a section of $\mathcal{F}_n^{\text{ext}}$ if q' takes all values of $\mathcal{F}_\alpha$, starting with $q' = 1/1$ and running down to $q' = 0/1$.

Proof We first note that

$$q_\alpha^* \geq 0 \quad \text{for all } \alpha \in \mathbb{N} \quad (9)$$

simply follows from $1 \leq b^* \leq b \leq n$.

Assume that $0 \leq q'' = a''/b'' < q' = a'/b' \leq 1$ are consecutive reduced fractions in $\mathcal{F}_\alpha$. Now we form the two fractions

$$\xi_+(q + q') = \frac{aa' + (a^* + aq)b'}{ba' + (b^* + bq)b'} < \xi_+(q + q'') = \frac{aa'' + (a^* + aq)b''}{ba'' + (b^* + bq)b''} \quad (10)$$

for an integer number $q \geq 0$, and note that all matrices in

$$\begin{pmatrix} b & b^* + bq \\ a & a^* + aq \end{pmatrix} \cdot \begin{pmatrix} a' & a'' \\ b' & b'' \end{pmatrix} = \begin{pmatrix} ba' + (b^* + bq)b' & ba'' + (b^* + bq)b'' \\ aa' + (a^* + aq)b' & aa'' + (a^* + aq)b'' \end{pmatrix}$$

have determinant one. We conclude that the fractions in (10) are already reduced. Using Theorem 2.4(a), we conclude that $\xi_+(q + q')$ and $\xi_+(q + q'')$ are consecutive fractions of $\mathcal{F}_n^{\text{ext}}$ if and only if the following three inequalities are satisfied:

$$ba' + (b^* + bq)b' \leq n, \quad (11)$$

$$ba'' + (b^* + bq)b'' \leq n, \quad (12)$$

$$b(a' + a'') + (b^* + bq)(b' + b'') \geq n + 1. \quad (13)$$

The left-hand side in (13) takes its smallest value for $a' = 1, b' = \alpha, a'' = 0, b'' = 1$, provided that $q'' < q'$ are any two consecutive fractions in $\mathcal{F}_\alpha$. Hence (13) is satisfied if and only if the integer q satisfies the inequality

$$q \geq q_{\alpha+1}^*. \quad (14)$$

The largest value on the left-hand side in (11) and (12) results for $\alpha = 1$ with $a' = b' = 1$ and $a'' = 0, b'' = 1$, respectively, and for $\alpha \geq 2$ with $a' = \alpha - 1, b' = \alpha$. In any case inequalities (11) and (12) are satisfied for the integer number q if and only if

$$q \leq Q_\alpha^* - 1, \quad (15)$$

provided that $q'' < q'$ are any two consecutive fractions in $\mathcal{F}_\alpha$. Inequalities (9), (14) and (15) conclude the proof of the lemma. $\square$

Lemma 2.6 Assume that $n, b, A \in \mathbb{N}$ are such that $n \geq 3b$ and $A \leq \sqrt{n/(2b)} - 1/4$. Using the quantities in Lemma 2.5, we obtain

$$0 \leq q_{\alpha+1}^* < Q_\alpha^* \quad \text{for all } \alpha = 1, \dots, A.$$

Proof For $\alpha = 1$ we have $Q_\alpha^* = \lfloor (n - b^*)/b \rfloor \geq 2$ due to $n \geq 3b$, and a simple estimation shows that $1 \leq q_2^* < Q_1^*$ in this case. Now assume that $\alpha \geq 2$. In order to show $q_{\alpha+1}^* < Q_\alpha^*$ for all $\alpha \leq A$ we only have to guarantee that

$$\frac{n+1-(\alpha+1)b^*-b}{(\alpha+1)b} + 1 < \frac{n-\alpha b^*+b}{\alpha b} - 1.$$

This inequality is satisfied if and only if

$$\alpha < \sqrt{\frac{n+b}{2b}} + \frac{1}{16b^2} - \frac{1}{4b}, \quad (16)$$

and the right-hand side in (16) is larger than $\sqrt{n/(2b)} - 1/4$. $\square$

Lemma 2.7 (transient sections of $\mathcal{F}_n^{\text{ext}}$) *Assume that $n, b, A \in \mathbb{N}$ are such that $n \geq 3b$ and $A \leq \sqrt{n/(2b)} - 1/4$. Then we have either $Q_{\alpha+1}^* = q_{\alpha+1}^*$ or $Q_{\alpha+1}^* = q_{\alpha+1}^* - 1$ for all $\alpha = 1, \dots, A-1$.*

In the latter case we call the section of $\mathcal{F}_n^{\text{ext}}$ in the interval

$$[\xi_+(q_{\alpha+1}^*), \xi_+(q_{\alpha+1}^* - 1))$$

a transient section (of $\mathcal{F}_n^{\text{ext}}$) with index $q = q_{\alpha+1}^ - 1$. The transient section with index q contains all fractions $\xi_+(q + q')$ with $q' \in \mathcal{F}_\alpha \cap (0, 1]$, and besides these only fractions of the form $\xi_+(q + \tilde{a}/(\alpha+1))$, where $\tilde{a} \in [1, \alpha]$ is relatively prime to $\alpha+1$.*

Proof For $1 \leq \alpha < A$ we put

$$\eta_+ = \frac{n - (\alpha+1)b^* + b}{(\alpha+1)b}, \quad \eta_- = \frac{n+1 - (\alpha+1)b^* - b}{(\alpha+1)b},$$

and obtain

$$Q_{\alpha+1}^* = \lfloor \eta_+ \rfloor, \quad q_{\alpha+1}^* = \lceil \eta_- \rceil, \quad \delta = \eta_+ - \eta_- = \frac{2b-1}{(\alpha+1)b} \in (0, 1),$$

because $b \geq 1$ and $\alpha+1 \geq 2$.

Case 1: There is an index $k \in \mathbb{N}$ such that

$$k-1 < \eta_- \leq k \leq \eta_+ < k+1.$$

In this case we have $k = \lceil \eta_- \rceil = \lfloor \eta_+ \rfloor$ and hence $Q_{\alpha+1}^* = q_{\alpha+1}^*$. Regarding $0 < \delta < 1$ we only have to study the following case.

Case 2: There is an integer number $k \geq 0$ with $k < \eta_- < \eta_+ < k+1$. Here we obtain that $\lceil \eta_- \rceil = k+1$ and $\lfloor \eta_+ \rfloor = k$, i.e. $Q_{\alpha+1}^* = q_{\alpha+1}^* - 1$. Only in this case we have to justify the stated properties of the transient section with index $q = q_{\alpha+1}^* - 1$.

From Lemma 2.6 we first obtain that $q_{\alpha+1}^* < Q_\alpha^*$ since $1 \leq \alpha < A$. It follows that $q < Q_\alpha^* - 1$, such that (15) in the proof of Lemma 2.5 is strictly satisfied. In this proof we also have concluded from (15) that $\xi_+(q+q') \in \mathcal{F}_n^{\text{ext}}$ for all $q' \in \mathcal{F}_\alpha \cap (0, 1]$.

Now we consider two consecutive fractions

$$0 \leq q'' = \frac{a''}{b''} < q' = \frac{a'}{b'} \leq 1$$

in $\mathcal{F}_\alpha$, and we will also admit that $q'' = 0/1$. Then the median $\tilde{\xi}$ formed with $\xi_+(q+q')$ and $\xi_+(q+q'')$ is given by the reduced fraction

$$\tilde{\xi} = \frac{a(a'+a'') + (a^*+aq)(b'+b'')}{b(a'+a'') + (b^*+bq)(b'+b'')}$$

with the denominator

$$B = b(a'+a'') + (b^*+bq)(b'+b'').$$

The transient section of $\mathcal{F}_n^{\text{ext}}$ with index $q = q_{\alpha+1}^* - 1$ violates (14) and thus contains at least one median $\tilde{\xi}$ of this form with

$$B \leq n, \quad (17)$$

such that $\tilde{\xi} \in \mathcal{F}_n^{\text{ext}}$ and

$$\xi_+(q_{\alpha+1}^*) \leq \tilde{\xi} < \xi_+(q_{\alpha+1}^* - 1).$$

For any such median $\tilde{\xi}$ we will prove next that

$$b' + b'' = \alpha + 1. \quad (18)$$

We assume that $b' + b'' \geq \alpha + 2$. Lemma 2.6 and $\alpha + 1 \leq A$ imply that

$$q = q_{\alpha+1}^* - 1 = Q_{\alpha+1}^* > q_{\alpha+2}^* \geq \frac{n+1-(\alpha+2)b^*-b}{(\alpha+2)b},$$

which contradicts (17):

$$\begin{aligned} B &\geq b(a'+a'') + (b^*+bq)(\alpha+2) \\ &> b(a'+a'') + b^*(\alpha+2) + n+1 - (\alpha+2)b^* - b \\ &= n+1 + b(a'+a''-1) \geq n+1. \end{aligned}$$

We have shown (18) and hence obtain

$$\begin{aligned} B &= b(a' + a'') + (b^* + bq)(\alpha + 1) \\ &\geq b + b^*(\alpha + 1) + n + 1 - (\alpha + 1)b^* - b - b(\alpha + 1) \\ &= n + 1 - b(\alpha + 1). \end{aligned}$$

We summarize these inequalities in

$$n + 1 - b(\alpha + 1) \leq B \leq n. \quad (19)$$

It remains to show that the transient section of $\mathcal{F}_n^{\text{ext}}$ with index q contains besides the fractions $\xi_+(q + q')$ with $q' \in \mathcal{F}_\alpha \cap (0, 1]$ only some mediant $\tilde{\xi} = \xi_+(q + \tilde{a}/(\alpha + 1))$, where $\tilde{a} = a' + a'' \in [1, \alpha]$ is relatively prime to $b' + b'' = \alpha + 1$.

For this purpose we will use the inequality

$$\sqrt{\frac{x}{2}} - \frac{1}{4} < \sqrt{x - \frac{3}{4}} - \frac{1}{2} \quad \text{for all } x \geq 3. \quad (20)$$

We regard that $q = q_{\alpha+1}^* - 1$ implies

$$b^* + bq \geq \frac{b^*(\alpha + 1) + n + 1 - (\alpha + 1)b^* - b}{\alpha + 1} - b = \frac{n + 1 - b(\alpha + 2)}{\alpha + 1},$$

and note the equivalence

$$\frac{n + 1 - b(\alpha + 2)}{\alpha + 1} \geq b(\alpha + 1) \iff \alpha + 1 \leq -\frac{1}{2} + \sqrt{\frac{n}{b} - \frac{3}{4} + \frac{1}{b}}.$$

Using (20) we obtain with $x = n/b \geq 3$ for all $\alpha = 1, \dots, A - 1$,

$$\alpha + 1 \leq \sqrt{\frac{n}{2b}} - \frac{1}{4} < \sqrt{\frac{n}{b} - \frac{3}{4}} - \frac{1}{2} < -\frac{1}{2} + \sqrt{\frac{n}{b} - \frac{3}{4} + \frac{1}{b}},$$

and hence

$$b^* + bq \geq b(\alpha + 1).$$

The denominator B' of the reduced fraction representing $\xi_+(q + q')$ satisfies

$$B' = ba' + (b^* + bq)b' \geq b(\alpha + 1),$$

and the denominator B'' of the reduced fraction representing $\xi_+(q + q'')$ the inequality

$$B'' = ba'' + (b^* + bq)b'' \geq b(\alpha + 1).$$

The last two inequalities and (19) imply that

$$B + B' > n, \quad B + B'' > n,$$

which concludes the proof of the lemma. $\square$

The following theorem is a slightly reformulated summary of the preceding lemmas.

Theorem 2.8 (local structure of $\mathcal{F}_n^{\text{ext}}$ right to a/b) *Assume that $n, b, A \in \mathbb{N}$ are such that $n \geq 3b$ and $A \leq \sqrt{n/(2b)} - 1/4$. We use the notations of the function ξ_+ in (6), of the quantities Q_α^* in (7) and q_α^* in (8). Then we have $(Q_{\alpha+1}^*, Q_\alpha^*) \supseteq (q_{\alpha+1}^*, Q_\alpha^*)$ for all $\alpha = 1, \dots, A-1$, and each interval $(q_{\alpha+1}^*, Q_\alpha^*)$ has at least length one. We define the section $\mathcal{F}_{n,\alpha}^{\text{ext}} = \mathcal{F}_n^{\text{ext}} \cap [\xi_+(Q_\alpha^*), \xi_+(Q_{\alpha+1}^*)]$.*

- (a) *In the case $Q_{\alpha+1}^* = q_{\alpha+1}^*$, the section $\mathcal{F}_{n,\alpha}^{\text{ext}}$ consists exactly of the fractions $\xi_+(\xi)$ with $\xi \in \mathcal{F}_\alpha^{\text{ext}} \cap (Q_{\alpha+1}^*, Q_\alpha^*)$.*
- (b) *In the case $Q_{\alpha+1}^* \neq q_{\alpha+1}^*$, we have $Q_{\alpha+1}^* = q_{\alpha+1}^* - 1$, and $\mathcal{F}_{n,\alpha}^{\text{ext}}$ contains all fractions $\xi_+(\xi)$ with $\xi \in \mathcal{F}_\alpha^{\text{ext}} \cap (Q_{\alpha+1}^*, Q_\alpha^*)$. For the index $q = q_{\alpha+1}^* - 1$ we obtain a transient section in $\mathcal{F}_{n,\alpha}^{\text{ext}}$ which also contains some fractions of the form $\xi_+(q + \tilde{a}/(\alpha+1))$, where $\tilde{a} \in [1, \alpha]$ is relatively prime to $\alpha+1$. Finally, besides these fractions of the form $\xi_+(\xi)$ and $\xi_+(q + \tilde{a}/(\alpha+1))$ there is no other fraction in $\mathcal{F}_{n,\alpha}^{\text{ext}}$.*

3 A family of functions Ψ_n^* , Ψ_n and f_n related to $\mathcal{F}_n$

Definition 3.1 We use Definition 2.1 and put for $a_{\alpha-1}/b_{\alpha-1} \leq t < a_\alpha/b_\alpha$, $\alpha = 1, \dots, N$:

$$\Psi_n^*(t) = -\frac{b_\alpha + b_{\alpha-1}}{2} \left[t - \frac{a_\alpha + a_{\alpha-1}}{b_\alpha + b_{\alpha-1}} \right],$$

and in addition $\Psi_n^*(1) = \Psi_n^*(0) = 1/2$. Then we have the 1-periodic extension $\Psi_n^*: \mathbb{R} \rightarrow \mathbb{R}$.

The function Ψ_n^* is linear between two subsequent Farey fractions in $\mathcal{F}_n$, and has a jump of height $1/b_\alpha$ at each Farey fraction $a_\alpha/b_\alpha \in \mathcal{F}_n$.

Theorem 3.2 (L^2 -estimate of Ψ_n^*)

$$\|\Psi_n^*\|_2^2 = \int_0^1 \Psi_n^*(t)^2 dt = \mathcal{O}\left(\frac{\log n}{n}\right), \quad n \rightarrow \infty.$$

Proof For $a \leq t \leq b$ and $s, q \in \mathbb{R}$ we consider $f(t) = s \cdot t + q$. Then

$$\int_a^b f(t)^2 dt = \frac{b-a}{3} [f(a)^2 + f(a)f(b) + f(b)^2] dt. \quad (21)$$

We put $\eta_\alpha = (1/b_{\alpha-1} + 1/b_\alpha)/2$ for $\alpha = 1, \dots, N$, use (21) and obtain

$$\|\Psi_n^*\|_2^2 = \frac{1}{3} \sum_{\alpha=1}^N \frac{\eta_\alpha^2}{b_\alpha b_{\alpha-1}} - \frac{1}{4} \sum_{\alpha=1}^N \frac{1}{b_\alpha^2 b_{\alpha-1}^2}. \quad (22)$$

From the symmetry relation $b_{N-\alpha} = b_\alpha$, which is valid for $\alpha = 1, \dots, N$, and from Theorem 2.4(a) it follows for $n > 1$ that

$$\sum_{\alpha=1}^N \frac{1}{b_\alpha^2 b_{\alpha-1}^2} = 2 \sum_{j=\lfloor (n+1)/2 \rfloor}^n \frac{1}{j^2} \sum_{\substack{n+1 \leq k < 2j \\ (k,j)=1}} \frac{1}{(k-j)^2}. \quad (23)$$

The following inequality is valid for all natural numbers $a \leq b$:

$$\sum_{a < m \leq b} \frac{1}{m^2} \leq \int_a^b \frac{dx}{x^2} = \frac{1}{a} - \frac{1}{b}.$$

We use it for $n > 1$ to estimate

$$\begin{aligned} \sum_{\alpha=1}^N \frac{1}{b_\alpha^2 b_{\alpha-1}^2} &\leq 2 \sum_{j=\lfloor (n+1)/2 \rfloor}^n \frac{1}{j^2} \sum_{m=n-j+1}^j \frac{1}{m^2} \\ &\leq 2 \sum_{n/2 \leq j < n} \frac{1}{j^2} \left(\frac{1}{n-j} - \frac{1}{j} \right) + \frac{4}{n^2} \\ &= 2 \sum_{n/2 \leq j < n} \frac{1}{j^3} \frac{2j-n}{n-j} + \frac{4}{n^2}. \end{aligned} \quad (24)$$

We approximate the right-hand side in (24) by the integral

$$2 \int_{n/2}^{n-1} \frac{2x-n}{n-x} \cdot \frac{dx}{x^3} = \frac{2}{n^2} \log(n-1) - \frac{1}{n^2} + \frac{1}{n^2(n-1)^2} = \mathcal{O}\left(\frac{\log n}{n^2}\right)$$

for $n \rightarrow \infty$. It follows from (23) that

$$\sum_{\alpha=1}^N \frac{1}{b_\alpha^2 b_{\alpha-1}^2} = \mathcal{O}\left(\frac{\log n}{n^2}\right) \quad \text{for } n \rightarrow \infty. \quad (25)$$

Finally we make use of the inequalities

$$\frac{n}{b_\alpha b_{\alpha-1}} < 2\eta_\alpha \leq 2$$

and

$$\frac{\eta_\alpha^2}{b_\alpha b_{\alpha-1}} \leq \frac{n^2}{b_\alpha^3 b_{\alpha-1}^3} \leq \frac{2n}{b_\alpha^2 b_{\alpha-1}^2}$$

in order to conclude with (25) and (22) the desired norm inequality. $\square$

Remark 3.3 The estimate of $\|\Psi_n^*\|_2^2$ is possible, because the jumps of Ψ_n^* are symmetrically centered around zero. This is illustrated for $n = 20$ in Fig. 1. Note that (25) also follows from [5, Theorem 1]. Next we define a sequence of functions Ψ_n , which have the same height of the jumps at each Farey fraction like Ψ_n^* .

Definition 3.4 For $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we make use of the 1-periodic function

$$q_k(t) = - \sum_{d|k} \mu(d) \left(\frac{kt}{d} - \left\lfloor \frac{kt}{d} \right\rfloor - \frac{1}{2} \right)$$

in order to define for $n \in \mathbb{N}$ the 1-periodic function

$$\Psi_n = \sum_{k=1}^n \frac{1}{k} q_k.$$

A plot of Ψ_{20} is presented in Fig. 2. We note that in general these jumps are not symmetrically centered around zero.

Note that $q_k(t)$ has a jump of height $+1$ at each reduced fraction $t = j/k$ but is continuous elsewhere. Comparing Ψ_n with Ψ_n^* , we obtain that the function Ψ_n is linear between two subsequent Farey fractions in $\mathcal{F}_n$ and also has a jump of height $1/b_\alpha$ at each Farey fraction $a_\alpha/b_\alpha \in \mathcal{F}_n$.

Therefore, the difference between Ψ_n^* and Ψ_n is free from jumps, which justifies the following notion.

Definition 3.5 For $n \in \mathbb{N}$ we define the *polygonal* function $f_n: \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_n(t) = \Psi_n^*(t) - \Psi_n(t).$$

A plot of f_{20} is depicted in Fig. 3.

Theorem 3.6 (elementary properties of f_n)

- (a) $f_n: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and odd.
- (b) $f_n(0) = f_n(1/2) = f_n(1) = 0$.
- (c) The function f_n is linear between two subsequent Farey fractions $\xi_{\alpha-1} = a_{\alpha-1}/b_{\alpha-1} < \xi_\alpha = a_\alpha/b_\alpha$ in $\mathcal{F}_n$ for $\alpha = 1, \dots, N$.
- (d) If we define the sum $s_n = \sum_{k=1}^n \varphi(k)/k$, $\eta_0 = 0$ and the quantities $\eta_\alpha = (1/b_{\alpha-1} + 1/b_\alpha)/2$ for $\alpha = 1, \dots, N$, then we obtain

$$f_n(\xi_\alpha) = s_n \xi_\alpha - \sum_{\beta=0}^{\alpha} \eta_\beta \quad \text{for all } \alpha = 0, \dots, N.$$

Proof Statement (c) simply follows from the definitions of Ψ_n^* , Ψ_n and f_n , and (b) from (a). To show (a) we first conclude that f_n is continuous, since Ψ_n^* and Ψ_n both have the

same height $1/b_\alpha$ of the jump at $a_\alpha/b_\alpha \in \mathcal{F}_n$. For $t \in (0, 1) \setminus \mathcal{F}_n$ we have $\Psi_n(1-t) = -\Psi_n(t)$. The Farey fractions $\xi_\alpha = a_\alpha/b_\alpha \in \mathcal{F}_n$ satisfy symmetry conditions

$$b_{N-\alpha} = b_\alpha, \quad a_{N-\alpha} = b_\alpha - a_\alpha, \quad \xi_{N-\alpha} = 1 - \xi_\alpha \quad \text{for all } \alpha = 0, \dots, N.$$

These conditions also imply that $\Psi_n^*(1-t) = -\Psi_n^*(t)$ for $t \in (0, 1) \setminus \mathcal{F}_n$. We obtain that $f_n(-t) = f_n(1-t) = -f_n(t)$ for $t \in [0, 1]$ and hence for all $t \in \mathbb{R}$, since f_n is continuous on $\mathbb{R}$. Finally note that statement (d) is equivalent to

$$f_n(\xi_\alpha) - f_n(\xi_{\alpha-1}) = \frac{s_n}{b_\alpha b_{\alpha-1}} - \eta_\alpha = \left(s_n - \frac{b_\alpha + b_{\alpha-1}}{2} \right) (\xi_\alpha - \xi_{\alpha-1}) \quad (26)$$

for all $\alpha = 1, \dots, N$, because f_n is continuous with $f_n(0) = 0$. Now (26) follows from $\Psi_n^{*'}(t) = -(b_\alpha + b_{\alpha-1})/2$ and $\Psi_n'(t) = -s_n$ for $\xi_{\alpha-1} < t < \xi_\alpha$. $\square$

The following result can be checked by using

$$\sum_{d|k} \mu(d) = \begin{cases} 1 & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Lemma 3.7 *Let $\lambda: \mathbb{N} \rightarrow \mathbb{C}$ be a completely multiplicative function, i.e. $\lambda(1) = 1$ and $\lambda(mk) = \lambda(m)\lambda(k)$ for all $m, k \in \mathbb{N}$. We define the two divisor matrices $A_{\lambda,n} = (a_{jk})_{j,k=1,\dots,n} \in \mathbb{C}^{n \times n}$ and $B_{\lambda,n} = (b_{km})_{k,m=1,\dots,n} \in \mathbb{C}^{n \times n}$ with entries*

$$a_{jk} = \begin{cases} \lambda\left(\frac{k}{j}\right) & \text{for } j|k, \\ 0 & \text{for } j \nmid k, \end{cases}$$

$$b_{km} = \begin{cases} \mu\left(\frac{m}{k}\right)\lambda\left(\frac{m}{k}\right) & \text{for } k|m, \\ 0 & \text{for } k \nmid m. \end{cases}$$

Then the matrix $B_{\lambda,n}$ is the inverse of $A_{\lambda,n}$, i.e.

$$\sum_{k=1}^n a_{jk} b_{km} = \delta_{jm} = \begin{cases} 1 & \text{for } j = m, \\ 0 & \text{for } j \neq m. \end{cases}$$

This lemma is needed for the proof of the next theorem.

Theorem 3.8 (L^2 -estimates for spaces of discrete functions) *There are constants $c_1, c_2 > 0$ such that the following two estimates are valid for all $\underline{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ and all $n \geq 3$:*

(a)

$$\frac{c_1 |\underline{x}|^2}{\log n \log \log n} \leq \left\| \sum_{k=1}^n x_k \beta(k \cdot) \right\|_2^2 \leq c_2 |\underline{x}|^2 \log n \log \log n$$

with the 1-periodic sawtooth function $\beta: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\beta(t) = t - [t] - \frac{1}{2}.$$

(b)

$$\frac{c_1 |B_{1,n} \underline{x}|^2}{\log n \log \log n} \leq \left\| \sum_{k=1}^n x_k q_k \right\|_2^2 \leq c_2 |B_{1,n} \underline{x}|^2 \log n \log \log n.$$

Here $|\underline{y}|$ denotes the Euclid-norm of any vector $\underline{y} \in \mathbb{R}^n$.

Proof We make use of Lemma 3.7. First we show that (b) follows from (a), then we prove (a). For a given vector $\underline{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we put

$$\underline{y} = (y_1, \dots, y_n)^T = -B_{1,n} \underline{x}$$

with

$$y_j = - \sum_{\alpha \leq n/j} \mu(\alpha) x_{\alpha j} \quad \text{for } j = 1, \dots, n,$$

and obtain

$$\sum_{j=1}^n y_j \beta(j \cdot) = - \sum_{j=1}^n \sum_{\alpha \leq n/j} \mu(\alpha) x_{\alpha j} \beta(j \cdot) = \sum_{k=1}^n x_k q_k.$$

Hence (b) follows from (a) if we replace $\underline{x}$ by $\underline{y}$ in (a).

To prove (a) we define the multiplicative function $h: \mathbb{N} \rightarrow \mathbb{N}$ by $h = |\mu| * I$, where $*$ denotes the Dirichlet convolution of arithmetical functions and $I: \mathbb{N} \rightarrow \mathbb{N}$ the identity, $I(k) = k$ for all $k \in \mathbb{N}$.

For every arithmetical function $\lambda: \mathbb{N} \rightarrow \mathbb{C}$ we define the diagonal matrix $D_{\lambda,n} = (\lambda(j) \delta_{jk})_{j,k=1,\dots,n} \in \mathbb{C}^{n \times n}$. An elementary, but lengthy calculation yields

$$\left\| \sum_{k=1}^n x_k \beta(k \cdot) \right\|_2^2 = \frac{1}{12} |D_{\sqrt{\varphi h}/I,n} A_{1/I,n} \underline{x}|^2, \quad \underline{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n. \quad (27)$$

Regarding that

$$|D_{\sqrt{\varphi h}/I,n} \underline{x}|^2 = \sum_{k=1}^n \frac{\varphi(k) h(k)}{k^2} x_k^2$$

for all $\underline{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ with

$$\frac{6}{\pi^2} \leq \frac{\varphi(k) h(k)}{k^2} = \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \leq 1$$

we obtain

$$\frac{6}{\pi^2} |\underline{x}|^2 \leq |D_{\sqrt{\varphi h}/I,n} \underline{x}|^2 \leq |\underline{x}|^2, \quad \underline{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n. \quad (28)$$

For the harmonic series we have

$$H(n) = \sum_{k=1}^n \frac{1}{k} = \mathcal{O}(\log n) \quad \text{for } n \rightarrow \infty. \quad (29)$$

Next we note the following inequality between multiplicative functions:

$$\left(\frac{1}{I} * 1\right)(k) = \sum_{d|k} \frac{1}{d} \leq \frac{k}{\varphi(k)} \quad \text{for all } k \in \mathbb{N}, \quad (30)$$

which follows from the following evaluation at prime powers $p^\alpha > 1$:

$$\left(\frac{1}{I} * 1\right)(p^\alpha) = \sum_{d|p^\alpha} \frac{1}{d} \leq \sum_{j=0}^{\infty} \frac{1}{p^j} = \frac{p}{p-1} = \frac{p^\alpha}{\varphi(p^\alpha)}.$$

From Prachar [15, Kapitel I, Satz 5.1] we obtain

$$\frac{k}{\varphi(k)} \leq C \log \log k \quad \text{for all } k \geq 3 \quad (31)$$

with an absolute constant $C > 0$. Now we apply the Cauchy–Schwarz inequality to

$$|A_{1/I,n}\underline{x}|^2 = \sum_{\alpha \leq n} \left(\sum_{\beta \leq n/\alpha} \frac{1}{\sqrt{\beta}} \cdot \frac{x_{\alpha\beta}}{\sqrt{\beta}} \right)^2,$$

and obtain

$$\begin{aligned} |A_{1/I,n}\underline{x}|^2 &\leq \sum_{\alpha \leq n} \sum_{\beta \leq n/\alpha} \frac{1}{\beta} \cdot \sum_{\beta \leq n/\alpha} \frac{x_{\alpha\beta}^2}{\beta} \\ &\leq H(n) \sum_{\alpha \leq n} \sum_{\beta \leq n/\alpha} \frac{x_{\alpha\beta}^2}{\beta} = H(n) \sum_{k \leq n} \sum_{d|k} \frac{1}{d} x_k^2. \end{aligned}$$

Using (29), (30), (31), we obtain a constant $\tilde{C} > 0$, such that we have for $n \geq 3$ and $\underline{x} \in \mathbb{R}^n$,

$$|A_{1/I,n}\underline{x}|^2 \leq \tilde{C} |\underline{x}|^2 \log n \log \log n. \quad (32)$$

We conclude the upper bound for $\|x_1\beta(1 \cdot) + \cdots + x_n\beta(n \cdot)\|_2^2$ in (a) from (27), (28) and (32).

Since $|\mu| \leq 1$, we have componentwise

$$|B_{1/I,n}(j, k)| \leq |A_{1/I,n}(j, k)| \quad \text{for all } j, k = 1, \dots, n,$$

and hence

$$|B_{1/I,n}\underline{x}|^2 \leq \tilde{C}|\underline{x}|^2 \log n \log \log n \quad (33)$$

for $n \geq 3$ and $\underline{x} \in \mathbb{R}^n$. The lower bound for $\|x_1\beta(1\cdot) + \dots + x_n\beta(n\cdot)\|_2^2$ in (a) follows from Lemma 3.7 with $B_{1/I,n}^{-1} = A_{1/I,n}$ and (33), (28), (27) for appropriate $c_1 > 0$ and all $n \geq 3$:

$$\begin{aligned} |\underline{x}|^2 &= |B_{1/I,n} A_{1/I,n} \underline{x}|^2 \leq \tilde{C} |A_{1/I,n} \underline{x}|^2 \log n \log \log n \\ &\leq \frac{1}{12c_1} |D_{\sqrt{\varphi h/I,n}} A_{1/I,n} \underline{x}|^2 \log n \log \log n \\ &= \frac{\log n \log \log n}{c_1} \left\| \sum_{k=1}^n x_k \beta(k\cdot) \right\|_2^2. \quad \square \end{aligned}$$

For the next two corollaries we make use of

$$\|f_n\|_2 \leq \|\Psi_n\|_2 + \mathcal{O}\left(\sqrt{\frac{\log n}{n}}\right) \quad \text{for } n \rightarrow \infty, \quad (34)$$

which follows from the definition of f_n and Theorem 3.2.

Corollary 3.9 (decay of L^2 -norms of functions f_n)

$$\lim_{n \rightarrow \infty} \|f_n\|_2 = 0.$$

Proof Put $x_k = 1/k$ for the components of $\underline{x}$ in Theorem 3.8(b) and form the vector

$$\underline{y}_n = (y_1, \dots, y_n)^T = B_{1,n} \underline{x} \quad (35)$$

with components

$$y_j = \frac{1}{j} \sum_{k \leq n/j} \frac{\mu(k)}{k} \quad \text{for } j = 1, \dots, n. \quad (36)$$

From Prachar [15, Kapitel III, Satz 5.1] we obtain uniformly in $x \geq 1$

$$\sum_{k \leq x} \frac{\mu(k)}{k} = \mathcal{O}(e^{-c\sqrt{\log x}}) + \mathcal{O}\left(\int_x^\infty e^{-c\sqrt{\log \xi}} \frac{d\xi}{\xi}\right) \quad (37)$$

with an absolute constant $c > 0$. The sums $\sum_{k \leq x} \mu(k)/k$ are bounded for $x \geq 1$, which gives for $n \rightarrow \infty$

$$|\underline{y}_n|^2 = \sum_{j \leq \sqrt{n}} \frac{1}{j^2} \left(\sum_{k \leq n/j} \frac{\mu(k)}{k} \right)^2 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

From (37) we obtain for any fixed $\alpha > 0$ and $x \geq 2$ that

$$\left| \sum_{k \leq x} \frac{\mu(k)}{k} \right| \leq \frac{c'(\alpha)}{(\log x)^\alpha},$$

and the right-hand side in Theorem 3.8(b) converges to zero,

$$\lim_{n \rightarrow \infty} \|\Psi_n\|_2^2 = 0.$$

Now (34) concludes the proof of the corollary. $\square$

Corollary 3.10 (variant of the Franel–Landau Theorem) *The Riemann hypothesis is equivalent to the following statement: For all $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that*

$$\|f_n\|_2 \leq \frac{c(\varepsilon)}{n^{1/2-\varepsilon}} \quad \text{for all } n \in \mathbb{N}.$$

Proof We put $M(x) = \sum_{k \leq x} \mu(k)$ and start with

$$\sum_{k \leq n} \frac{\mu(k)}{k} = \frac{M(n)}{n} + \sum_{k=1}^{n-1} \frac{M(k)}{k(k+1)} \quad \text{for all } n \in \mathbb{N}. \quad (38)$$

The prime number theorem,

$$\lim_{n \rightarrow \infty} \frac{M(n)}{n} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \quad \text{for all } n \in \mathbb{N},$$

allows us to form the limit $n \rightarrow \infty$ in (38),

$$\sum_{k=1}^{\infty} \frac{M(k)}{k(k+1)} = 0,$$

which gives the following representation with an absolutely convergent series on the right-hand side:

$$\sum_{k \leq n} \frac{\mu(k)}{k} = \frac{M(n)}{n} - \sum_{k=n}^{\infty} \frac{M(k)}{k(k+1)} \quad \text{for all } n \in \mathbb{N}. \quad (39)$$

From [14] it is known that Riemann's hypothesis is equivalent to the statement $|M(x)| \leq \tilde{c}(\varepsilon)x^{1/2+\varepsilon}$ for $x \geq 1$ and all fixed $\varepsilon > 0$, and the proof follows with (39), Theorem 3.8(b) and (34) using (35) and (36) in the proof of Corollary 3.9. $\square$

4 Behaviour of functions f_n near Farey fractions

Our aim is to describe the behaviour of f_n in the vicinity of a fixed reduced fraction $a/b \in \mathcal{F}_n^{\text{ext}}$ for large values of the Farey index $n \in \mathbb{N}$.

Lemma 4.1

$$\frac{6}{\pi^2} - \frac{1}{t} \sum_{k \leq t} \frac{\varphi(k)}{k} = \mathcal{O}\left(\frac{\log t}{t}\right) \quad \text{for } t \rightarrow \infty.$$

Proof Using that

$$\frac{\varphi(k)}{k} = \prod_{p|k} \frac{p-1}{p} = \sum_{d|k} \frac{\mu(d)}{d},$$

we have for all $n \in \mathbb{N}$

$$\sum_{k \leq n} \frac{\varphi(k)}{k} = \sum_{k \leq n} \sum_{d|k} \frac{\mu(d)}{d} = \sum_{d=1}^n \left\lfloor \frac{n}{d} \right\rfloor \frac{\mu(d)}{d} = \frac{6n}{\pi^2} - R_n$$

with the error term

$$R_n = n \sum_{d > n} \frac{\mu(d)}{d^2} + \sum_{d \leq n} \left(\frac{n}{d} - \left\lfloor \frac{n}{d} \right\rfloor \right) \frac{\mu(d)}{d}$$

and the estimate

$$|R_n| \leq n \int_n^\infty \frac{dx}{x^2} + \sum_{d \leq n} \frac{1}{d} \leq 2 + \log n \quad \text{for all } n \in \mathbb{N}, \quad (40)$$

which concludes the proof of the lemma. $\square$

Lemma 4.2 Assume that the continuous function $f: [1, \infty) \rightarrow (0, \infty)$ is convex and monotonically decreasing. Define the primitive function $F: [1, \infty) \rightarrow (0, \infty)$ by $F(x) = \int_1^x f(t) dt$. Then we have for all $q_{\pm} \in \mathbb{N}$ with $q_- \leq q_+$,

$$\begin{aligned} F(q_+) - F(q_-) + \frac{1}{2} (f(q_-) - f(q_+)) &\leq \sum_{q_- \leq q < q_+} f(q) \\ &\leq F(q_+) - F(q_-) + f(q_-) - f(q_+). \end{aligned}$$

Proof We prove the equivalent inequalities

$$F(q_+) - F(q_-) - \frac{1}{2} (f(q_-) - f(q_+)) \leq \sum_{q_- < q \leq q_+} f(q) \leq F(q_+) - F(q_-).$$

Summation with respect to q shows that it is sufficient to consider the special case $q_+ = q_- + 1$: We obtain from the monotonicity of f that

$$F(q_- + 1) - F(q_-) \geq f(q_- + 1) = \sum_{q_- < q \leq q_- + 1} f(q).$$

From the convexity of f we have

$$F(q_- + 1) - F(q_-) \leq \frac{1}{2} (f(q_-) + f(q_- + 1)),$$

and hence

$$F(q_- + 1) - F(q_-) - \frac{1}{2} (f(q_-) + f(q_- + 1)) \leq f(q_- + 1) = \sum_{q \leq q_- + 1} f(q). \quad \square$$

Lemma 4.3 *With the conditions and notations from Theorem 2.8 we define for $\alpha = 1, \dots, A - 1$ the section*

$$\mathcal{F}_{n,\alpha}^{\text{ext}} = \mathcal{F}_n^{\text{ext}} \cap [\xi_+(Q_\alpha^*), \xi_+(Q_{\alpha+1}^*)]$$

and the sum

$$S_{\alpha,n} \left(\frac{a}{b} \right) = \sum_{a'/b' \in \mathcal{F}_{n,\alpha}^{\text{ext}}} \frac{1}{b'}.$$

Put $x_+(q) = n/(b^* + bq)$ for $q \geq 0$, $x_\alpha = x_+(Q_\alpha^*)$, $x_{\alpha+1} = x_+(Q_{\alpha+1}^*)$. Then we obtain

$$\begin{aligned} -\frac{(\alpha+1)}{2} \frac{b^2}{n^2} (x_{\alpha+1}^2 - x_\alpha^2) - \frac{b}{2n} (x_{\alpha+1} - x_\alpha) &\leq b S_{\alpha,n} \left(\frac{a}{b} \right) - \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \log \frac{x_{\alpha+1}}{x_\alpha} \\ &\leq \frac{b\alpha}{n} (x_{\alpha+1} - x_\alpha) + \frac{b\alpha}{n+1-b(\alpha+1)}. \end{aligned}$$

Proof We make use of Theorem 2.8. For the partial sum

$$\tilde{S}_{\alpha,n} \left(\frac{a}{b} \right) = \sum_{\substack{a'/b' = \xi_+(\xi) \\ \xi \in \mathcal{F}_\alpha^{\text{ext}} \cap (Q_{\alpha+1}^*, Q_\alpha^*]}} \frac{1}{b'}$$

of $S_{\alpha,n}(a/b)$ and with $\mathcal{F}_\alpha^+ = \{\xi \in \mathcal{F}_\alpha : 0 < \xi \leq 1\}$ we obtain the representation

$$\tilde{S}_{\alpha,n}\left(\frac{a}{b}\right) = \sum_{a'/b' \in \mathcal{F}_{\alpha}^{+}} \sum_{q=Q_{\alpha+1}^{*}-1}^{Q_{\alpha}^{*}-1} \frac{1}{ba' + (b^{*} + bq)b'} \quad (41)$$

and the estimates

$$\tilde{S}_{\alpha,n}\left(\frac{a}{b}\right) \leq S_{\alpha,n}\left(\frac{a}{b}\right) \leq \tilde{S}_{\alpha,n}\left(\frac{a}{b}\right) + \frac{\alpha}{n+1-b(\alpha+1)}. \quad (42)$$

The right-hand side of (42) results from the fractions $\xi_{+}(q + \tilde{a}/(\alpha+1))$ in the transient sections, using (19) in the proof of Lemma 2.7 and the fact that $\varphi(\alpha+1) \leq \alpha$ is the number of integers $\tilde{a} \in [1, \alpha]$, which are relatively prime to $\alpha+1$.

We obtain from (41) and (42) that

$$\begin{aligned} S_{\alpha,n}\left(\frac{a}{b}\right) &\leq \sum_{a'/b' \in \mathcal{F}_{\alpha}^{+}} \frac{1}{b'} \sum_{q=Q_{\alpha+1}^{*}-1}^{Q_{\alpha}^{*}-1} \frac{1}{b^{*} + bq} + \frac{\alpha}{n+1-b(\alpha+1)} \\ &= \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \sum_{q=Q_{\alpha+1}^{*}-1}^{Q_{\alpha}^{*}-1} \frac{1}{b^{*} + bq} + \frac{\alpha}{n+1-b(\alpha+1)}. \end{aligned}$$

Now we apply Lemma 4.2 with $f(q) = 1/(b^{*} + bq)$ and get

$$\begin{aligned} S_{\alpha,n}\left(\frac{a}{b}\right) &\leq \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \left\{ \frac{1}{b} \log \frac{b^{*} + bQ_{\alpha}^{*}}{b^{*} + bQ_{\alpha+1}^{*}} + \frac{1}{b^{*} + bQ_{\alpha+1}^{*}} - \frac{1}{b^{*} + bQ_{\alpha}^{*}} \right\} \\ &\quad + \frac{\alpha}{n+1-b(\alpha+1)}. \end{aligned} \quad (43)$$

Regarding $x_{+}(q) = n/(b^{*} + bq)$ we obtain from (43) the right-hand side of the desired inequality.

For the proof of the left-hand side we start from

$$\begin{aligned} \frac{1}{ba' + (b^{*} + bq)b'} &= \frac{1}{b'(b^{*} + bq)} \cdot \frac{1}{1 + a'/b' \cdot b/(b^{*} + bq)} \\ &\geq \frac{1}{b'(b^{*} + bq)} - \frac{a'}{b'^2} \cdot \frac{b}{(b^{*} + bq)^2}. \end{aligned}$$

Then (41) and (42) yield

$$\begin{aligned} S_{\alpha,n}\left(\frac{a}{b}\right) &\geq \tilde{S}_{\alpha,n}\left(\frac{a}{b}\right) \\ &\geq \sum_{a'/b' \in \mathcal{F}_{\alpha}^{+}} \frac{1}{b'} \sum_{q=Q_{\alpha+1}^{*}-1}^{Q_{\alpha}^{*}-1} \frac{1}{b^{*} + bq} - \sum_{a'/b' \in \mathcal{F}_{\alpha}^{+}} \frac{a'}{b'^2} \sum_{q=Q_{\alpha+1}^{*}-1}^{Q_{\alpha}^{*}-1} \frac{b}{(b^{*} + bq)^2}. \end{aligned} \quad (44)$$

We apply again Lemma 4.2 and infer that

$$\sum_{a'/b' \in \mathcal{F}_\alpha^+} \frac{1}{b'} \sum_{q=Q_{\alpha+1}^*}^{Q_\alpha^*-1} \frac{1}{b^*+bq} \geq \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \left\{ \frac{1}{b} \log \frac{x_{\alpha+1}}{x_\alpha} + \frac{x_{\alpha+1} - x_\alpha}{2n} \right\} \quad (45)$$

as well as

$$\begin{aligned} \sum_{a'/b' \in \mathcal{F}_\alpha^+} \frac{a'}{b'^2} \sum_{q=Q_{\alpha+1}^*}^{Q_\alpha^*-1} \frac{b}{(b^*+bq)^2} &\leq \left(\frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \right) \cdot \left\{ -\frac{x_\alpha}{n} + \frac{x_{\alpha+1}}{n} \right. \\ &\quad \left. + \frac{b}{(b^*+bQ_{\alpha+1}^*)^2} - \frac{b}{(b^*+bQ_\alpha^*)^2} \right\}. \end{aligned} \quad (46)$$

For (46) we note that

$$\sum_{a'/b' \in \mathcal{F}_\alpha^+} \frac{a'}{b'^2} = \sum_{a'/b' \in \mathcal{F}_\alpha} \frac{b' - a'}{b'^2} = \frac{1}{2} \sum_{a'/b' \in \mathcal{F}_\alpha} \frac{1}{b'} = \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k}.$$

We use

$$\begin{aligned} \left(\frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \right) \cdot \left\{ \frac{b}{(b^*+bQ_{\alpha+1}^*)^2} - \frac{b}{(b^*+bQ_\alpha^*)^2} \right\} \\ \leq \frac{\alpha+1}{2} \cdot \frac{b}{n^2} (x_{\alpha+1}^2 - x_\alpha^2), \end{aligned}$$

subtract (46) from (45), and establish with (44) the left-hand side of the desired inequality. $\square$

Lemma 4.4 *Using the conditions and notations from Lemma 4.3 and Theorem 2.8 we obtain the following estimates:*

(i)

$$\left| \log \frac{x_\alpha}{\alpha} \right| \leq \frac{b\alpha}{n} + \left(\frac{b\alpha}{n} \right)^2, \quad \alpha = 1, \dots, A,$$

(ii) $x_{\alpha+1} + x_\alpha \leq 2(2\alpha+1)$, $\alpha = 1, \dots, A-1$,

(iii) $0 < x_{\alpha+1} - x_\alpha \leq 3$, $\alpha = 1, \dots, A-1$,

(iv) $-b\alpha/n \leq x_\alpha - \alpha \leq 2\alpha^2 b/n$, $\alpha = 1, \dots, A$.

Proof Recall the definition of Q_α^* in (7). For $\alpha = 1$ we obtain

$$n - b \leq b^* + bQ_\alpha^* \leq n,$$

and for $\alpha \geq 2$,

$$\frac{n - b(\alpha - 1)}{\alpha} \leq b^* + bQ_\alpha^* \leq \frac{n + b}{\alpha}.$$

This gives

$$\alpha \frac{n}{n + b} \leq x_\alpha \leq \alpha \frac{n}{n - \alpha b}, \quad \alpha = 1, \dots, A, \quad (47)$$

$$(\alpha + 1) \frac{n}{n + b} \leq x_{\alpha+1} \leq (\alpha + 1) \frac{n}{n - \alpha b}, \quad \alpha = 1, \dots, A - 1. \quad (48)$$

Regarding $b \leq n/3$, we have for $\alpha = 1, \dots, A$:

$$\alpha b < \sqrt{\frac{nb}{2}} \leq \frac{n}{\sqrt{6}} < \frac{n}{2}, \quad (49)$$

$$\frac{n}{n - \alpha b} < 2. \quad (50)$$

This implies (ii), $x_\alpha + x_{\alpha+1} \leq 2(2\alpha + 1)$, $\alpha = 1, \dots, A - 1$.

We use (47), (49) and (50) to derive (i):

$$\begin{aligned} \log \frac{x_\alpha}{\alpha} &\geq \log \frac{n}{n + b} = -\log \left(1 + \frac{b}{n} \right) \geq -\frac{b}{n} \geq -\frac{b\alpha}{n} - \left(\frac{b\alpha}{n} \right)^2, \\ \log \frac{x_\alpha}{\alpha} &\leq \log \frac{1}{1 - \alpha b/n} = \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{\alpha b}{n} \right)^j \\ &\leq \frac{\alpha b}{n} + \frac{1}{2} \left[\left(\frac{\alpha b}{n} \right)^2 + \left(\frac{\alpha b}{n} \right)^3 + \dots \right] \\ &= \frac{\alpha b}{n} + \frac{1}{2} \left(\frac{\alpha b}{n} \right)^2 \frac{n}{n - \alpha b} < \frac{\alpha b}{n} + \left(\frac{\alpha b}{n} \right)^2. \end{aligned}$$

We show (iii) and take care for $\alpha(\alpha + 1) \leq A^2 \leq n/(2b)$:

$$\begin{aligned} x_{\alpha+1} - x_\alpha &\leq (\alpha + 1) \frac{n}{n - \alpha b} - \frac{\alpha n}{n + b} = 1 + \alpha b \left(\frac{1}{b + n} + \frac{1 + \alpha}{n - \alpha b} \right) \\ &< 1 + \frac{1}{2} + b \frac{\alpha(\alpha + 1)}{n - \alpha b} < \frac{3}{2} + b \frac{n/(2b)}{n - \alpha b} = \frac{3}{2} + \frac{1}{2} \frac{n}{n - \alpha b} < 3. \end{aligned}$$

Finally, (iv) easily follows from (47). $\square$

The following function g plays a crucial role in the present paper, especially as a limit function in Theorem 4.6 below.

Lemma 4.5 Define the continuous, piecewise smooth and odd function $g: \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} \left(\frac{6}{\pi^2} - \frac{1}{2}\right)x & \text{for } 0 \leq x \leq 1, \\ \frac{6}{\pi^2} - \frac{1}{2} + \int_1^x \left(\frac{6}{\pi^2} - \frac{1}{t} \sum_{k \leq t} \frac{\varphi(k)}{k}\right) dt & \text{for } x > 1, \\ -g(-x) & \text{for } x < 0. \end{cases} \quad (51)$$

Then we have for $k \in \mathbb{N}$ and $k < x \leq k+1$,

$$g(x) = g(k) + \frac{6}{\pi^2}(x-k) - \sum_{m=1}^k \frac{\varphi(m)}{m} \cdot \log \frac{x}{k}, \quad (52)$$

and especially

$$g(k) = \frac{6}{\pi^2}k - \frac{1}{2} + \sum_{m=1}^{k-1} \frac{\varphi(m)}{m} \cdot \log \frac{m}{k} \quad \text{for all } k \in \mathbb{N}. \quad (53)$$

Proof Since g is continuous and piecewise smooth, (52) can be obtained by forming the derivatives on each interval $(k, k+1)$, whereas (53) results from (52) by Abelian summation. $\square$

Theorem 4.6 (scaling property of f_n near Farey fractions) Assume that $a/b \in \mathcal{F}_n^{\text{ext}}$ and put

$$g_{a,b}(n, x) = b \left(f_n \left(\frac{a}{b} + \frac{x}{bn} \right) - f_n \left(\frac{a}{b} \right) \right), \quad x \in \mathbb{R}.$$

Then for $n \rightarrow \infty$ the sequence of functions $g_{a,b}(n, \cdot)$ converges uniformly on each interval $[-x_*, x_*]$, $x_* > 0$ fixed, towards the limit function g from Lemma 4.5.

More precisely, if $n, b, A \in \mathbb{N}$ such that $n \geq 3b$ and $A \leq \sqrt{n/(2b)} - 1/4$, then we obtain for all $x \in \mathbb{R}$ with $|x| \leq An/(n+b)$,

$$|g_{a,b}(n, x) - g(x)| \leq A \frac{2 + \log n}{n} + cA^2 \frac{b}{n}. \quad (54)$$

Here $c > 0$ is an absolute constant.

Proof For given $n \in \mathbb{N}$ and given reduced fraction $a/b \in \mathcal{F}_n^{\text{ext}}$ there holds the symmetry relation

$$g_{a,b}(n, x) = -g_{b-a,b}(n, -x), \quad x \in \mathbb{R}.$$

Thus it is sufficient to prove the theorem under the restriction that $x \geq 0$, and inequality (54) results from the statement

$$\begin{aligned} x \left(\frac{s_n}{n} - \frac{6}{\pi^2} \right) - cA^2 \frac{b}{n} &\leq g_{a,b}(n, x) - g(x) \\ &\leq x \left(\frac{s_n}{n} - \frac{6}{\pi^2} \right) + cA^2 \frac{b}{n}, \quad 0 \leq x \leq \frac{An}{n+b}, \end{aligned} \quad (55)$$

see Lemma 4.1, inequality (40), and recall that $s_n = \sum_{k \leq n} \varphi(k)/k$.

We use the conditions and notations from Lemma 2.5 and recall

$$\begin{aligned} x_+(q) &= nb \left(\xi_+(q) - \frac{a}{b} \right) = \frac{n}{b^* + bq}, \quad q \geq 0, \\ x_j &= x_+(Q_j^*), \quad j = 1, \dots, A. \end{aligned}$$

Note that $x_+(Q_A^*) = x_A \geq An/(n+b)$, see (48) in the proof of Lemma 4.4 with $\alpha + 1 = A$, and that $Q_A^* > 0$ due to Lemma 2.6. For $x \geq 0$ we define the functions

$$\begin{aligned} \tilde{g}_{a,b}(n, x) &= x \frac{s_n}{n} - g_{a,b}(n, x), \\ \tilde{g}(x) &= x \frac{6}{\pi^2} - g(x) \end{aligned} \quad (56)$$

and will show that

$$|\tilde{g}_{a,b}(n, x) - \tilde{g}(x)| \leq cA^2 \frac{b}{n} \quad \text{for all } x \in [0, x_A] \quad (57)$$

with a sufficiently large constant $c > 0$ independent of n, a, b, A, x , such that (57) implies (55).

From the definition $Q_1^* = \lfloor (n - b^*)/b \rfloor$ it follows that $x_1 \geq 1$. We also obtain from the definition of x_1 that

$$1 \leq x_1 < 2, \quad \frac{1}{x_1} > 1 - \frac{b}{n}. \quad (58)$$

The following equations result from the definitions of $f_n, g, g_{a,b}$, from (56), (58) and Theorem 2.4(b):

$$\tilde{g}(x) = \frac{x}{2}, \quad \tilde{g}_{a,b}(n, x) = \frac{x}{2} \left(\frac{1}{x_1} + \frac{b}{n} \right), \quad 0 \leq x \leq 1, \quad (59)$$

$$\tilde{g}(x) = \frac{1}{2} + \log x, \quad \tilde{g}_{a,b}(n, x) = \frac{x}{2} \left(\frac{1}{x_1} + \frac{b}{n} \right), \quad 1 \leq x \leq x_1. \quad (60)$$

From (58), (59) and (60) we conclude that

$$|\tilde{g}_{a,b}(n, x) - \tilde{g}(x)| \leq \frac{b}{n}, \quad 0 \leq x \leq x_1.$$

Thus we may assume in the sequel that x is a fixed real number which satisfies $x_1 < x \leq x_A$ with $A > 1$. Due to Theorem 2.8, there is exactly one index $\alpha \in \{1, \dots, A-1\}$ with

$$x_\alpha < x \leq x_{\alpha+1}, \quad (61)$$

and it is sufficient to prove (57) for $A = \alpha + 1$ with the index α in (61). From x we also obtain an index $q_x \in \mathbb{N}$ which satisfies the two inequalities

$$Q_{\alpha+1}^* \leq q_x < Q_\alpha^*, \quad x_+(q_x+1) \leq x \leq x_+(q_x). \quad (62)$$

We supplement the definition of the sums $S_{\alpha,n}(a/b)$ in Lemma 4.3 by

$$S_{0,n}\left(\frac{a}{b}\right) = \frac{1}{2b}. \quad (63)$$

We obtain for the index α in (61) that

$$\tilde{g}_{a,b}(n, x_\alpha) = b \sum_{\beta=0}^{\alpha-1} S_{\beta,n}\left(\frac{a}{b}\right) + \frac{1}{2} \frac{b}{b^* + bQ_\alpha^*}, \quad (64)$$

and from Lemma 4.4, (47), (50),

$$\frac{1}{2} \frac{b}{b^* + bQ_\alpha^*} \leq \frac{1}{2} \frac{b\alpha}{n - \alpha b} \leq \frac{b\alpha}{n}. \quad (65)$$

Due to Lemmas 4.3 and 4.4, there is an absolute constant $c_1 > 0$ such that for $\beta = 1, \dots, \alpha - 1$,

$$-c_1 \frac{b\beta}{n} \leq bS_{\beta,n}\left(\frac{a}{b}\right) - \sum_{k=1}^{\beta} \frac{\varphi(k)}{k} \log \frac{x_{\beta+1}}{x_\beta} \leq c_1 \frac{b\beta}{n}.$$

Summation with respect to β and (63), (64), (65) yields with an absolute constant $c_2 > 0$

$$-c_2 \frac{\alpha^2 b}{n} \leq \tilde{g}_{a,b}(n, x_\alpha) - \frac{1}{2} - \sum_{\beta=1}^{\alpha-1} \sum_{k=1}^{\beta} \frac{\varphi(k)}{k} \log \frac{x_{\beta+1}}{x_\beta} \leq c_2 \frac{\alpha^2 b}{n}. \quad (66)$$

Abelian summation gives

$$\sum_{\beta=1}^{\alpha-1} \sum_{k=1}^{\beta} \frac{\varphi(k)}{k} \log \frac{x_{\beta+1}}{x_\beta} = \log x_\alpha \sum_{k=1}^{\alpha-1} \frac{\varphi(k)}{k} - \sum_{k=1}^{\alpha-1} \frac{\varphi(k)}{k} \log x_k. \quad (67)$$

Now we use (56), (67) and (53) in Lemma 4.5 and get

$$\begin{aligned} & \sum_{\beta=1}^{\alpha-1} \sum_{k=1}^{\beta} \frac{\varphi(k)}{k} \log \frac{x_{\beta+1}}{x_{\beta}} \\ &= \tilde{g}(\alpha) - \frac{1}{2} + \log \frac{x_{\alpha}}{\alpha} \sum_{k=1}^{\alpha-1} \frac{\varphi(k)}{k} - \sum_{k=1}^{\alpha-1} \frac{\varphi(k)}{k} \log \frac{x_k}{k}. \end{aligned} \quad (68)$$

Since $\tilde{g}'$ is bounded, $\sup_{y>0} |\tilde{g}'(y)| = c_3$, we obtain from Lemma 4.4(iv)

$$|\tilde{g}(x_{\alpha}) - \tilde{g}(\alpha)| \leq c_3 |x_{\alpha} - \alpha| \leq \frac{2c_3}{n} \alpha^2 b, \quad (69)$$

and from (68), (69) and Lemma 4.4(i) with some constant $c_4 > 0$ we obtain

$$-c_4 \frac{\alpha^2 b}{n} \leq \sum_{\beta=1}^{\alpha-1} \sum_{k=1}^{\beta} \frac{\varphi(k)}{k} \log \frac{x_{\beta+1}}{x_{\beta}} - \left(\tilde{g}(x_{\alpha}) - \frac{1}{2} \right) \leq c_4 \frac{\alpha^2 b}{n}. \quad (70)$$

The sum of inequalities (66), (70) gives with $c_5 = c_2 + c_4$,

$$|\tilde{g}_{a,b}(n, x_{\alpha}) - \tilde{g}(x_{\alpha})| \leq c_5 \frac{\alpha^2 b}{n}. \quad (71)$$

Due to (61), (62) there holds

$$x_{\alpha} \leq x_{+}(q_x + 1) \leq x \leq x_{+}(q_x) \leq x_{\alpha+1}. \quad (72)$$

We define

$$\Sigma_1 = \sum_{a'/b' \in \mathcal{F}_n^{\text{ext}} \cap [\xi_{+}(Q_{\alpha}^{*}), \xi_{+}(q_x + 1))} \frac{1}{b'}, \quad \Sigma_2 = \sum_{a'/b' \in \mathcal{F}_n^{\text{ext}} \cap [\xi_{+}(q_x + 1), \xi_{+}(q_x)]} \frac{1}{b'},$$

and regard that the sum Σ_2 also contains the inverse denominator of the boundary fraction $\xi_{+}(q_x)$. We have

$$\begin{aligned} \tilde{g}_{a,b}(n, x_{+}(q_x + 1)) - \tilde{g}_{a,b}(n, x_{\alpha}) &= b \Sigma_1 + \frac{b}{2n} \cdot (x_{+}(q_x + 1) - x_{\alpha}), \\ 0 &\leq \frac{b}{2n} \cdot (x_{+}(q_x + 1) - x_{\alpha}) \leq \frac{3}{2} \frac{b}{n}, \end{aligned} \quad (73)$$

because $x_{+}(q_x + 1) - x_{\alpha} \leq x_{\alpha+1} - x_{\alpha} \leq 3$ from Lemma 4.4(iii).

For the estimation of Σ_1 we use the same method like in the proof of Lemma 4.3, see (41)–(46),

$$\begin{aligned}\Sigma_1 &\leq \sum_{a'/b' \in \mathcal{F}_\alpha^+} \sum_{q=q_x+1}^{Q_\alpha^*-1} \frac{1}{ba' + (b^* + bq)b'} + \frac{\alpha}{n+1-b(\alpha+1)} \\ &\leq \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \sum_{q=q_x+1}^{Q_\alpha^*-1} \frac{1}{b^* + bq} + \frac{\alpha}{n+1-b(\alpha+1)} \\ &\leq \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \left\{ \frac{1}{b} \log \frac{x_+(q_x+1)}{x_\alpha} + \frac{x_+(q_x+1) - x_\alpha}{n} \right\} + \frac{\alpha}{n+1-b(\alpha+1)}.\end{aligned}\quad (74)$$

For the lower bound of Σ_1 we have the estimation

$$\begin{aligned}\Sigma_1 &\geq \sum_{a'/b' \in \mathcal{F}_\alpha^+} \frac{1}{b'} \sum_{q=q_x+1}^{Q_\alpha^*-1} \frac{1}{b^* + bq} - \sum_{a'/b' \in \mathcal{F}_\alpha^+} \frac{a'}{b'^2} \sum_{q=q_x+1}^{Q_\alpha^*-1} \frac{b}{(b^* + bq)^2} \\ &\geq \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \left\{ \frac{1}{b} \log \frac{x_+(q_x+1)}{x_\alpha} + \frac{x_+(q_x+1) - x_\alpha}{2n} \right\} \\ &\quad - \left(\frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \right) \left\{ \frac{x_+(q_x+1) - x_\alpha}{n} + \frac{b}{n^2} (x_+(q_x+1)^2 - x_\alpha^2) \right\}.\end{aligned}$$

This can be simplified if we note that

$$\frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \leq \frac{\alpha+1}{2} = \frac{A}{2},$$

then

$$\begin{aligned}\Sigma_1 &\geq \frac{1}{b} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \log \frac{x_+(q_x+1)}{x_\alpha} \\ &\quad - \frac{x_+(q_x+1) - x_\alpha}{2n} - \frac{bA}{2n^2} (x_+(q_x+1)^2 - x_\alpha^2).\end{aligned}\quad (75)$$

We put

$$\lambda = \frac{x_+(q_x+1) - x_\alpha}{x_{\alpha+1} - x_\alpha}, \quad y = \alpha + \lambda,$$

and regard that $0 \leq \lambda \leq 1$, see (72). Then

$$\frac{x_+(q_x+1)}{y} = \lambda' \frac{x_{\alpha+1}}{\alpha+1} + (1-\lambda') \frac{x_\alpha}{\alpha}$$

with the quantity

$$\lambda' = \lambda \frac{\alpha + 1}{\alpha + \lambda} \in [0, 1].$$

Using α in (61), we obtain from (47) and (48) that

$$\frac{n}{n+b} \leq \frac{x_+(q_x+1)}{y} \leq \frac{n}{n-\alpha b}. \quad (76)$$

Exactly as in the proof of Lemma 4.4(i) we conclude from (76) that

$$\left| \log \frac{x_+(q_x+1)}{y} \right| \leq \frac{\alpha b}{n} + \left(\frac{\alpha b}{n} \right)^2. \quad (77)$$

Recall that $A = \alpha + 1$. Another consequence of (76) is

$$|x_+(q_x+1) - y| \leq \frac{2b}{n} A^2. \quad (78)$$

Using that $c_3 = \sup_{y>0} |\tilde{g}'(y)|$ we obtain from (78) that

$$|\tilde{g}(x_+(q_x+1)) - \tilde{g}(y)| \leq 2c_3 \frac{b}{n} A^2. \quad (79)$$

For (74), (75) we note the identity

$$\log \frac{x_+(q_x+1)}{x_\alpha} = \log \frac{y}{\alpha} + \log \frac{x_+(q_x+1)}{y} - \log \frac{x_\alpha}{\alpha}$$

with the main term $\log(y/\alpha)$ and the two error terms $\log((x_+(q_x+1))/y)$, $\log(x_\alpha/\alpha)$ on the right-hand side. We use (73), (74), (75), (77) and Lemma 4.4(i), and conclude for an appropriate constant $c_6 > 0$ that

$$\begin{aligned} -c_6 \frac{A^2 b}{n} &\leq \tilde{g}_{a,b}(n, x_+(q_x+1)) - \tilde{g}_{a,b}(n, x_\alpha) \\ &\quad - \log \frac{y}{\alpha} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} \leq c_6 \frac{A^2 b}{n}. \end{aligned} \quad (80)$$

Regarding $y = \alpha + \lambda \in [\alpha, \alpha + 1]$ and Lemma 4.5 we have

$$\log \frac{y}{\alpha} \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k} = \tilde{g}(y) - \tilde{g}(\alpha). \quad (81)$$

From (69), (79), (80), (81) we get an absolute constant $c_7 > 0$ such that

$$\begin{aligned} -c_7 \frac{A^2 b}{n} &\leq \tilde{g}_{a,b}(n, x_+(q_x+1)) - \tilde{g}(x_+(q_x+1)) \\ &\quad - \tilde{g}_{a,b}(n, x_\alpha) + \tilde{g}(x_\alpha) \leq c_7 \frac{A^2 b}{n}, \end{aligned}$$

and regarding (71) with $c_8 = c_5 + c_7$,

$$|\tilde{g}_{a,b}(n, x_+(q_x+1)) - \tilde{g}(x_+(q_x+1))| \leq c_8 \frac{A^2 b}{n}. \quad (82)$$

Finally we use the two simple inequalities

$$|\tilde{g}_{a,b}(n, x) - \tilde{g}_{a,b}(n, x_+(q_x+1))| \leq b \Sigma_2, \quad (83)$$

$$|\tilde{g}(x) - \tilde{g}(x_+(q_x+1))| \leq c_3 (x_+(q_x) - x_+(q_x+1)) \leq c_3 \frac{A^2 b}{n} \quad (84)$$

and the following estimates for $b \Sigma_2$ with an absolute constant $c_9 > 0$:

$$\begin{aligned} b \Sigma_2 &\leq \sum_{a'/b' \in \mathcal{F}_a} \frac{b}{ba' + (b^* + bq_x)b'} + \frac{b\alpha}{n+1-b(\alpha+1)} \\ &\leq \frac{b}{b^* + bq_x} \cdot \left(1 + \sum_{k=1}^{\alpha} \frac{\varphi(k)}{k}\right) + \frac{b\alpha}{n+1-b(\alpha+1)} \\ &\leq \frac{bx_{\alpha+1}}{n} A + \frac{b\alpha}{n+1-b(\alpha+1)} \leq c_9 \frac{bA^2}{n}. \end{aligned} \quad (85)$$

With $c = c_3 + c_8 + c_9$ and (82)–(85) we obtain inequality (57). $\square$

The plot in Fig. 5 illustrates the good agreement between the limit function g , see Fig. 4, and the function $g_{2,5}(1000, \cdot)$ in the range $x = 5000 \cdot (\xi - 2/5) \in [0, 10]$.

5 Asymptotic behaviour of the limit function g

In this section we need some input from the analytic theory of numbers and the theory of the Riemann zeta function, for details we refer to the standard textbooks [3, 9, 11, 15].

Lemma 5.1 For $\operatorname{Re} s > 0$,

$$G(s) = \int_1^\infty \frac{g(x)}{x^{s+1}} dx = \frac{1}{s} \left[\frac{6}{\pi^2} - \frac{1}{2} + \frac{6}{\pi^2} \cdot \frac{1}{s-1} - \frac{\zeta(s)}{s\zeta(s+1)} \right] \quad (86)$$

Fig. 4 The limit function $g(x)$ in the range $|x| \leq 30$

Fig. 5 Comparison between $g_{2,5}(1000, \cdot)$ and g

defines a holomorphic function G , which has an analytic continuation on the axis $\operatorname{Re} s = 0$. The Mellin integral converges absolutely for $\operatorname{Re} s > 0$. Regard a limit for $s \rightarrow 1$ on the right-hand side in (5.1).

Proof Due to (40), Lemma 4.1 and (51) we have uniformly in $x \geq 2$

$$g'(x) = \mathcal{O}\left(\frac{\log x}{x}\right) \quad \text{for } x \rightarrow \infty.$$

This implies the rough estimate $g(x) = \mathcal{O}(\log^2 x)$ for $x \rightarrow \infty$, such that the Mellin integral in (86) is absolutely convergent for $\operatorname{Re} s > 0$. The Mellin integral defines a holomorphic function in the half plane $\operatorname{Re} s > 0$. Representation (86) is valid for $\operatorname{Re} s > 1$ because

$$\begin{aligned} \int_1^\infty \frac{\sum_{k \leq x} \varphi(k)/k}{x^{s+1}} dx &= \frac{1}{s} \frac{\zeta(s)}{\zeta(s+1)}, \\ \int_1^\infty \frac{\int_1^x 1/t \cdot \sum_{k \leq t} \varphi(k)/k dt}{x^{s+1}} dx &= \frac{1}{s^2} \frac{\zeta(s)}{\zeta(s+1)}. \end{aligned}$$

Hence it is also valid for $\operatorname{Re} s > 0$ by analytic continuation. Due to Hadamard, $1/\zeta(s+1)$ has an analytic continuation on the axis $\operatorname{Re} s = 0$. From

$$\zeta(0) = -\frac{1}{2}, \quad \lim_{s \rightarrow 0} s \zeta(s+1) = 1$$

we obtain that $\lim_{s \rightarrow 0} [6/\pi^2 - 1/2 + 6/\pi^2 \cdot 1/(s-1) - \zeta(s)/(s \zeta(s+1))] = 0$. $\square$

Lemma 5.2 *There is an absolute constant $\tilde{C} > 0$ with*

$$\left| \frac{\zeta(s)}{s} \right| \leq \tilde{C} \quad (87)$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ and $|s| \geq 2$.

Proof From Ingham [9, Chapter II, Theorem 9] we use (6) for $\delta = 1/2$,

$$|\zeta(\sigma + it)| \leq C_1 \sqrt{|t|}, \quad (88)$$

which is valid for all $\sigma \geq 1/2$ and all $t \in \mathbb{R}$ with $|t| \geq 1$. Here C_1 denotes a sufficiently large absolute constant.

We also note, with an absolute constant $C_2 > 0$, the trivial estimate

$$|\zeta(\sigma + it)| \leq C_2 \quad (89)$$

for all $\sigma \geq 3/2$ and all $t \in \mathbb{R}$.

Due to (88), (89) it is sufficient to prove (87) in the domain

$$0 \leq \operatorname{Re} s \leq \frac{1}{2}, \quad |\operatorname{Im} s| \geq 1. \quad (90)$$

For values of s in this domain we use the functional equation of the Riemann zeta function in the form

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s). \quad (91)$$

From the textbook [1, Kapitel 1.4, Corollary 1.4.4] we derive that

$$\left| \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \right| = \left(\frac{|t|}{2\pi} \right)^{1/2-\sigma} \left[1 + \mathcal{O}\left(\frac{1}{|t|} \right) \right] \quad (92)$$

with $\sigma = \operatorname{Re} s$, $t = \operatorname{Im} s$. From (88), (91) and (92) we obtain (87) in the region (90) and hence for all s with $\operatorname{Re} s \geq 0$ and $|s| \geq 2$. $\square$

Lemma 5.3 *The holomorphic function G in Lemma 5.1 is a member of the Hardy space $H_0^2 = H^2(\operatorname{Re} s > 0)$. Here any member $h \in H_0^2$ is equipped with the norm*

$$\|h\|_{H_0^2} = \left(\frac{1}{2\pi} \sup_{\sigma > 0} \int_{-\infty}^{\infty} |h(\sigma + it)|^2 dt \right)^{1/2}.$$

Proof From Ingham [9, Chapter II, Theorem 10] we obtain the following estimate with an absolute constant $C > 0$:

$$\frac{1}{|\zeta(s+1)|} \leq C \log^7(2 + |\operatorname{Im} s|), \quad \operatorname{Re} s \geq 0, \quad (93)$$

see also Ingham's general discussion before Theorem 9 concerning the use of the big- $\mathcal{O}$ symbols for the correct interpretation of his result. Using (93) and Lemma 5.2 we obtain that

$$\left| \frac{\zeta(s)}{s^2 \zeta(s+1)} \right| \leq C \tilde{C} \frac{\log^7(2 + |\operatorname{Im} s|)}{|s|}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ and $|s| \geq 2$. Finally we use the analytical continuation of G from Lemma 5.1 and conclude

$$|G(s)| \leq C' \frac{\log^7(2 + |\operatorname{Im} s|)}{|s| + 1}$$

for all $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$ and a new constant $C' > 0$. $\square$

A good introduction to the theory of Hardy spaces is presented in the textbooks [7, 12, 16].

Theorem 5.4 (integral representation of the limit function g) *For $x = e^u > 1$, i.e. for $u > 0$, we have*

$$g(x) = g_0(x) + \left(\frac{6}{\pi^2} - \frac{1}{2} \right) \cdot \frac{1}{x}$$

with

$$g_0(x) = g_0(e^u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itu} \left[\frac{1}{it - t^2} \left(\frac{6}{\pi^2} - \frac{1}{2} \right) - \frac{6}{\pi^2} \cdot \frac{1}{t^2 + it} + \frac{\pi^{it-1/2}}{t^2} \cdot \frac{\Gamma((1-it)/2)}{\Gamma(it/2)} \cdot \frac{\zeta(1-it)}{\zeta(1+it)} \right] dt.$$

The integral above is absolutely convergent.

Proof We also have $G_0 \in H_0^2 = H^2(\operatorname{Re} s > 0)$, from Lemma 5.3 for

$$G_0(s) = G(s) - \left(\frac{6}{\pi^2} - \frac{1}{2} \right) \frac{1}{s+1},$$

or from (86) in Lemma 5.1, regarding a limit for $s \rightarrow 1$,

$$G_0(s) = \frac{1}{s(s+1)} \left(\frac{6}{\pi^2} - \frac{1}{2} \right) + \frac{6}{\pi^2} \frac{1}{s(s-1)} - \frac{\zeta(s)}{s^2 \zeta(s+1)}. \quad (94)$$

Using the functional equation (91) for $\zeta(s)$ in the nominator of the last fraction in (94), we obtain that $G_0(s)$ decays faster than $G(s)$ along $\operatorname{Re} s = \sigma \geq 0$. Especially on the boundary $\operatorname{Re} s = 0$ there holds the following representation for $t \in \mathbb{R} \setminus \{0\}$, regarding a limit for $t \rightarrow 0$:

$$G_0(it) = \frac{1}{it - t^2} \left(\frac{6}{\pi^2} - \frac{1}{2} \right) - \frac{6}{\pi^2} \frac{1}{t^2 + it} + \frac{\pi^{it-1/2}}{t^2} \frac{\Gamma((1-it)/2)}{\Gamma(it/2)} \frac{\zeta(1-it)}{\zeta(1+it)},$$

and from (92) the boundary function $G_0(i \cdot)$ satisfies

$$G_0(it) = \mathcal{O}(|t|^{-3/2}) \quad \text{for } |t| \rightarrow \infty$$

with $G_0(i \cdot) \in (L^1 \cap L^2)(\mathbb{R})$. Thus we can apply Mellin's inversion formula and obtain for $x = e^u > 1$,

$$g_0(x) = g_0(e^u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itu} G_0(it) dt,$$

$$g(x) = g_0(x) + \left(\frac{6}{\pi^2} - \frac{1}{2} \right) \frac{1}{x}.$$

□

Corollary 5.5 $\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow -\infty} g(x) = 0$.

Proof Apply the Riemann–Lebesgue lemma of Fourier analysis on the absolutely convergent integral in Theorem 5.4. □

6 Conclusions and outlook

In Sect. 3 we have introduced 1-periodic polygonal functions f_n which carry important number theoretical information about Farey sequences, see especially Corollary 3.10. The main result Theorem 4.6 was obtained by using only elementary properties of Farey fractions, namely from Theorem 2.8. The limit function g plays a major role in Theorem 4.6, such that we have studied its asymptotic behaviour in Sect. 5.

An appropriate complex integration technique may refine Theorem 5.4 and can afford quantitative remainder term estimates for Corollary 5.5 when this is needed for later purposes.

Theorem 4.6 describes the behaviour of the Farey sequence and the function f_n with large order n in the vicinity of any fixed fraction. It would be an interesting future task to describe the local behaviour of f_n for almost all irrational numbers in terms of appropriate distribution functions for the Farey sequences. Distribution functions for the Farey sequences were already studied in [2], see also the references therein.

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