Rational connectedness and order of non-degenerate meromorphic maps from $\mathbb{C}^n$

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Abstract We show that an n -dimensional compact Kähler manifold X admitting a non-degenerate meromorphic map $f: \mathbb{C}^n \rightarrow X$ of order $\rho_f < 2$ is rationally connected.

Keywords Rational connectedness · Order of a non-degenerate meromorphic map

Mathematics Subject Classification 32A22 · 14E08 · 32H04 · 14J40 · 32Q45

1 Introduction

The purpose of this paper is the following result.

Theorem 1.1 *Let X be a compact Kähler manifold of fixed dimension n . Let $f: \mathbb{C}^n \dashrightarrow X$ be a non-degenerate meromorphic map of order $\rho_f < 2$ (see the next section for the definition of ρ_f). Then X is rationally connected, hence projective.*

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(Here a meromorphic map $f: \mathbb{C}^n \dashrightarrow X$ is called *non-degenerate* if there is a point v such that f is defined in v and $DF_v: T_v\mathbb{C}^n \rightarrow T_{f(v)}X$ is surjective.)

This result belongs to a series of similar statements relating the existence and growth of maps from $\mathbb{C}^n$ to algebro-geometric properties of the target space X . These statements are better expressed by introducing the following invariant $\rho(X)$, which suggests many questions, some of which are raised in the last section.

Definition 1.2 Let X be an n -dimension connected compact complex manifold. Define $\rho(X) = \inf\{\rho_f : f: \mathbb{C}^m \dashrightarrow X, m \geq n, f \text{ non-degenerate}\}$. It is understood that $\rho(X) \in [0, +\infty]$, and that $\rho_f = +\infty$ if there exists no non-degenerate meromorphic map $f: \mathbb{C}^n \dashrightarrow X$.

The invariant ρ is easily seen to be bimeromorphic, preserved by finite étale covers, and *increasing* (i.e., $\rho(X) \geq \rho(Y)$ if there exists a dominant meromorphic map $g: X \rightarrow Y$). It is obvious that $\rho(X) = 0$ if X is unirational, because $\rho_f = 0$ for every rational map f .

Kobayashi and Ochiai proved that the existence of a non-degenerate meromorphic map from $\mathbb{C}^n$ to a projective manifold X implies that X is not of general type [9]. Thus $\rho(X) = +\infty$ if X is of general type. It is proved in [4], more generally, that X is ‘special’ if there exists a non-degenerate meromorphic map $f: \mathbb{C}^n \dashrightarrow X$. In particular, X must be special if $\rho(X) < +\infty$.

If X is Kähler and K_X is pseudo-effective of numerical dimension $\nu \in \{0, 1, \dots, n\}$, then $\rho(X) \geq 2(1 - \nu/n)^{-1} \geq 2$ [6]. This implies the previous result of Kobayashi–Ochiai (since X is of general type if and only if $\nu = n$). Using [2], it also implies that if X is projective, then X is uniruled if $\rho(X) < 2$.

If $h^0(X, \text{Sym}^k(\Omega_X^p)) \neq 0$, for some $k, p > 0$, then $\rho(X) \geq 2$ [11]. The Kähler condition is not required here. If, in addition, X is assumed to be a Kähler surface, then the condition $h^0(X, \text{Sym}^k(\Omega_X^p)) = 0$ for all p, k implies that X is rational. Thus, if X is a Kähler surface and if $\rho(X) < 2$, then X must be rational [11]. On the other hand, some Hopf (thus non-rational, non-Kähler) surfaces have $\rho(X) = 1$ [11].

Mumford conjectured that for a projective manifold (of arbitrary dimension) the condition “ $h^0(X, \text{Sym}^k(\Omega_X^p)) \neq 0$ for some $k, p > 0$ ” is equivalent to X being not rationally connected. However, it is still unknown whether this conjecture holds.

The present result generalizes these two results from [6, 11], avoiding the deep methods of [6]. The estimate $\rho_f < 2$ is optimal, since for an Abelian variety A we have $\rho_A = 2$ for the universal covering map $\tau: \mathbb{C}^g \rightarrow A$.

All these results provide lower bounds for $\rho(X)$ deduced from the geometry of X . Producing upper bounds for $\rho(X)$ (i.e., the existence of non-degenerate f) turns out to be a completely open topic, except in the trivial case of X unirational or a torus.

The case of rationally connected vs unirational manifolds when $n \geq 3$ is of great interest. For example, what is $\rho(X)$ if X is a ‘general’ smooth quartic in $\mathbb{P}^4$? If $\rho(X) > 0$, then X is not unirational. More generally, are there projective manifolds X such that $\rho(X) \in (0, 2)$? Such a manifold X would necessarily be rationally connected, but not unirational. See the last section for some more questions.

2 Characteristic function and order of a non-degenerate meromorphic map

We start with some preparations. Let X be a compact complex manifold and let $f: \mathbb{C}^n \dashrightarrow X$ be a meromorphic map. The map f is said to be (differentiably) *non-degenerate* if there is a point $p \in \mathbb{C}^n$ such that f is holomorphic at p and $(Df)_p: T_p \mathbb{C}^n \rightarrow T_{f(p)} X$ is surjective.

Let $\alpha = dd^c \|z\|^2$ on $\mathbb{C}^n$, and let ω be a positive $(1, 1)$ -form on X . The characteristic function of f (with respect to ω) is defined as

$$T_f(r; \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} (f^* \omega) \wedge \alpha^{n-1}.$$

Here $B_t = \{z \in \mathbb{C}^n : \|z\| < t\}$. Observe that the integral over B_t is well defined even if f is only meromorphic, not necessarily holomorphic.

If ω and $\tilde{\omega}$ are any two positive $(1, 1)$ -forms on X , then (by the compactness of X) there are constants $C_1, C_2 > 0$ such that

$$C_1 \omega < \tilde{\omega} < C_2 \omega \tag{1}$$

and, consequently,

$$C_1 T_f(r, \omega) < T_f(r, \tilde{\omega}) < C_2 T_f(r, \omega), \quad r > 1.$$

The characteristic function $T_f(r, \omega)$ is a very fine tool for measuring the growth of a map. A coarser invariant is the *order* ρ_f which is defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T_f(r, \omega)}{\log r},$$

where ω is a positive $(1, 1)$ -form on X . Due to inequalities (1) this number $\rho_f \in \mathbb{R} \cup \{+\infty\}$ is independent of the choice of the positive $(1, 1)$ -form ω used in the definition of $T_f(r, \omega)$.

The well-known Crofton's formula (see e.g. [8]) implies that $T_f(r)$ equals the average of $T_{f|L}(r)$ taken over all complex lines $L \subset \mathbb{C}^n$. This permits to extend many properties of characteristic functions from the case of entire curves to the higher-dimensional domains. For instance, let $\tau: X' \rightarrow X$ be a bimeromorphic holomorphic map between compact complex manifolds. Then $\tilde{f} = \tau^{-1} \circ f: \mathbb{C}^n \rightarrow X'$ is again a non-degenerate meromorphic map, and $\rho_f = \rho_{\tilde{f}}$.

Now let $\alpha: X \rightarrow Y$ be a dominant holomorphic map. Then it follows directly from the definitions that $T_{\alpha \circ f}(r) \leq T_f(r)$ (if $\omega_X \geq \alpha^*(\omega_Y)$) and, consequently, $\rho_{\alpha \circ f} \leq \rho_f$.

The order is easily seen to behave nicely with respect to products. Let $X = X_1 \times X_2$, ω_i be positive $(1, 1)$ -forms in X_i and let $f_i: \mathbb{C}^{n_i} \rightarrow X_i$ be non-degenerate meromorphic maps. Define $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$, $n = n_1 + n_2$ and $f: \mathbb{C}^n \rightarrow X$ as $f(v_1, v_2) = (f_1(v_1), f_2(v_2))$. Then $T_f(r) = T_{f_1}(r) + T_{f_2}(r)$. This implies $\rho_f = \max\{\rho_{f_1}, \rho_{f_2}\}$, because for any $t_1, t_2 > 0$ we have

$$\max_i \log t_i \leq \log(t_1 + t_2) \leq \log(2 \max_i t_i) = \log 2 + \log \max_i t_i.$$

3 Rational connectedness

As usual, a compact complex manifold X (usually Kähler) is called *rationally connected* (RC) if every two points can be linked by a (possibly singular) irreducible rational curve. (Equivalently in the Kähler case: can be linked by a chain of rational curves). Unirational manifolds are RC. There are unirational threefolds which are not rational, e.g. smooth cubic hypersurfaces in $\mathbb{P}_4$. However, it is not known whether there exist RC manifolds which are not unirational, although it is expected that these should exist (the case of ‘general’ quartics in $\mathbb{P}_4$ being one of the first open cases).

Rationally connected compact Kähler manifolds are projective (since $h^{2,0} = 0$, using Kodaira’s criterion).

Let X be a compact connected Kähler manifold. Then there exists an “almost holomorphic” rational dominant map $\rho: X \dashrightarrow Y$ (called the *RC-reduction*, or *rational quotient* [3], or the *MRC-fibration* [10] if X is projective), such that the fibers are RC, and maximal with this property.

When X is projective, it is known (by [7]) that the base Y is not uniruled. In fact, [7] shows that if $f: X \dashrightarrow Y$ is a surjective meromorphic map with fibres and base Y which are both RC, then X is RC if it is compact Kähler (remark first that $h^{2,0}(X) = 0$, and that X is thus projective). The base Y of the RC-reduction $\rho: X \dashrightarrow Y$ is not uniruled also when X is compact Kähler. Let indeed $r: Y \dashrightarrow Z$ be the RC-reduction of Y . The fibres of $r \circ \rho: X \dashrightarrow Z$ are thus RC, and so $Y = Z$, which means that Y is not uniruled.

Due to [2] a projective manifold is uniruled if and only if K_X is not pseudo-effective. Based on this, another recent criterion is the following (see [5]): *Let X be a compact Kähler manifold. Then X is rationally connected if and only if there is no pseudo-effective invertible subsheaf $F \subset \Omega_X^p$ (for some $p \in \mathbb{N}$).*

The proof consists in observing that X is not RC precisely if its RC-reduction $\rho: X \dashrightarrow Y$ has $\dim Y = p > 0$. Define then $F = \rho^*(K_Y)$, which is pseudo-effective since Y is not uniruled.

It is conjectured that a compact Kähler (or equivalently, projective) manifold X is RC if (and only if) there is no $\mathbb{Q}$ -effective (instead of pseudo-effective) invertible subsheaf $F \subset \Omega_X^p$ (for some $p \in \mathbb{N}$). By means of the RC-reduction as above, this conjecture is equivalent to the ‘non-vanishing conjecture’, claiming that if K_X is pseudo-effective, it is $\mathbb{Q}$ -effective.

4 Pseudo-effective line bundles

A singular hermitian metric h on a complex line bundle is given in the form $|\cdot|_h = e^{-\phi} |\cdot|_s$, where s is the standard metric for some local holomorphic trivialization and ϕ is a L_{loc}^1 -function. The L_{loc}^1 -condition ensures that the curvature $\Theta_h = -dd^c \log e^{-\phi} = dd^c \phi$ makes sense (in the sense of currents) and represents the Chern class of the line bundle.

A line bundle L on a compact complex manifold is called *pseudo-effective* if there is a singular hermitian metric h with semipositive curvature $\Theta_h \geq 0$. This condition means that the metric is locally given via a weight function $e^{-\phi}$ with ϕ being plurisub-

harmonic. Now let F^* be the dual line bundle equipped with the *dual metric* h^* , i.e., the metric with weight factor e^ϕ . If F is pseudo-effective and s is a holomorphic section in F^* , plurisubharmonicity of ϕ implies that $\log \|s\|_{h^*}$ is plurisubharmonic.

5 Measuring the derivative

Let V, W be complex vector spaces equipped with hermitian inner products. The norm $\|F\|$ of a complex linear map $F: V \rightarrow W$ is defined by $\|F\|^2 = \text{trace}(F^\dagger \circ F)$ where $F^\dagger$ denotes the adjoint of F . If A is the matrix describing F with respect to orthonormal bases on V and W , then

$$\|F\|^2 = \sum_{i,j} |A_{ij}|^2.$$

The inner products on V and W naturally induce inner products on the dual vector spaces $V^*,$ respectively, W^* . Observe that, given a linear map $F: V \rightarrow W$, the dual map $F^*: W^* \rightarrow V^*$ fulfills $\|F^*\| = \|F\|$.

Let $k \in \mathbb{N}$. Then F induces linear maps $\otimes^k F: \otimes^k V \rightarrow \otimes^k W$ and $\Lambda^k: \Lambda^k V \rightarrow \Lambda^k W$. An orthonormal base $e_1, \dots, e_n$ of V induces an orthonormal base of $\otimes^k V$ given by all tensors of the form $e_{i_1} \otimes \dots \otimes e_{i_k}$. Using such bases, it is clear that $\|\otimes^k F\| = \|F\|^k$. As a consequence,

$$\|\Lambda^k F\| \leq \|F\|^k, \quad (2)$$

because $\Lambda^k V \subset \otimes^k V$.

We now compare this norm with the *operator norm* which we denote as $\|F\|_{\text{op}}$. It is defined as

$$\|F\|_{\text{op}} = \sup_{v \neq 0} \frac{\|F(v)\|}{\|v\|}.$$

Clearly, $\|F(e_j)\| \leq \|F\|_{\text{op}}$ if e_j form an orthonormal base. Since $\|F\|^2 = \sum_i \|F(e_i)\|^2$, we have $\|F\|^2 \leq \dim V \cdot \|F\|_{\text{op}}^2$. On the other hand, evidently $\|F(e_j)\|^2 \leq \|F\|^2$ for all j and therefore $\|F(v)\| \leq \|F\| \cdot \|v\|$ for all $v \in V$. Thus $\|F\|_{\text{op}} \leq \|F\|$.

We continue with a local observation. Let U, V be open subsets in $\mathbb{C}^n$, let $\alpha = dd^c \|z\|^2 = \sum_j i/\pi \cdot dz_j \wedge d\bar{z}_j$ be the Kähler form for the euclidean metric and let $f: U \rightarrow V$ be a holomorphic map. Then

$$(f^* \alpha) \wedge \alpha^{n-1} = \frac{1}{n} \|Df\|^2 \alpha^n,$$

where

$$\|Df\|^2 = \text{trace}((Df)^* \circ (Df)) = \sum_{j,k} \left| \frac{\partial f_j}{\partial z_k} \right|^2.$$

Proposition 5.1 *Let $U \subset \mathbb{C}^n$ be an open subset, X an n -dimensional compact complex manifold equipped with a hermitian metric h and a positive $(1, 1)$ -form ω and let $f: U \rightarrow X$ be a holomorphic map. Let $\alpha = dd^c \|z\|^2$ on $\mathbb{C}^n$. Then there is a constant $C > 0$ such that*

$$f^* \omega \wedge \alpha^{n-1} \geq C \|Df\|^2 \alpha^n.$$

Here the norm $\|Df\|$ is calculated in each point $x \in U$ with respect to h on $T_{f(x)}X$ and the euclidean norm on $T_x \mathbb{C}^n$.

Proof We cover X with finitely many open subsets V_k such that each V_k admits an embedding $j_k: V_k \rightarrow \mathbb{C}^n$ and each V_k contains a relatively compact open subset $W_k \subset V_k$ such that $X = \bigcup_k W_k$. Let h_k denote the hermitian metric on V_k induced by the euclidean metric via its embedding in $\mathbb{C}^n$. We choose $C_1 > 0$ such that $h_k \leq C_1 h$ everywhere on each W_k and $C_2 > 0$ such that $\omega \geq C_2 j_k^* \alpha$ on each W_k . Then the claim (with $C = C_1 C_2$) follows from the preceding local observation. $\square$

6 Proof of Theorem 1.1

Assume by contradiction that X is not rationally connected. Then there is a pseudo-effective invertible subsheaf $\mathcal{F} \subset \Omega_X^p$ (for some $p \in \mathbb{N}$). We fix a hermitian metric h on X . The hermitian metric on T_X induces a hermitian metric h_p on Ω_X^p .

The injection of sheaves $\mathcal{F} \hookrightarrow \Omega_X^p$ corresponds to a non-zero vector bundle homomorphism $\xi_0: F \rightarrow \Omega_X^p$, where F is a pseudo-effective line bundle on X . Since F is pseudo-effective, there is a singular hermitian metric g on F such that $\Theta_g \geq 0$, i.e., with positive curvature current.

We claim: *There is a constant $K \in \mathbb{R}^+$ such that*

$$\|\xi_0(v)\|_{h_p} \leq K \|v\|_g \quad (3)$$

for all $v \in F_x$, $x \in X$.

Indeed, locally $\|\cdot\|_g$ may be written as $e^{-u} |\cdot|$ for some plurisubharmonic function u . Since u is upper semicontinuous with values in $[-\infty, +\infty)$, the map $v \mapsto \|v\|_g$ is lower semicontinuous with values in $(0, +\infty]$ for $v \neq 0$. Hence

$$v \mapsto \frac{\|\xi_0(v)\|_{h_p}}{\|v\|_g}$$

is upper semicontinuous with values in $[0, +\infty)$ for $v \neq 0$. Since upper semicontinuous functions with values in $\mathbb{R}$ assume a maximum on compact sets, compactness of X now implies the claim.

Next we consider the pull-back bundles f^*F and $f^*\Omega_X^p$. They are defined on $\mathbb{C}^n \setminus I(f)$, where $I(f)$ denoting the indeterminacy set of the meromorphic map f . We consider these bundles equipped with the pull-backed metrics, which we denote by $\tilde{g} = f^*g$ and $\tilde{h}_p = f^*h_p$.

The meromorphic map $f: \mathbb{C}^n \dashrightarrow X$ and the vector bundle homomorphism $\xi_0: F \rightarrow \Omega_X^p$ induce vector bundle homomorphisms

$$f^*F \xrightarrow{\xi} f^*\Omega_X^p \xrightarrow{\Lambda^p Df^*} \Omega_{\mathbb{C}^n}^p$$

on $\mathbb{C}^n \setminus I(f)$. Let $\beta = (\Lambda^p Df^*) \circ \xi$. Now β is a holomorphic section in

$$\mathrm{Hom}(f^*F, \Omega_{\mathbb{C}^n}^p) \simeq f^*(F^*) \otimes \Omega_{\mathbb{C}^n}^p \simeq \bigoplus \binom{n}{p} f^*(F^*)$$

and f^*F^* is the dual of a pseudo-effective line bundle. Therefore $\log \|\beta\|_{\tilde{g}}$ defines a plurisubharmonic function on $\mathbb{C}^n \setminus I(f)$. Plurisubharmonic functions extend through closed analytic subsets of codimension at least two. Hence $\log \|\beta\|_{\tilde{g}}$ extends to a plurisubharmonic function ζ_0 defined on the whole $\mathbb{C}^n$.

Let $x \in \mathbb{C}^n \setminus I(f)$ and $v \in (f^*F)_x \setminus \{0\}$. Let $w = \xi(v)$. Then $\beta = (\Lambda^p Df^*) \circ \xi$ combined with inequality (3) implies

$$\|(\Lambda^p Df^*)(w)\| \geq \|w\|_{\tilde{h}_p} \frac{1}{K} \exp(\zeta_0(x)).$$

Observe also (see inequality (2)) that

$$\|(Df^*)(w)\|^p \geq \|(\Lambda^p Df^*)(w)\|.$$

Define $\zeta = \exp((\zeta_0 - \log K)/p)$. The plurisubharmonicity of ζ_0 implies the plurisubharmonicity of ζ . Thus we obtain the existence of a plurisubharmonic function ζ on $\mathbb{C}^n$ such that $\|Df\|_h \geq \zeta$.

Using Proposition 5.1, we may deduce that $(f^*\omega) \wedge \alpha^{n-1} \geq \zeta \alpha^n$. It follows that

$$T_f(r; \omega) \geq \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \zeta \alpha^n.$$

Since moving the origin, i.e., replacing f by $f \circ \tau$, where τ denotes a translation, does not affect the order ρ_f , we may assume that $c = \zeta(0, \dots, 0) > 0$. Using the sub-mean value property of plurisubharmonic functions, it follows that

$$T_f(r; \omega) \geq c \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \alpha^n = cv \int_1^r \frac{dt}{t^{2n-1}} t^{2n} = \frac{cv}{2} r^2 + O(1),$$

where v denotes the volume of the unit ball. Therefore,

$$\rho_f = \limsup \frac{\log T_f(r)}{\log r} \geq \limsup \frac{\log r^2}{\log r} = 2.$$

7 Non-Kähler manifolds

Our result is not valid for non-Kähler manifolds. In fact, there are Hopf surfaces X admitting a non-degenerate holomorphic map $f: \mathbb{C}^n \rightarrow X$ of order $\rho_f = 1$ (see [11]). Of course, these Hopf surfaces are non-Kähler and not rationally connected; they do not contain any rational curve at all. More precisely, in [11] the following is proved

Theorem 7.1 *Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and let $\sim$ denote the equivalence relation on $\mathbb{C}^2 \setminus \{(0, 0)\}$ given by*

$$v \sim w \iff \text{there exists } k \in \mathbb{Z} \text{ such that } \lambda^k v = w.$$

Let $X = \mathbb{C}^2 \setminus \{(0, 0)\} / \sim$ and let $f: \mathbb{C}^2 \rightarrow X$ be the map induced by $(z, w) \mapsto (z, 1 + zw)$. Then $\rho_f = 1$.

This result has been generalized by Amemiya [1] to the class of Hopf surfaces defined by an equivalence $(z, w) \sim (\lambda^k z, \mu^k w)$, $k \in \mathbb{Z}$, where λ may be different from μ (but $|\lambda|, |\mu| > 1$).

8 Questions

The following questions are not expected to have necessarily positive answers. Let X be an n -dimensional compact Kähler manifold, and let $f: \mathbb{C}^m \dashrightarrow X$ be a meromorphic non-degenerate map.

1. If there exists such $f: \mathbb{C}^m \dashrightarrow X$, can it be chosen so that $\rho_f < +\infty$? In other words, if there exists an f as above, is $\rho(X) < +\infty$?
2. If $\rho(X) < +\infty$, does there exist some $f: \mathbb{C}^m \dashrightarrow X$ with $\rho_f = \rho(X)$?
3. If X is rationally connected, does there exist a non-degenerate meromorphic map $f: \mathbb{C}^n \dashrightarrow X$?
4. If X is RC, and if there exists a non-degenerate $f: \mathbb{C}^m \dashrightarrow X$, is X unirational (i.e., can f be chosen algebraic)? A positive answer would imply that there exists no X with $\rho(X) \in (0, 2)$.
5. Is X unirational if $\rho_f = 0$? Is X unirational if $\rho(X) = 0$? (It should be remarked that $\rho_f = 0$ for every algebraic map, but the condition $\rho_f = 0$ is substantially weaker than algebraicity, as seen by appropriate power series in one variable.)
6. Is the estimate $\rho(X) \geq 2(1 - \nu/n)^{-1}$ in [6] optimal if K_X is pseudo-effective with $\nu(X) = \nu$? In other words, does there exist X_n with $\nu(X) = \nu$ (or better, with $\kappa(X) = \nu$) and with $\rho(X) \geq 2(1 - \nu/n)^{-1}$ for any $n > 0$ and $\nu \in \{0, 1, \dots, n\}$?

The questions 3 and 4 were raised for $n = 3$ in [4, Question 9.5].

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