Generalized Dehn–Sommerville relations for hypergraphs

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Abstract The generalized Dehn–Sommerville relations determine the odd subalgebra of the combinatorial Hopf algebra. We introduce a class of eulerian hypergraphs that satisfy the generalized Dehn–Sommerville relations for the combinatorial Hopf algebra of hypergraphs. We characterize a wide class of eulerian hypergraphs in terms of the combinatorics of underlying clutters. The analogous results hold for simplicial complexes by the isomorphism which is induced by the correspondence between clutters and simplicial complexes.

Keywords Hopf algebra · Hypergraph · Clutter · Simplicial complex · Generalized Dehn–Sommerville relations

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1 Introduction

Hopf algebras in combinatorics first appeared in the fundamental papers by Joni and Rota [8] and Schmitt [9]. The Hopf algebra structure often arises on linear basis indexed by combinatorial objects with a coproduct given as the disassembly of these objects. A specific enumerative information about combinatorial objects is provided by multiplicative functionals called characters. A graded Hopf algebra with a distinguished character is called a combinatorial Hopf algebra. The theory of combinatorial Hopf algebras is formalized in [3]. The quasisymmetric functions serve as the generating functions in combinatorics. They are the terminal object in the category of combinatorial Hopf algebras.

In this paper we study combinatorial Hopf algebras of hypergraphs and simplicial complexes. The combinatorial Hopf algebra of hypergraphs extends the chromatic Hopf algebra of graphs. The generating quasisymmetric function appears to be the chromatic symmetric function of a hypergraph, introduced in [12]. The classes of hypergraphs closed under disjoint sums and restrictions form Hopf subalgebras of hypergraphs. Clutters and building sets are objects of this type. These objects play an important role in combinatorial algebra and geometry. The ring of building sets was introduced in [5]. The combinatorial Hopf algebra of building sets is extensively studied in [7]. All main properties of a building set, hypergraph and simplicial complex essentially depend on the underlying clutters. We use this fact to elaborate the results from [7] for building sets to hypergraphs and simplicial complexes.

The classical Dehn–Sommerville relations state the symmetry of the h -vector of a simplicial polytope. Their extension to flag f -vectors of eulerian posets proved by Bayer and Billera [4] is called the generalized Dehn–Sommerville relations. Similar relations are associated to any combinatorial Hopf algebra. These relations characterize the odd subalgebra defined as the maximal Hopf subalgebra on which the character is odd. We solve the generalized Dehn–Sommerville relations in the Hopf algebra of hypergraphs.

We say that a hypergraph is eulerian if the Euler character vanishes at all induced subhypergraphs. The eulerian hypergraphs satisfy the generalized Dehn–Sommerville relations and generate the proper subalgebra of the odd subalgebra of hypergraphs. We prove that an odd clutter whose intersection complex is a clique complex of a chordal graph is eulerian. The converse is true under additional condition that any proper subclutter is also a proper restriction. This corrects the deficiency in the formulation of [7, Theorem 8.11], where the characterization of eulerian building sets is given in terms of combinatorics of underlying clutters. We obtain another characterization which follows from Stanley’s expansion of the chromatic symmetric function of a hypergraph into the power sum symmetric functions.

The ring of simplicial complexes appears in [2]. The correspondence of clutters and their independence complexes induces an isomorphism of Hopf algebras of clutters and simplicial complexes. Under this isomorphism eulerian simplicial complexes correspond to independence complexes of eulerian clutters.

2 Preliminaries from combinatorial Hopf algebras

We fix some notations and review basic definitions and facts from the theory of combinatorial Hopf algebras developed in [3]. An extensive exposure on Hopf algebras in combinatorics can be found in [6]. A *composition* $\alpha \models n$ of an integer $n \in \mathbb{N}$ is a sequence $\alpha = (a_1, \dots, a_k)$ of positive integers with $a_1 + \dots + a_k = n$. Let $k(\alpha) = k$ be the length of α . A *partition* $\lambda \vdash n$ is a composition $\lambda \models n$ with $a_1 \geq a_2 \geq \dots \geq a_k$.

The *combinatorial Hopf algebra* $(\mathcal{H}, \zeta)$ (CHA for short) over a field $\mathbf{k}$ is a graded connected Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ equipped with the multiplicative linear functional $\zeta: \mathcal{H} \rightarrow \mathbf{k}$ called the *character*. The terminal object in the category of combinatorial Hopf algebras is the CHA of quasisymmetric functions (QSym, ζ_Q) . The basic reference for quasisymmetric functions is [10]. Let $\{M_\alpha\}_{\alpha \models n, n \in \mathbb{N}}$ be the linear basis of monomial quasisymmetric functions, where

$$M_{(a_1, \dots, a_k)} = \sum_{i_1 < \dots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$

The character ζ_Q is determined by

$$\zeta_Q(M_\alpha) = \begin{cases} 1, & \alpha = (n) \text{ or } (), \\ 0, & \text{otherwise.} \end{cases}$$

The unique canonical morphism $\Psi: (\mathcal{H}, \zeta) \rightarrow (\text{QSym}, \zeta_Q)$ is given on homogeneous elements by

$$\Psi(h) = \sum_{\alpha \models n} \zeta_\alpha(h) M_\alpha, \quad h \in \mathcal{H}_n, \quad (1)$$

where ζ_α is the convolution product

$$\zeta_\alpha = \zeta_{a_1} \cdots \zeta_{a_k}: \mathcal{H} \xrightarrow{\Delta^{(k-1)}} \mathcal{H}^{\otimes k} \xrightarrow{\text{proj}} \mathcal{H}_{a_1} \otimes \cdots \otimes \mathcal{H}_{a_k} \xrightarrow{\zeta^{\otimes k}} \mathbf{k}.$$

The algebra of symmetric functions Sym is the subalgebra of QSym and it is the terminal object in the category of cocommutative combinatorial Hopf algebras.

The antipode of a graded connected bialgebra $\mathcal{H}$ is determined by Takeuchi's formula ([14, Lemma 14], see also [6, Proposition 1.44])

$$S = \sum_{k \geq 0} (-1)^k m^{(k-1)} \circ \pi^{\otimes k} \circ \Delta^{(k-1)}, \quad (2)$$

where $\pi = \text{I} - u\epsilon$ and I, u, ϵ are the identity map, unit and counit of $\mathcal{H}$, respectively. The convolution inverse ζ^{-1} , called the *Möbius character* of $\mathcal{H}$, is the composition $\zeta^{-1} = \zeta \circ S$.

Let $\mathbf{k}[m]$ be the polynomial Hopf algebra with comultiplication and antipode given by $\Delta(m) = 1 \otimes m + m \otimes 1$ and $S(m) = -m$. Then the *principal specialization* $\text{ps}(f)(m) = f|_{x_1 = \dots = x_m = 1, x_{m+1} = \dots = 0}$ is a Hopf morphism $\text{ps}: \text{QSym} \rightarrow \mathbf{k}[m]$. For

elements of CHA $(\mathcal{H}, \zeta)$ it defines a polynomial invariant $\chi(h, m) = \text{ps}(\Psi(h))(m)$, where Ψ is the canonical morphism (1). It satisfies the equality

$$\chi(h, m) = \zeta^{*m}(h), \quad m \in \mathbb{Z}. \quad (3)$$

For details, see [6, Proposition 7.7].

Let $\bar{\zeta}$ be the *conjugate character* defined on homogeneous elements by $\bar{\zeta}(h) = (-1)^n \zeta(h)$, $h \in \mathcal{H}_n$. Then the *Euler character* χ of CHA $(\mathcal{H}, \zeta)$ is defined by

$$\chi = \bar{\zeta}\zeta: \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\bar{\zeta} \otimes \zeta} \mathbf{k} \otimes \mathbf{k} \xrightarrow{m} \mathbf{k}.$$

The *odd subalgebra* $S_-(\mathcal{H}, \zeta)$ is the largest graded Hopf subalgebra of $\mathcal{H}$ on which the character is odd $\zeta^{-1} = \bar{\zeta}$. A morphism $\phi: (\mathcal{H}_1, \zeta_1) \rightarrow (\mathcal{H}_2, \zeta_2)$ preserves the odd subalgebra $\phi(S_-(\mathcal{H}_1, \zeta_1)) \subset S_-(\mathcal{H}_2, \zeta_2)$. The *generalized Dehn–Sommerville relations* for CHA $(\mathcal{H}, \zeta)$

$$(\text{id} \otimes (\bar{\zeta} - \zeta^{-1}) \otimes \text{id}) \circ \Delta^{(2)}(h) = 0 \quad (4)$$

describe the sufficient and necessary conditions for homogeneous elements to belong to the odd subalgebra $S_-(\mathcal{H}, \zeta)$. Given a homogeneous element $h \in S_-(\mathcal{H}, \zeta)$ of rank n , it follows from (1) and the generalized Dehn–Sommerville relations for quasi-symmetric functions [3, Example 5.10] that

$$\sum_{j=0}^{a_i} (-1)^j \zeta_{(a_1, \dots, a_{i-1}, j, a_i - j, a_{i+1}, \dots, a_k)}(h) = 0, \quad (5)$$

for each $\alpha = (a_1, \dots, a_k) \models n$ and $i \in \{1, \dots, k\}$.

Rota's Hopf algebra of posets. The set of isomorphism classes of finite graded posets $\mathbf{P}$ spans the Hopf algebra of posets $\mathcal{H}(\mathbf{P})$ with the product $P_1 \cdot P_2 = P_1 \times P_2$ and the coproduct

$$\Delta(P) = \sum_{0_P \leq x \leq 1_P} [0, x] \otimes [x, 1].$$

The unit is the poset with one element and the counit is given by

$$\epsilon(P) = \begin{cases} 1, & 0_P = 1_P, \\ 0, & \text{otherwise.} \end{cases}$$

For a poset P with $0_P \neq 1_P$ the antipode formula (2) gives

$$S(P) = \sum_{k \geq 1} (-1)^k \sum_{0 < x_1 < \dots < x_{k-1} < 1} [0, x_1] \otimes \dots \otimes [x_{k-1}, 1].$$

Let $\zeta(P) = 1$ for any poset P . The Möbius character $\zeta^{-1} = \zeta \circ S$ is the classical Möbius function on posets. A poset P is eulerian if and only if $\chi([x, y]) = \epsilon([x, y])$, $x \leq y \in P$, which particularly means that the Euler character vanishes at all closed intervals $\chi([x, y]) = 0$, $x < y \in P$.

The canonical morphism $\Psi: (\mathcal{H}(\mathbf{P}), \zeta) \rightarrow (\text{QSym}, \zeta_Q)$ to a poset P assigns the generating function of its flag f -vector

$$\Psi(P) = \sum_{\alpha \models n} f_\alpha(P) M_\alpha.$$

Relations (5) are precisely the Bayer–Billera relations for the flag f -vector of an eulerian poset P .

3 Hopf algebra of hypergraphs

A *hypergraph* H on the finite vertex set V is a collection $H \subset \mathcal{P}(V)$ of non-empty subsets of V . The subsets $e \in H$ are called edges and we require that any edge has at least two vertices. The hypergraph D with no edges is the discrete hypergraph on V . Two hypergraphs H_1 and H_2 are isomorphic if there is a bijection $f: V(H_1) \rightarrow V(H_2)$ such that $e \in H_1$ if and only if $f(e) \in H_2$. A restriction of the hypergraph H on the vertex set $I \subset V(H)$ is the hypergraph $H|_I = \{e \in H : e \subset I\}$.

Let $\mathcal{H}(\mathbf{H}) = \bigoplus_{n \geq 0} \mathcal{H}_n(\mathbf{H})$ be the graded $\mathbf{k}$ -vector space linearly spanned by the set $\mathbf{H}$ of all isomorphism classes of finite hypergraphs, where the grading is given by the number of vertices. Define the product and the coproduct by

$$\begin{aligned} H_1 \sqcup H_2 &= \{e \subset V(H_1) \sqcup V(H_2) : e \in H_1 \text{ or } e \in H_2\}, \\ \Delta(H) &= \sum_{I \subset V(H)} H|_I \otimes H|_{V(H) \setminus I}. \end{aligned}$$

Then the space $\mathcal{H}(\mathbf{H})$ is a graded connected commutative and cocommutative Hopf algebra. The unit element is the hypergraph $H_\emptyset$ on the empty set of vertices and the counit is defined by

$$\epsilon(H) = \begin{cases} 1, & H = H_\emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The antipode is determined on $H \neq H_\emptyset$ by (2) with

$$S(H) = \sum_{k \geq 1} (-1)^k \sum_{I_1 \sqcup \dots \sqcup I_k = V} H|_{I_1} \sqcup \dots \sqcup H|_{I_k}, \quad (6)$$

where the sum goes over all ordered decompositions of the vertex set. Let

$$\zeta_{\mathbf{H}}(H) = \begin{cases} 1, & H \text{ is discrete,} \\ 0, & \text{otherwise.} \end{cases}$$

We obtain the CHA of hypergraphs $(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$. The canonical morphism (1) of $\mathcal{H}(\mathbf{H})$ to quasisymmetric functions is determined by

$$\Psi(H) = \sum_{\alpha \models n} (\zeta_{\mathbf{H}})_{\alpha}(H) M_{\alpha}, \quad H \in \mathcal{H}_n(\mathbf{H}).$$

For a composition $\alpha = \{a_1, \dots, a_k\} \models n$ the coefficient $(\zeta_{\mathbf{H}})_{\alpha}(H)$ is the number of ordered decompositions $(I_1, \dots, I_k)$ of the vertex set V such that $H|_{I_j}$ is discrete of rank a_j for all $j = 1, \dots, k$. Since $\mathcal{H}(\mathbf{H})$ is a cocommutative Hopf algebra, the function Ψ is symmetric. It is easily seen to be the generating function $\Psi(H) = \sum_f \prod_{v \in V} x_{f(v)}$ of *proper colorings* $f: V \rightarrow \mathbb{N}$ of a hypergraph H . Recall that a coloring $f: V \rightarrow \mathbb{N}$ is proper if it is not monochromatic on edges of H .

The principal specialization $\chi(H, m) = \text{ps}(\Psi(H))(m)$ defines the *chromatic polynomial* of a hypergraph H . The value $\chi(H, m)$ is the number of proper colorings of the hypergraph H by at most m colors. By definition of the character $\zeta_{\mathbf{H}}$ we see that $\chi(H, m) = \zeta_{\mathbf{H}}^m(H)$, which holds for any integer $m \in \mathbb{Z}$ by (3). Therefore the antipode formula (6) gives

$$\chi(H, -1) = \zeta_{\mathbf{H}}^{-1}(H) = \zeta_{\mathbf{H}} \circ S(H) = \sum_{\alpha \models n} (-1)^{k(\alpha)} (\zeta_{\mathbf{H}})_{\alpha}(H). \quad (7)$$

Recall that if H is a simple graph, the chromatic polynomial $\chi(H, m)$ calculated at $m = -1$ gives the number of acyclic orientations of H [13].

Let $\mathbf{S}$ be a family of isomorphism classes of hypergraphs closed under taking disjoint sums and restrictions. The $\mathbf{k}$ -vector space $\mathcal{H}(\mathbf{S})$ with the basis $\mathbf{S}$ and with the character $\zeta_{\mathbf{S}} = \zeta|_{\mathcal{H}(\mathbf{S})}$ is a combinatorial Hopf subalgebra of the algebra of hypergraphs $(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$.

Example 3.1 Let $\mathbf{D} = \{D_n\}_{n \geq 0}$ be a family of discrete hypergraphs. Then $\mathcal{H}(\mathbf{D})$ is isomorphic to the polynomial Hopf algebra $\mathbf{k}[x]$, where $x = D_1$. The character $\zeta_1 = \zeta|_{\mathcal{H}(\mathbf{D})}$ is defined by $\zeta_1(p) = p(1)$, $p \in \mathbf{k}[x]$.

Example 3.2 Let $\mathbf{G}$ be the class of simple graphs considered as 2-uniform hypergraphs. Then $(\mathcal{H}(\mathbf{G}), \zeta_{\mathbf{G}})$ is the chromatic Hopf algebra of graphs.

We distinguish two important classes of hypergraphs that are closed under restrictions and disjoint unions.

Clutters. A clutter C on the finite set V is a hypergraph which is an antichain in the boolean poset $\mathcal{P}(V)$ of all subsets of V . For a hypergraph H denote by $C(H)$ the clutter of minimal edges of H . The clutters are naturally identified with simplicial complexes. A simplicial complex K defines the clutter of its maximal faces. The family of all isomorphism classes of clutters $\mathbf{C}$ defines the combinatorial Hopf algebra of clutters $(\mathcal{H}(\mathbf{C}), \zeta_{\mathbf{C}})$. The obvious inclusions are monomorphisms of CHA

$$(\mathcal{H}(\mathbf{D}), \zeta_{\mathbf{D}}) \subset (\mathcal{H}(\mathbf{G}), \zeta_{\mathbf{G}}) \subset (\mathcal{H}(\mathbf{C}), \zeta_{\mathbf{C}}).$$

Building sets. A collection B of non-empty subsets of the vertex set V is a *building set* on V if it satisfies the following properties:

- if $e, e' \in B$ and $e \cap e' \neq \emptyset$ then $e \cup e' \in B$,
- $\{v\} \in B$ for all $v \in V$.

We identify a building set B with the hypergraph $B \setminus \{\{v\} : v \in V\}$. The family of all isomorphism classes of building sets $\mathbf{B}$ defines the combinatorial Hopf algebra of building sets $(\mathcal{H}(\mathbf{B}), \zeta_{\mathbf{B}})$. This algebra is studied in [7].

Let $p: \mathcal{H}(\mathbf{H}) \rightarrow \mathcal{H}(\mathbf{C})$ be the projection map which assigns the clutter $C(H)$ of minimal edges to a hypergraph H . Since $C(H_1 \sqcup H_2) = C(H_1) \sqcup C(H_2)$ for $H_1, H_2 \in \mathbf{H}$ and $C(H|_I) = C(H)|_I$ for $H \in \mathbf{H}$ and any $I \subset V(H)$, the map p is a Hopf algebra epimorphism. It holds that $p \circ i = 1_{\mathcal{H}(\mathbf{C})}$, where $i: \mathcal{H}(\mathbf{C}) \rightarrow \mathcal{H}(\mathbf{H})$ is the inclusion map. Thus i is a splitting monomorphism.

The odd subalgebra of hypergraphs $S_-(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$ is determined by the generalized Dehn–Sommerville relations (4). Since the inclusion $i: (\mathcal{H}(\mathbf{C}), \zeta_{\mathbf{C}}) \rightarrow (\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$ is a morphism of combinatorial Hopf algebras, we have $S_-(\mathcal{H}(\mathbf{C}), \zeta_{\mathbf{C}}) = S_-(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}}) \cap \mathcal{H}(\mathbf{C})$.

4 Eulerian hypergraphs

Definition 4.1 A hypergraph H is *eulerian* if $\chi_{\mathbf{H}}(H|_I) = \epsilon(H|_I)$ for all $I \subset V(H)$, where $\chi_{\mathbf{H}} = \bar{\zeta}_{\mathbf{H}} \zeta_{\mathbf{H}}$ is the Euler character of $\text{CHA}(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$.

The following characterization is just a reformulation of the definition.

Lemma 4.2 A hypergraph $H \neq H_{\emptyset}$ is eulerian if and only if either $H|_I$ is discrete or $\zeta_{\mathbf{H}}^{-1}(H|_I) = 0$ for all $\emptyset \neq I \subset V(H)$.

Proof The condition $\chi_{\mathbf{H}}(H) = \epsilon(H)$ is equivalent to the condition $\zeta_{\mathbf{H}}^{-1}(H) = \bar{\zeta}_{\mathbf{H}}(H)$. Let $\chi(D_n, m) = m^n$ be the chromatic polynomial of the discrete hypergraph D_n on n vertices. By (7) we have $\zeta^{-1}(D_n) = \chi(D_n, -1) = (-1)^n = \bar{\zeta}(D_n)$. As $\bar{\zeta}_{\mathbf{H}}(H) = 0$ for the non-discrete H , the lemma follows. $\square$

Disjoint sums and restrictions of eulerian hypergraphs are eulerian as well, so the family of all eulerian hypergraphs forms the Hopf subalgebra $\mathcal{E}(\mathbf{H})$ of $\text{CHA}(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$, which we call the *eulerian subalgebra* of hypergraphs.

Proposition 4.3 The eulerian subalgebra $\mathcal{E}(\mathbf{H})$ is a subalgebra of the odd algebra $S_-(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$. The coefficients $(\zeta_{\mathbf{H}})_{\alpha}(H)$, $\alpha \models V(H)$ of an eulerian hypergraph $H \in \mathcal{E}(\mathbf{H})$ satisfy relations (5).

Proof The odd subalgebra $S_-(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$ is characterized by the generalized Dehn–Sommerville relations (4)

$$\begin{aligned} (\text{id} \otimes (\zeta^{-1} - \bar{\zeta}) \otimes \text{id}) \circ \Delta^{(2)}(H) &= \sum_{I \sqcup J \sqcup K = V(H)} H|_I \otimes (\zeta^{-1} - \bar{\zeta})(H|_J) \otimes H|_K \\ &= \sum_{J \subset V(H)} (\zeta^{-1} - \bar{\zeta})(H|_J) \Delta(H|_{J^c}) = 0. \end{aligned}$$

It follows that $H \in S_-(\mathcal{H}(\mathbf{H}), \zeta_{\mathbf{H}})$ if $(\zeta^{-1} - \bar{\zeta})(H|_J) = 0$ for all $J \subset V(H)$, which is equivalent to $H \in \mathcal{E}(\mathbf{H})$ by Lemma 4.2. Relations (4) imply (5). $\square$

Let $\mathcal{E}(\mathbf{C}) = \mathcal{E}(\mathbf{H}) \cap \mathcal{H}(\mathbf{C})$ be the eulerian subalgebra of clutters.

Proposition 4.4 *A hypergraph H is eulerian if and only if the clutter $C(H)$ is eulerian.*

Proof The proposition follows from the identity $\chi_{\mathbf{H}}(H) = \chi_{\mathbf{C}}(C(H))$, $H \in \mathbf{H}$, which is a consequence of the fact that the projection p is a morphism of combinatorial Hopf algebras, i.e. $\zeta_{\mathbf{H}} = \zeta_{\mathbf{C}} \circ p$. $\square$

The main computational tool is the *deletion–contraction property*. For the chromatic polynomial of a hypergraph it is proved in [15, Proposition 3.1], see also [7, Theorem 4.5] for the corresponding formula for building sets. The chromatic symmetric function of a hypergraph H depends only on the clutter of minimal edges, i.e. $\Psi(H) = \Psi(C(H))$. Therefore $\chi(H, m) = \chi(C(H), m)$ for chromatic polynomials. For a clutter C on the vertices V and an edge $e \in C$ define $C - e = C \setminus \{e\}$ to be *deletion* of e from C and $C/e = \{e'/e : e' \in C - e\}$ to be *contraction* of C by e , where C/e is the clutter on $V \setminus e \cup \{e\}$ and

$$e'/e = \begin{cases} e', & e' \cap e = \emptyset, \\ e' \setminus e \cup \{e\}, & e' \cap e \neq \emptyset. \end{cases}$$

The similar counting argument as for the chromatic polynomial of a graph shows that there is a one-to-one correspondence between proper m -colorings of $C - e$ which fail to be proper m -colorings of C and proper m -colorings of C/e . Therefore

$$\chi(C, m) = \chi(C - e, m) - \chi(C/e, m)$$

and specially for $m = -1$ we obtain by (7)

$$\zeta^{-1}(C) = \zeta^{-1}(C - e) - \zeta^{-1}(C/e).$$

To a clutter C is associated the abstract simplicial complex on the vertex set C , called the *nerve* $N(C) = \{C' \subset C : \bigcap C' \neq \emptyset\}$. The *intersection graph* $G(C)$ of a clutter is the 1-skeleton $N(C)^{(1)}$ of the nerve. We say that C is an *odd clutter* if $\bigcap C'$ is an odd element set for all faces $C' \in N(C)$ of the nerve. The wide class of eulerian clutters is identified by the following theorem which is proved for building sets in [7]. Recall, a graph is called *chordal* if each its n -cycle for $n > 3$ has a chord (a diagonal in the cycle). The *clique complex* of a graph is a simplicial complex whose faces are cliques, i.e. induced complete subgraphs. The clique complexes coincide with the flag complexes. A simplicial complex is a *flag complex* if all its minimal missing faces are edges.

Theorem 4.5 ([7, Proposition 8.10]) *Let C be an odd clutter such that*

- (i) *the graph $G(C)$ is a chordal graph,*
- (ii) *the nerve $N(C)$ is the clique complex of $G(C)$.*

Then C is an eulerian clutter.

Proof We may suppose that $\bigcup C = V$. For $\{V\}$ we have $\zeta^{-1}(\{V\}) = (-1)^{|V|} + 1$ by deletion–contraction, so $\{V\}$ is eulerian if and only if $|V|$ is odd. Suppose the statement is true for all clutters with $n \geq 1$ edges and let $C = \{e_0, e_1, \dots, e_n\}$ be a clutter with $n + 1$ edges. Since the property of being a clique complex of a chordal graph is hereditary on full subcomplexes it is sufficient to prove that $\zeta^{-1}(C) = 0$. But

$$\zeta^{-1}(C) = -\zeta^{-1}(C/e_0),$$

by deletion–contraction since $\zeta^{-1}(C - e_0) = 0$ by induction.

Consider a map $\pi: C - e_0 \rightarrow C/e_0$ defined by $\pi(e) = e/e_0$. Let $S = \{e \in C : e \cap e_0 \neq \emptyset\}$. Then $N(S) = \text{star}_{N(C)}(\{e_0\})$ and $N(S - e_0) = \text{link}_{N(C)}(\{e_0\})$ are the star and the link of the vertex e_0 in $N(C)$. The complex $N(C/e_0)$ is obtained from $N(C)$ by replacing $N(S)$ with the simplex $\sigma = N(S/e_0)$

$$N(C/e_0) = N(C - e_0) \cup \sigma = N(C) \setminus \text{star}_{N(C)}(\{e_0\}) \cup \sigma.$$

The map π induces a simplicial embedding $N(C - e_0) \hookrightarrow N(C/e_0)$. The proof follows from the fact that C/e_0 satisfies conditions of theorem.

First we show that C/e_0 is an odd clutter. The edges of C/e_0 are of two kinds $C/e_0 = C \setminus S \sqcup S/e_0$. For a face $C' \in N(C/e_0)$ we have

$$|\bigcap C'| = \begin{cases} |\bigcap \pi^{-1}(C')| & \text{if } C' \not\subseteq S/e_0, \\ |\bigcap \pi^{-1}(C')| - |\bigcap \pi^{-1}(C') \cap e_0| + 1 & \text{if } C' \subseteq S/e_0, \end{cases}$$

which is odd since C is an odd clutter.

Suppose there is a cycle L of the graph $G(C/e_0)$ with no chords. Then L must contain at least two vertices from S/e_0 since in the opposite case $\pi^{-1}(L)$ would be a cycle without chords in a chordal graph $G(C - e_0)$. On the other hand, the number of vertices from S/e_0 in the cycle L is at most two since otherwise L would have a chord because all such vertices are connected in $N(C/e_0)$. Let $L \cap S/e_0 = \{e_1/e_0, e_2/e_0\}$. We conclude that either $\pi^{-1}(L)$ or $\pi^{-1}(L) \cup \{e_0\}$, depending on whether e_1 and e_2 intersect, will be a cycle in $G(C)$ without chords.

Let $C' \subset C/e_0$ be a clique in the graph $G(C/e_0)$ such that $\bigcap C' = \emptyset$. Obviously $\pi^{-1}(C')$ is not contained in S . If $\pi^{-1}(C') \cap S = \emptyset$ or $\bigcap (\pi^{-1}(C') \cap S) \neq \emptyset$ then $\pi^{-1}(C')$ is a clique in $G(C)$ which contradicts condition (ii) for C . So we can find $e_1, e_2 \in \pi^{-1}(C') \cap S$ with $e_1 \cap e_2 = \emptyset$. For an arbitrary $e_3 \in \pi^{-1}(C') \setminus S$ the family $\{e_0, e_1, e_2, e_3\}$ is a chordless cycle in $G(C)$ which contradicts (i). $\square$

The converse of Theorem 4.5 is not true in general. The following example is due to Nikolai Erokhovets.

Example 4.6 Consider the clutter $C = \{\{1, 2, 3, 6, 7\}, \{3, 4, 5, 7, 8\}, \{1, 2, 3, 4, 5\}\}$ or, more generally, any clutter $C = \{e_1, e_2, e_3\}$ such that $|e_i|$ -odd for all i , $e_3 \subsetneq e_1 \cup e_2$, $|e_1 \cap e_2|$ -even, $|e_1 \cap e_2 \cap e_3|$, $|e_1 \cap e_3|$, $|e_2 \cap e_3|$ -odd. C is not odd since $|e_1 \cap e_2|$ -even. For any proper set $I \subsetneq V$ we have $C|_I$ is either discrete, or it consists of one odd

set, or it consists of two odd sets with odd intersection. So C is eulerian if and only if $\zeta^{-1}(C) = 0$. We have

$$\zeta^{-1}(C) = \zeta^{-1}(C - e_3) - \zeta^{-1}(C/e_3) = 0,$$

since both deletion and contraction is a clutter of two odd sets with even intersection.

The following examples show that even conditions (i) and (ii) of Theorem 4.5 do not need to be satisfied by eulerian clutters.

Example 4.7 Let $C_1 = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{1, 2, 3, 4, 5\}$, $e_2 = \{5, 6, 7, 8, 9\}$, $e_3 = \{3, 4, 5, 6, 7\}$ and $e_4 = \{1, 9, 10\}$ be a clutter on 10 vertices. The nerve $N(C_1)$ is not a clique complex since $\{e_1, e_2, e_4\}$ is its missing face. By simple inspection we see that C_1 is eulerian if and only if $\zeta^{-1}(C_1) = 0$, which is satisfied, since

$$\zeta^{-1}(C_1) = \zeta^{-1}(C - e_3) - \zeta^{-1}(C/e_3) = 0.$$

Similarly, let $C_2 = \{e_1, e_2, e_3, e_5, e_6\}$, where $e_5 = \{1, 10, 11\}$ and $e_6 = \{9, 11, 12\}$, be a clutter on 12 vertices. Then it is easy to see that C_2 is eulerian, but $G(C_2)$ is not a chordal graph.

Nevertheless the converse of Theorem 4.5 holds for eulerian clutters with an additional condition.

Theorem 4.8 *Let C be an eulerian clutter such that*

$$e \not\subseteq \bigcup (C - e), \quad e \in C. \quad (\star)$$

Then C is odd and the nerve $N(C)$ is the clique complex of the chordal graph $G(C)$.

Proof Note that $(\star)$ is equivalent to the condition that any subclutter $C' \subsetneq C$ is a proper restriction of C , i.e. for the sets of vertices there holds $V(C') \subsetneq V(C)$.

Let $C' \subset C$ be a minimal face of $N(C)$ such that $\bigcap C'$ is even. Then by induction and deletion–contraction it follows that $\zeta^{-1}(C') = (-1)^{|C'| - 1} 2$. We have that either $C' = C$ or $C' \subsetneq C$. In both cases C is not eulerian.

Case 1: Suppose $C' = \{e_1, \dots, e_n\}$ is a minimal non-face of $N(C)$ and $n > 2$. Then $C' - e_1$ satisfies conditions of Theorem 4.5, so $\zeta^{-1}(C') = -\zeta^{-1}(C'/e_1)$ by deletion–contraction. But $N(C'/e_1)$ is a minimal face of $N(C/e_1)$ such that $\bigcap (C'/e_1)$ is even, which implies $\zeta^{-1}(C'/e_1) = (-1)^{n-2} 2$.

Case 2: Let $C' = \{e_1, \dots, e_n\} \subset C$ be an odd clutter such that $G(C')$ is the n -cycle with no chords where $n > 3$. We find by induction and deletion–contraction that $\zeta^{-1}(C') = (-1)^{n-1} 2$.

In both cases it follows from $(\star)$ that either $C' = C$ or C' is a proper restriction of C , which contradicts the condition that C is eulerian. $\square$

Remark 4.9 The restriction $(\star)$ from Theorem 4.8 is quite strong. For a clutter C it is equivalent to the fact that the Taylor resolution of the Stanley–Reisner ring of the independence complex of C is minimal [1, Lemma 4.14].

Recall that the power sum basis of symmetric functions is given by all p_λ , where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ for a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\}$ and $p_j = x_1^j + x_2^j + x_3^j + \cdots$ for a positive integer j .

Let C be a clutter on the vertex set V . For a subclutter $S \subset C$, let $\lambda(S) \vdash |V|$ be the partition of the number of vertices $|V|$ whose parts are equal to the numbers of vertices of connected components of S . The chromatic symmetric function Ψ of a clutter C has the following expansion in the power sum basis of symmetric functions [12, Theorem 3.5]:

$$\Psi(C) = \sum_{S \subset C} (-1)^{|S|} p_{\lambda(S)}.$$

Remark 4.10 In the paper by Stanley [12] it is only mentioned that the proof of this equality for hypergraphs is similar to that for graphs which is given in [11]. By repeating arguments for graphs we checked that the formula is indeed correct.

Proposition 4.11 *The canonical morphism $\Psi: \mathcal{H}(C) \rightarrow \text{Sym}$ is an epimorphism.*

Proof Let $D_{[n]} = \{[n]\}$ be a clutter on $[n] = \{1, \dots, n\}$. For a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k\}$ set $D_\lambda = D_{[\lambda_1]} \sqcup D_{[\lambda_2]} \sqcup \cdots \sqcup D_{[\lambda_k]}$. We have

$$p_\lambda = \prod_{i=1}^k p_{\lambda_i} = \prod_{i=1}^j (\Psi(D_{\lambda_i}) - \Psi(D_{[\lambda_i]})) \cdot \Psi(D_{k-j}),$$

where $j = \max\{i : \lambda_i > 1\}$. On the other hand,

$$\Psi(D_\lambda) = \prod_{i=1}^k \Psi(D_{[\lambda_i]}) = \prod_{i=1}^j (p_{1^{\lambda_i}} - p_{\lambda_i}) \cdot p_{1^{k-j}},$$

which shows that the transition matrix of bases $\{p_\lambda\}$ and $\{\Psi(D_\lambda)\}$ is an involution. $\square$

If a clutter C satisfies the generalized Dehn–Sommerville relations, i.e. $C \in S_-(\mathcal{H}(C), \zeta_C)$ then $\Psi(C) \in S_-(\text{Sym}, \zeta_S)$. We say that a partition λ is odd if all its parts are odd. The odd subalgebra of symmetric functions is linearly spanned by the set of all odd power sum symmetric functions [3, Proposition 7.1], $S_-(\text{Sym}, \zeta_S) = \text{span}\{p_\lambda : \lambda \text{ is odd}\}$.

Theorem 4.12 *The clutter C is eulerian if the partition $\lambda(S)$ is odd for any subclutter $S \subset C$.*

Proof Particular, we have that $\lambda(C')$ is an odd partition for any simplex $C' \in N(C)$ which is equivalent by the inclusion–exclusion property to the fact that the clutter C is odd. We show that C satisfies conditions of Theorem 4.5:

- (i) Suppose there is a nonchordal cycle $e_1, \dots, e_k$, $k > 3$, in the intersection graph $G(C)$. As $\lambda(\{e_1, \dots, e_k\})$ is an odd partition, we conclude that $|e_1 \cup \cdots \cup e_k|$ is

odd. On the other hand,

$$|e_1 \cup \dots \cup e_k| = |e_1| + \dots + |e_k| - |e_1 \cap e_2| - |e_2 \cap e_3| - \dots - |e_k \cap e_1|,$$

which is even.

(ii) Suppose that $\{e_1, \dots, e_k\}, k > 2$, is a minimal non-face of the nerve $N(C)$. Then

$$0 = |e_1 \cap \dots \cap e_k| = |e_1 \cup \dots \cup e_k| - \sum_{j=1}^{k-1} (-1)^{j-1} \sum_{i_1 < \dots < i_j} |e_{i_1} \cap \dots \cap e_{i_j}|.$$

But any of $2^k - 1$ summands on the righthand side are odd since the clutter C is odd.

Hence the nerve $N(C)$ is a flag complex, i.e. it is the clique complex of the graph $G(C)$. $\square$

Remark 4.13 The eulerian clutters generate the proper subalgebra of the odd subalgebra of clutters. There are clutters which satisfy the generalized Dehn–Sommerville relations, but which are not eulerian. The simple example is given by the clutter on five vertices $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4, 5\}\}$. The clutter C and the eulerian clutter $C' = \{\{1, 2, 3\}, \{3, 4, 5\}\}$ have the same chromatic symmetric function

$$\Psi(C) = \Psi(C') = p_{1,1,1,1,1} - 2p_{3,1,1} + p_5.$$

5 Hopf algebra of simplicial complexes

An abstract simplicial complex K on the vertex set $V = V(K)$ is a lower ideal in the boolean poset $\mathcal{P}(V)$ of all subsets of V . An element $\sigma \in K$ is called a *face* of K . We require that $\{v\} \in K$ for all vertices $v \in V$. The induced subcomplex $K|_I$ of K on the vertex set $I \subset V(K)$ is a family $K|_I = \{\sigma \subset I : \sigma \in K\}$. The join of simplicial complexes K_1 and K_2 is the complex $K_1 * K_2 = \{\sigma \cup \tau \subset V(K_1) \sqcup V(K_2) : \sigma \in K_1, \tau \in K_2\}$. A simplicial isomorphism is a bijection $f: V(K_1) \rightarrow V(K_2)$ such that $\sigma \in K_1$ if and only if $f(\sigma) \in K_2$.

The *independence complex* $\text{Ind}(C)$ of a clutter C is a simplicial complex $\text{Ind}(C) = \{I \subset V(C) : C|_I \text{ is discrete}\}$. A simplicial complex K determines the clutter $C(K)$ of minimal non-faces of K . These assignments are mutually inverse $\text{Ind}(C(K)) = K$, $C(\text{Ind}(C)) = C$ and satisfy the following properties:

$$\text{Ind}(C_1 \sqcup C_2) = \text{Ind}(C_1) * \text{Ind}(C_2) \quad \text{and} \quad \text{Ind}(C)|_I = \text{Ind}(C|_I), \quad (8)$$

for all clutters C_1 and C_2 and for any $I \subset V(C)$.

Let $\mathcal{H}(\mathbf{K}) = \bigoplus_{n \geq 0} \mathcal{H}_n(\mathbf{K})$ be the graded $\mathbf{k}$ -vector space linearly spanned by the set $\mathbf{K}$ of all isomorphism classes of finite simplicial complexes. The grading is given by the number of vertices. Let $\text{Ind}: \mathcal{H}(C) \rightarrow \mathcal{H}(\mathbf{K})$ be the linear extension of the map that associates the independence complex $\text{Ind}(C)$ to a clutter C . By properties (8) the

space $\mathcal{H}(\mathbf{K})$ is equipped with the Hopf algebra structure with respect to which the map Ind is a Hopf algebra isomorphism. The product on $\mathcal{H}(\mathbf{K})$ is defined by the join $K_1 * K_2$ and the coproduct by induced subcomplexes $\Delta(K) = \sum_{I \subset V(K)} K|_I \otimes K|_{V(K) \setminus I}$. The space $\mathcal{H}(\mathbf{K})$ is a graded connected commutative and cocommutative Hopf algebra. The unit element is the simplicial complex $K_\emptyset$ on the empty set of vertices and the counit is

$$\epsilon(K) = \begin{cases} 1, & K = K_\emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The antipode is determined on $K \neq K_\emptyset$ by (2)

$$S(K) = \sum_{k \geq 1} (-1)^k \sum_{I_1 \sqcup \dots \sqcup I_k = V(K)} K|_{I_1} * \dots * K|_{I_k}, \quad (9)$$

where the sum goes over all ordered decompositions of the vertex set. The map Ind becomes an isomorphism of combinatorial Hopf algebras if we take

$$\zeta_{\mathbf{K}}(K) = \begin{cases} 1, & K \text{ is a simplex,} \\ 0, & \text{otherwise.} \end{cases}$$

A *partition* of the simplicial complex K is the ordered decomposition $(I_1, \dots, I_k)$ of the vertex set $V(K)$ such that all induced subcomplexes $K|_{I_j}$ are simplexes. For a composition $\alpha = (a_1, \dots, a_k) \models n$ the coefficient $(\zeta_{\mathbf{K}})_\alpha(K)$ is the number of partitions of K whose parts have the prescribed numbers of elements $|I_j| = a_j$, $1 \leq j \leq k$. The canonical morphism $\Psi: (\mathcal{H}(\mathbf{K}), \zeta_{\mathbf{K}}) \rightarrow (\text{QSym}, \zeta_Q)$, given by (1) with

$$\Psi(K) = \sum_{\alpha \models n} (\zeta_{\mathbf{K}})_\alpha(K) M_\alpha, \quad K \in \mathcal{H}_n(\mathbf{K}),$$

assigns the generating function of all partition functions to a simplicial complex K . A *partition function* of a simplicial complex K is the map $f: V(K) \rightarrow \mathbb{N}$ such that $f^{-1}(\{i\})$ is a face of K for all $i \in f(V(K))$. Then $\Psi(K) = \sum_f \prod_{v \in V(K)} x_{f(v)}$, where the sum is over all partition functions.

The principal specialization $\chi(K, m) = \text{ps}(\Psi(K))(m)$ defines the *partition polynomial* of a simplicial complex K which counts the number of partition functions $f: V(K) \rightarrow [m]$. The value $\chi(K, -1)$ is of special interest and by (3) and the antipode formula (9) we have

$$\chi(K, -1) = \zeta_{\mathbf{K}}^{-1}(K) = \zeta_{\mathbf{K}} \circ S(K) = \sum_{\alpha \models n} (-1)^{k(\alpha)} (\zeta_{\mathbf{K}})_\alpha(K).$$

The value of the Euler character $\chi_K = \bar{\zeta}_K \zeta_K$ at a simplicial complex K is given by

$$\chi_K(K) = \sum_{(I, I^c) \text{ partition}} (-1)^{|I|}.$$

Definition 5.1 A simplicial complex K is *eulerian* if $\chi_K(K|_I) = \epsilon(K|_I)$ for all $I \subset V(K)$.

As in the case of hypergraphs we have an equivalent characterization.

Lemma 5.2 A simplicial complex $K \neq K_\emptyset$ is eulerian if and only if either $K|_I$ is a simplex or $\zeta_K^{-1}(K|_I) = 0$ for all $\emptyset \neq I \subset V(K)$.

The eulerian subalgebra $\mathcal{E}(\mathbf{K})$ is generated by all isomorphism classes of eulerian simplicial complexes. It is a subalgebra $\mathcal{E}(\mathbf{K}) \subset S_-(\mathcal{H}(\mathbf{K}), \zeta_K)$ of the odd algebra of CHA of simplicial complexes. Consequently relations (5) are satisfied by coefficients $(\zeta_K)_\alpha$, $\alpha \models |V(K)|$ of the eulerian simplicial complexes $K \in \mathcal{E}(\mathbf{K})$.

The isomorphism Ind gives the complete characterization of eulerian simplicial complexes.

Theorem 5.3 A simplicial complex K is eulerian, $K \in \mathcal{E}(\mathbf{K})$, if and only if it is the independence complex $K = \text{Ind}(C)$ of an eulerian clutter $C \in \mathcal{E}(\mathbf{C})$ or equivalently if and only if its clutter $C(K)$ of minimal non-faces is eulerian $C(K) \in \mathcal{E}(\mathbf{C})$.

Example 5.4 1. Suppose that $\{i, j\}$ is a minimal non-face of a simplicial complex K . Then $\chi(K|_{\{i, j\}}) = -2$, so K is not eulerian. More generally the necessary condition for K to be eulerian is that its minimal non-faces have odd cardinalities.
2. Let $\Delta[n]$ be a simplex on n vertices. The boundary complex $\partial\Delta[n]$ is eulerian if and only if n is odd.
3. Let C be a clutter such that the nerve complex $N(C)$ is a tree. Then the independence complex $\text{Ind}(C)$ is eulerian if and only if C is odd. Indeed if C is odd then Theorem 4.5 applies. For the converse it is sufficient to note that each $e \in C$ is odd as well as $e \cap e'$ with $e \cap e' \neq \emptyset$.

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