Reflecting properties in continuous images of small weight

Ofelia T. Alas¹ · Lucia R. Junqueira¹ ·
Richard G. Wilson²

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Abstract A topological property $\mathcal{P}$ is reflected in continuous images of weight at most ω_1 if a space X has $\mathcal{P}$ whenever every continuous image of X of weight at most ω_1 has $\mathcal{P}$. When X is a generalized ordered space, we consider a number of topological properties including feeble Lindelöfness and κ -monolithicity. In the final section we study small images of pseudocompact and countably compact spaces; we give a condition on continuous images of a pseudocompact space in order that it be compact and show that it is consistently true that a countably compact space of countable projective tightness is countably tight.

Keywords GO-space · Pseudocompact space · Countably compact space · ω -bounded space · Pseudocharacter · Countable projective tightness

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✉ Lucia R. Junqueira
lucia@ime.usp.br
Ofelia T. Alas
alas@ime.usp.br
Richard G. Wilson
rgw@xanum.uam.mx

¹ Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo 05314-970, Brazil

² Departamento de Matemáticas, Universidad Autónoma Metropolitana, Unidad Iztapalapa, Avenida San Rafael Atlixco, # 186, Apartado Postal 55-532, 09340 Mexico, D.F., Mexico

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1 Introduction and terminology

Following the notation and terminology of [10, 11, 13], we say that a topological property $\mathcal{P}$ is *reflectable* in continuous images of weight κ in a class of topological spaces $\mathcal{C}$ if a space $X \in \mathcal{C}$ has the property $\mathcal{P}$ whenever every continuous image of X of weight at most κ has $\mathcal{P}$. This paper is a sequel to [13] in which $\mathcal{C}$ is the class of generalized ordered (or GO-) spaces, that is, the class of all subspaces of ordered spaces. It was shown in [13] that in GO-spaces, unlike in the class of all Tychonoff spaces, density, network weight and tightness are all reflected in small continuous images. Among the properties considered here are that of being κ -monolithic, having countable extent, pseudocompactness and countable compactness. We answer a number of questions posed in [13].

Recall that for an arbitrary cardinal κ , a space (X, τ) has *extent* equal to κ , written $e(X) = \kappa$, if $\sup\{|D| : D \subseteq X \text{ is closed and discrete}\} = \kappa$; a space X is κ -*monolithic* (respectively, *strongly* κ -*monolithic*) if for each $A \subseteq X$ with $|A| \leq \kappa$, we have $\text{nw}(\text{cl}(A)) \leq \kappa$ (respectively, $w(\text{cl}(A)) \leq \kappa$) (see [12, Section 1.2]). A space is *monolithic* if it is κ -monolithic for each κ . Since for each subset S of a GO-space $w(S) = \text{nw}(S)$, it follows that monolithicity and strong monolithicity coincide in this class. A space X is *strongly collectionwise Hausdorff* if every closed discrete subset of X can be separated by a discrete family of open sets.

If X is a GO-space, then $p \in X$ is a *left* (respectively, *right*) *pseudogap* if $[p, \rightarrow)$ is open and p has no immediate predecessor (respectively, $(\leftarrow, p]$ is open and p has no immediate successor). A point $p \in X$ is a *pseudogap* of X if it is either a right or a left pseudogap. A *jump* of X is a pair of consecutive elements of X . Let X^* denote the (minimal) Dedekind completion of X (obtained by adding one point in each gap and in each pseudogap); intervals in X^* are denoted by $(a, c)^*$, while suprema and infima in X^* are denoted by $\sup^*$ and $\inf^*$ and closures by cl^* .

The remaining notation and terminology is standard and follows [3, 6]; basic information regarding GO-spaces can be found in [7]. Throughout the paper, all spaces are assumed to be Tychonoff.

2 Continuous images of GO-spaces

In the sequel we shall use the following known result a number of times; the proof is left to the reader.

Lemma 2.1 *If X is a Tychonoff space and $a, b \in X$, then the quotient space Y obtained from X by identifying a and b is Tychonoff.*

The following theorem answers [13, Question 4.7]; the much simplified proof was communicated to us by the referee.

Theorem 2.2 *For any infinite cardinal κ , if every continuous image of a GO space X of weight at most κ^+ is κ -monolithic, then X is κ -monolithic.*

Proof Suppose that the set $D \subseteq X$ is such that with $|D| \leq \kappa$. Then $Y = \text{cl}(D)$ is also a GO space and we need to prove that $\text{nw}(Y) \leq \kappa$. Suppose that $f: Y \rightarrow Z$ is a continuous surjective map and $\text{w}(Z) \leq \kappa^+$; there is no loss of generality if we assume that $Z \subseteq [0, 1]^{\kappa^+}$.

The set Y is closed in X and so it follows from the normality of X that there exists a continuous map $g: X \rightarrow [0, 1]^{\kappa^+}$ such that $g|_Y = f$; then $Z' = g[X] \supseteq Z$. The space Z' must be κ -monolithic being a continuous image of X of weight at most κ^+ , and so since $\text{d}(Z) \leq \text{d}(Y) \leq \kappa$ it follows that $\text{nw}(Z) \leq \kappa$. Therefore we have proved that every continuous image of Y of weight not exceeding κ^+ must have network weight at most κ . Since Y is a GO space, we can apply [13, Theorem 3.11] to see that $\text{w}(Y) \leq \kappa$ and hence $\text{nw}(Y) \leq \kappa$, that is, X is κ -monolithic. $\square$

Recall that a space X is feebly κ -Lindelöf if every locally finite family of non-empty open sets is of cardinality at most κ . In case $\kappa = \omega$, then X is said to be a feebly Lindelöf space. It is well known that if X is a regular space, then for each κ , being feebly κ Lindelöf is equivalent to the discrete κ chain condition, that is, every discrete family of non-empty open sets is of cardinality at most κ .

[11, Example 2.4] shows that in the class of Tychonoff spaces having countable extent does not reflect in continuous images of weight ω_1 . However, the situation is different in the class of GO-spaces. If $D = \{d_\alpha : \alpha \in \kappa\}$ is a closed discrete subset of a GO-space X , then by collectionwise normality of X there is a discrete family of open sets $\mathcal{U} = \{U_\alpha : \alpha \in \kappa\}$ such that $d_\alpha \in U_\alpha$ for each $\alpha \in \kappa$. It follows immediately that a GO-space X is feebly κ -Lindelöf if and only if $\text{e}(X) \leq \kappa$.

Theorem 2.3 *The property of being feebly κ -Lindelöf is reflected in continuous images of weight κ^+ .*

Proof Suppose that $\{F_\alpha : \alpha \in \kappa^+\}$ is a discrete family of non-empty open sets in X and for each $\alpha \in \kappa^+$, pick $x_\alpha \in F_\alpha$. A simple well known argument can be used to show that there is a continuous function $f: X \rightarrow J(\kappa^+)$, the hedgehog with κ^+ spines (see [3, Example 4.1.5]) with the property that f maps x_α to the end point of the α 'th spine and $X \setminus \bigcup \{F_\alpha : \alpha \in \kappa^+\}$ to the vertex. Thus $f[X]$ has weight κ^+ and is not feebly κ -Lindelöf. $\square$

The following result is now an immediate consequence of the lemma and the fact that if a surjective map $g: X \rightarrow Y$ is continuous and $\text{e}(Y) \geq \kappa$, then $\text{e}(X) \geq \kappa$ as well.

Theorem 2.4 *If X is a strongly collectionwise Hausdorff space, then $\text{e}(X) \leq \kappa$ if and only if every continuous image of X of weight at most κ^+ has extent at most κ .*

In the class of GO-spaces even more is true, and the following result seems to be of interest in its own right.

Theorem 2.5 *Suppose that κ is an infinite cardinal; if X is a GO-space, and $\text{e}(X) \geq \kappa^+$, then X can be continuously mapped onto the discrete space of cardinality κ^+ .*

Proof Suppose that $\text{e}(X) \geq \kappa^+$ and that D is a closed discrete subset of $(X, <)$ of cardinality κ^+ . Note that if J is a countably compact interval in $(X, <)$, then $J \cap D$ is

finite and hence any Dedekind complete subinterval of X can contain at most countably many elements of D . We define an equivalence relation $\sim$ on D as follows:

$$d \sim d' \iff \begin{array}{l} \text{there is no element } p \in X^* \setminus X \\ \text{such that } p \in (d, d')^* \text{ or } p \in (d', d)^*. \end{array}$$

Clearly then, there are κ^+ -many equivalence classes, each of which is countable and without loss of generality, by passing to a subset of D , we may assume each equivalence class contains a unique element of D ; in other words, between any two elements of D there is a gap or a pseudogap. Well order D as $\{d_\alpha : \alpha < \kappa^+\}$ and consider $d_0 \in D$. Let $c_0 = \sup^*\{d \in D : d < d_0\}$ and $e_0 = \inf^*\{d \in D : d > d_0\}$. Since D is closed and discrete in X , if $z \in \{c_0, d_0\}$, then either

$$z \in X^* \setminus X \quad \text{or} \quad z \in D \setminus \{d_0\}.$$

If $c_0 \in X^* \setminus X$, then put $p_0 = c_0$ and if $c_0 \in D$, then pick an element $p_0 \in X^* \setminus X$ such that $c_0 < p_0 < d_0$. Similarly we may find $q_0 \in X^* \setminus X$ such that $d_0 < q_0 \leq e_0$ and the resulting clopen subinterval $I_0 = (p_0, q_0)^* \cap X$ of X is such that $I_0 \cap D = \{d_0\}$. Suppose now that $\alpha < \kappa^+$ and that we have constructed disjoint clopen intervals $I_\beta = (p_\beta, q_\beta)^* \cap X$, where $p_\beta, q_\beta \in X^* \setminus X$ and $I_\beta \cap D = \{d_\beta\}$ for each $\beta < \alpha$; consider $d_\alpha \in D$. It is now easy to see that since D is closed and discrete, it follows that $d_\alpha \notin \text{cl}(\bigcup\{I_\beta : \beta < \alpha\})$. Thus

$$\begin{aligned} c_\alpha &= \sup^*\{d_\beta : \beta < \alpha \text{ and } d_\beta < d_\alpha\} < d_\alpha \\ &< \inf^*\{d_\beta : \beta < \alpha \text{ and } d_\beta > d_\alpha\} = e_\alpha. \end{aligned}$$

As before, if $c_\alpha \in X^* \setminus X$, then clearly $c_\alpha \geq q_\beta$ for all q_β such that $\beta < \alpha$ and $d_\beta < d_\alpha$ and we put $p_\alpha = c_\alpha$. If on the other hand, $c_\alpha \in D \setminus \{d_\alpha\}$, then there is $p_\alpha \in X^* \setminus X$ such that $c_\alpha < p_\alpha < d_\alpha$. A similar argument can be used to find $q_\alpha \in X^* \setminus X$ such that $I_\alpha = (p_\alpha, q_\alpha)^* \cap X$ is a clopen interval in X disjoint from all the intervals I_β and $I_\alpha \cap D = \{d_\alpha\}$.

Now for each element $d \in D$ suppose that we have chosen $p_d, q_d \in X^* \setminus X$ such that $(p_d, q_d)^* \cap D = \{d\}$ and the intervals $I_d = (p_d, q_d)^* \cap X$ are mutually disjoint; we claim that the family $\{I_d : d \in D\}$ is discrete. To prove the claim, notice that since D is closed and discrete and X is collectionwise normal, there is a discrete family of open sets $\mathcal{U} = \{U_d : d \in D\}$ such that $d \in U_d$ and without loss of generality we may assume that $U_d \subseteq I_d$. Suppose, contrary to our claim, that $x \in X$ is an accumulation point of the family $\{I_d : d \in D\}$; then each open interval J in X with $x \in J$ meets infinitely many of the sets I_d . However, this implies that J contains infinitely many intervals I_d , and hence infinitely many intervals U_d ; this contradicts the fact that $\mathcal{U}$ is a discrete family.

Since each interval I_d is open and closed in X , it now follows that $\bigcup\{I_d : d \in D\}$ is also open and closed (in X) and we may define a map $f : X \rightarrow D$ by $f[I_d] = \{d\}$ and $f[X \setminus \bigcup\{I_d : d \in D\}] = \{d_0\}$, where $d_0 \in D$; clearly, f is a continuous map from X onto the discrete space of cardinality κ^+ . $\square$

Following [13], we say that a space X has *countable projective tightness* if every continuous image Y of X with $w(Y) \leq \omega_1$ has countable tightness. It is easy to see that any hereditarily separable space has countable projective tightness. If X is a space which is not feebly Lindelöf, then it contains a closed, discrete, C -embedded subset A of cardinality ω_1 which can be mapped continuously onto the one-point Lindelöfication of a discrete space of the same cardinality which we assume to be embedded in $\mathbb{R}^{\omega_1}$. As in [10, Lemma 3.3] this map can be extended continuously to X , showing that X does not have countable projective tightness. The following result is an immediate consequence.

Proposition 2.6 *If X has countable projective tightness, then X is feebly Lindelöf and if X is normal, in particular, if X is a GO-space, then $e(X) = \omega$.*

In [13] it was shown that a GO-space of countable projective tightness has countable tightness and Proposition 2.6 shows that it must have countable extent. However, even more is true.

Theorem 2.7 *If X is a GO-space which has countable projective tightness, then X is a Lindelöf space.*

Proof Note first that if there is some point $p \in X^* \setminus X$ such that $t(p, X^*) > \omega$, then the quotient topology on X obtained by identifying p and a point $x \in X$ in the space $X \cup \{p\}$ is, by Lemma 2.1, a Tychonoff space and has uncountable tightness. It follows that every gap and pseudogap of X must have countable coinitality and countable cofinality; it now follows from [8, Theorem E], that X is paracompact. Then, if X is not Lindelöf, there must exist a locally finite uncountable open cover of the space X and by choosing a point in each element of this cover we obtain an uncountable closed, discrete subset of X . This contradicts Proposition 2.6, and so X is Lindelöf. $\square$

Since in the class of compact spaces, countable tightness is preserved under continuous images, the lexicographically ordered square $[0, 1]^2$ shows that a LOTS of countable projective tightness need be neither separable nor hereditarily Lindelöf.

The question was posed in [13] as to whether, in the class of GO-spaces, σ -pseudocompactness (or equivalently, σ -countable compactness) is reflected in continuous images of weight ω_1 . We note here only that a σ -pseudocompact space is feebly Lindelöf, but the converse is false even in the class of first countable GO-spaces, as the space of irrationals illustrates.

3 Small images of pseudocompact spaces

If X is a non-pseudocompact space, then it contains a closed, discrete, C -embedded, countably infinite subspace D . The set D can be mapped continuously onto a countable dense subspace of $\mathbb{R}^{\omega_1}$ and then (as in [10, Lemma 3.3]) this function can be extended continuously to all of X ; thus X has a continuous image of weight at most ω_1 which does not have any point of countable character. The next theorem considers the case in which X is pseudocompact and answers positively [13, Question 4.4].

Theorem 3.1 *A pseudocompact Tychonoff space X has the property that every continuous image Y of X with $w(Y) \leq \omega_1$ has countable pseudocharacter if and only if X is compact and perfect.*

Proof To simplify the terminology, let us say that a space X has property P if it is a pseudocompact Tychonoff space such that every continuous image Y of X with $w(Y) \leq \omega_1$ has countable pseudocharacter.

It follows immediately from [13, Proposition 3.3 (b)] that if X has property P , then it is perfect and so $\psi(X) = \omega$. Since both X and Y are pseudocompact, a folklore result which appears in a slightly stronger form as [1, Lemma 3.7] then implies that $\chi(X) = \chi(Y) = \omega$ also.

Suppose now that the space X has property P and Y is a continuous image of X of weight at most ω_1 which is not compact. Then since Y is pseudocompact, it is not realcompact, and so it can be embedded in $\mathbb{R}^{\omega_1}$ as a non-closed subspace (Z, τ) . Furthermore, the closure of Z in $\mathbb{R}^{\omega_1}$ is a realcompact, pseudocompact space and is therefore compact. Let $z \in Z$, $p \in \text{cl}(Z) \setminus Z$ and consider the quotient space obtained by identifying p and z in the space $Z \cup \{p\}$; for convenience we denote this space by (Z, τ') . Note that since $\chi(p, Z \cup \{p\}) \leq w(\mathbb{R}^{\omega_1}) = \omega_1$ it follows that $w(Z, \tau') \leq \omega_1$. By the previous lemma, (Z, τ') is a Tychonoff space and since $\tau' \subseteq \tau$, (Z, τ') is pseudocompact. Furthermore, since (Z, τ) is pseudocompact, it is G_δ -dense in $Z \cup \{p\}$ and so $\{p\}$ is not a G_δ -set in $Z \cup \{p\}$ which implies that (Z, τ') does not have countable pseudocharacter, a contradiction, since it is a continuous image of X which has property P . Thus we have proved that if X has property P , then every continuous image of weight ω_1 of X is a first countable compact space.

Again suppose that X has property P ; we require to show that X is compact. To prove this, suppose to the contrary that $\beta X \setminus X \neq \emptyset$. Let $f: \beta X \rightarrow T$ be a continuous surjection of βX onto a space of weight at most ω_1 . Then the map $f|X: X \rightarrow f[X]$ is a continuous surjection of X onto a space of weight at most ω_1 and hence by the result of the previous paragraph, $f[X]$ is compact. However, X is dense in βX and so $f[X]$ is dense in T and it follows that $f[X] = T$. It follows from the property P of the space X , and $w(T) \leq \omega_1$, that $\psi(T) = \chi(T) = \omega$. Since T was assumed to be an arbitrary image of βX of weight ω_1 , the space βX also has the property P . However, as we saw in the second paragraph of this proof, this implies that βX is first countable and as a consequence, X is compact.

Conversely, suppose that X is a perfectly normal compact space and $f: X \rightarrow Y$ is a continuous map onto a Hausdorff space Y . A standard argument shows that Y is first countable. $\square$

It is clear from the comment at the beginning of this section, that if every continuous image of weight ω_1 of a space X has character ω , then X is pseudocompact. This observation allows us to formulate the following result.

Corollary 3.2 *A space X has the property that every continuous image of weight at most ω_1 of X has character ω if and only if X is a perfectly normal compact space.*

[13, Question 4.3] asks whether a space with property P must have countable extent. However, if X is any Tychonoff space with property P , then by [13, Proposition 3.3 (a)],

every closed, discrete and C -embedded subset of X is countable and hence X is feebly Lindelöf. Thus there is a positive answer to [13, Question 4.3] in case X is strongly collectionwise Hausdorff.

Recall that a space X is ω -bounded if the closure of every countable subset of X is compact. The next theorem has a proof similar to that of Theorem 3.1; it gives a partial answer to [13, Question 4.11] and generalizes [11, Theorem 2.11] in ZFC.

Theorem 3.3 *If X is an ω -bounded space which has countable projective tightness, then X is compact and $t(X) = \omega$.*

Proof Suppose that Y is a continuous image of X of weight at most ω_1 ; we claim that Y is compact. To prove this, note first that if $w(Y) = \omega$, then Y is metrizable and hence compact. Now suppose that $w(Y) = \omega_1$; note that Y is ω -bounded and we may consider Y to be a subspace of $\mathbb{R}^{\omega_1}$. If Y is not compact, then it is not realcompact and hence Y is not closed in $\mathbb{R}^{\omega_1}$. As in Theorem 3.1, there is a point $p \in \text{cl}(Y) \setminus Y$ and by identifying p with a point $y \in Y$, we obtain a continuous image Z of X of weight ω_1 . Furthermore, since Y is ω -bounded, there is no countable subset of Y with p in its closure. However there is some, necessarily uncountable subset $A \subseteq Y$ such that $p \in \text{cl}(A)$ and $y \notin \text{cl}(A)$ and hence as a subset of Z , the set A witnesses that Z does not have countable tightness. This contradicts the assumption that X has countable projective tightness and so Y is compact.

Again suppose that the ω -bounded space X has countable projective tightness and suppose that X is not compact, that is, $\beta X \setminus X \neq \emptyset$. Let $f: \beta X \rightarrow T$ be a continuous surjection onto a space T of weight at most ω_1 . Then $f|X: X \rightarrow f[X]$ has the same properties as f and hence from the previous paragraph, we may conclude that $f[X]$ is compact and dense in T and so $f[X] = T$. Since X has countable projective tightness, it follows that $t(T) = \omega$ and since T is an arbitrary continuous image of βX of weight ω_1 , βX has countable projective tightness. It now follows from [11, Theorem 2.11], that βX , and hence X , has countable tightness. Finally, since X is ω -bounded, the closure in βX of every countable subset of X is contained in X ; this combined with the countable tightness of βX implies that $X = \beta X$, and so X is compact. $\square$

The technique used in the previous two theorems can be used to give a consistent positive answer to [13, Question 4.11]. However we do not know if the next result is a theorem in ZFC. The inequality $\omega_1 < \mathfrak{p}$ is a well known consequence of $\text{MA} + \neg \text{CH}$.

Theorem 3.4 $(\omega_1 < \mathfrak{p})$ *If X is a countably compact space of countable projective tightness, then $t(X) = t(\beta X) = \omega$.*

Proof Let X be a countably compact space of countable projective tightness and Y a continuous image of X with $w(Y) \leq \omega_1$; note that Y is countably compact. In the sequel we note only the differences between this and the proof of the previous theorem. Suppose that $q \in \text{cl}(Y) \setminus Y \subseteq \mathbb{R}^{\omega_1}$; if $t(q, Y \cup \{q\}) = \omega$, there is a countable set $A \subseteq Y$ such that $q \in \text{cl}(A)$. It is then a consequence of $\omega_1 < \mathfrak{p}$ that there is an injective sequence S in A converging to q (see [9]). However this is impossible since S is closed and discrete in the countably compact space Y . Thus as before, we may take $q \in \text{cl}(Y) \setminus Y$ such that $t(q, Y \cup \{q\}) = \omega_1$ and so prove that Y is compact. The

proof continues as in the final paragraph of Theorem 3.3 and we conclude that βX , and hence X , has countable tightness. $\square$

We note that unlike Theorem 3.3, in Theorem 3.4, we cannot conclude from this proof that X is compact. However, the Proper Forcing Axiom (PFA), which is consistent with ZFC and the existence of a supercompact cardinal, implies MA_{ω_1} and $\mathfrak{p} = \mathfrak{c} = \omega_2$ (see [5]). Furthermore, PFA also implies that every compact space of countable tightness is sequential (see [2]). However, if βX is sequential and X is countably compact, then clearly, $X = \beta X$, and so X is compact. On the other hand, under Jensen's $\diamond$, there exists a compact hereditarily separable space K of cardinality $\mathfrak{c}$ with no non-trivial convergent sequences (see [4]). If $x \in K$, then $K \setminus \{x\}$ is a countably compact, hereditarily separable space which is not compact but which has countable projective tightness.

Question 3.5 Is it true in ZFC that if X is a countably compact space of countable projective tightness, then X has countable tightness?

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