On Baire classification of mappings with values in connected spaces

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Abstract We generalize the Lebesgue–Hausdorff theorem on the characterization of Baire-one functions for σ -strongly functionally discrete mappings defined on arbitrary topological spaces.

Keywords σ -strongly functionally discrete mapping · Baire-one mapping · Lebesgue–Hausdorff theorem

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1 Introduction

A subset A of a topological space X is *functionally* F_σ -set (G_δ -set) if A is a union (intersection) of a sequence of functionally closed (functionally open) subsets of X .

Let X and Y be topological spaces and $f: X \rightarrow Y$ a mapping. We say that f belongs to

- the *first Baire class*, $f \in B_1(X, Y)$, if f is a pointwise limit of a sequence of continuous mappings from X to Y ;
- the *first Lebesgue class*, $f \in H_1(X, Y)$, if $f^{-1}(V)$ is an F_σ -set in X for any open subset V of Y ;
- the *first functional Lebesgue class*, $f \in K_1(X, Y)$, if $f^{-1}(V)$ is a functionally F_σ -set in X for any open subset V of Y ;

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- the first weak functional Lebesgue class, $f \in K_1^w(X, Y)$, if $f^{-1}(V)$ is a functionally F_σ -set in X for any functionally open subset V of Y .

The classical Lebesgue–Hausdorff theorem [6, 11] tells us that the classes $B_1(X, Y)$ and $H_1(X, Y)$ coincide if X is a metrizable space and $Y = [0, 1]^\omega$. This theorem was generalized by many mathematicians (see [3, 5, 7, 10, 13–15, 17] and the literature therein). For classification of mappings with values in non-separable metric spaces, Hansell in [4] introduced a notion of σ -discrete mapping. Namely, a mapping $f: X \rightarrow Y$ is σ -discrete if there exists a family $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ of subsets of X such that every family $\mathcal{B}_n$ is discrete in X and the preimage $f^{-1}(V)$ of any open set $V \subseteq Y$ is a union of sets from $\mathcal{B}$. The class of all σ -discrete mappings from X to Y is denoted by $\Sigma(X, Y)$. Among others we note the following generalizations of the Lebesgue–Hausdorff theorem.

Theorem A (Hansell [5]) *Let X be a collectionwise normal space and Y be a closed convex subset of a Banach space. Then $B_1(X, Y) = H_1(X, Y) \cap \Sigma(X, Y)$.*

Theorem B (Fosgerau [3]) *Let Y be a complete metric space. Then the following conditions are equivalent:*

- (i) Y is connected and locally connected;
- (ii) $H_1([0, 1], Y) = B_1([0, 1], Y)$;
- (iii) $H_1(X, Y) \cap \Sigma(X, Y) = B_1(X, Y)$ for any metric space X .

In order to weaken the condition on X in Theorem B, Veselý [17] considered strongly σ -discrete mappings. A mapping $f: X \rightarrow Y$ is *strongly σ -discrete* if there exists a family $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n$ of subsets of X such that for every family $\mathcal{B}_n = (B_i : i \in I_n)$ there exists a discrete family $(U_i : i \in I_n)$ of open sets in X with $\overline{B_i} \subseteq U_i$ for all $i \in I_n$ and, moreover, the preimage $f^{-1}(V)$ of any open set V in Y is a union of sets from $\mathcal{B}$. Veselý [17] denoted the collection of all such mappings by $\Sigma^*(X, Y)$ and proved the following theorem.

Theorem C (Veselý [17]) *Let X be a normal space and Y a connected and locally arcwise connected metrizable space. Then $B_1(X, Y) = H_1(X, Y) \cap \Sigma^*(X, Y)$.*

The aim of this paper is to generalize Theorems B and C for mappings defined on an arbitrary topological space. With this purpose in Sect. 2 we shall introduce a class $\Sigma^f(X, Y)$ of σ -strongly functionally discrete mappings (which coincides with the class $\Sigma^*(X, Y)$ if the space X is normal).

Theorem D *Let X be a topological space and Y a connected and locally arcwise connected metrizable space. Then $B_1(X, Y) = K_1(X, Y) \cap \Sigma^f(X, Y)$.*

Theorem E *For a completely metrizable space Y the following conditions are equivalent:*

- (i) Y is connected and locally arcwise connected;
- (ii) $K_1(X, Y) \cap \Sigma^f(X, Y) = B_1(X, Y)$ for any topological space X ;
- (iii) $H_1([0, 1], Y) = B_1([0, 1], Y)$.

2 Relations between Baire-one, Lebesgue-one and σ -discrete mappings

Definition 2.1 A family $\mathcal{A} = (A_i : i \in I)$ of subsets of a topological space X is called

- (a) *discrete*, if every point of X has an open neighborhood which intersects at most one set from $\mathcal{A}$;
- (b) *strongly discrete*, if there exists a discrete family $(U_i : i \in I)$ of open subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$;
- (c) *strongly functionally discrete* or *sfd-family*, if there exists a discrete family $(U_i : i \in I)$ of functionally open subsets of X such that $\overline{A_i} \subseteq U_i$ for every $i \in I$.

Notice that (c) $\Rightarrow$ (b) $\Rightarrow$ (a) for any X ; a topological space X is collectionwise normal if and only if every discrete family in X is strongly discrete; if X is normal then (b) $\Leftrightarrow$ (c).

Definition 2.2 Let $\mathcal{P}$ be a property of a family of subsets of topological space. A family $\mathcal{A}$ is called a σ - $\mathcal{P}$ -family if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where every family $\mathcal{A}_n$ has the property $\mathcal{P}$.

Definition 2.3 A family $\mathcal{B}$ of sets of a topological space X is called a *base* for a mapping $f : X \rightarrow Y$ if the preimage $f^{-1}(V)$ of an arbitrary open set V in Y is a union of sets from $\mathcal{B}$.

Definition 2.4 If a mapping $f : X \rightarrow Y$ has a base which is a σ - $\mathcal{P}$ -family, then we say that f is a σ - $\mathcal{P}$ mapping.

The collection of all σ - $\mathcal{P}$ mappings from X to Y will be denoted by

- $\Sigma(X, Y)$ if $\mathcal{P}$ is the property of discreteness;
- $\Sigma^*(X, Y)$ if $\mathcal{P}$ is the property of a strong discreteness;
- $\Sigma^f(X, Y)$ if $\mathcal{P}$ is the property of a strong functional discreteness.

By $\Sigma_0^{f*}(X, Y)$ we denote the collection of all mappings from X to Y which has a σ -sfd base of functionally closed subsets of X .

Notice that every mapping into a second-countable space is σ -discrete. Hansell proved in [4] that every Borel measurable mapping (in particular, every H_1 -mapping) $f : X \rightarrow Y$ is σ -discrete if X is a complete metric space and Y is a metric space.

Theorem 2.5 Let X be a topological space and Y be a metrizable space. Then

$$B_1(X, Y) \subseteq \Sigma_0^{f*}(X, Y) = K_1(X, Y) \cap \Sigma^f(X, Y).$$

Proof The equality $\Sigma_0^{f*}(X, Y) = K_1(X, Y) \cap \Sigma^f(X, Y)$ is proved in [8, Theorem 6].

Let $f \in B_1(X, Y)$ and $(f_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $f_n : X \rightarrow Y$ such that $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for every $x \in X$. Fix any σ -discrete open base $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ of Y , where each family $\mathcal{V}_m$ is discrete. For every $V \in \mathcal{V}$ we take

a sequence $(G_{k,V})_{k=1}^{\infty}$ of open sets such that $\overline{G_{k,V}} \subseteq G_{k+1,V}$ for every $k \in \mathbb{N}$ and $V = \bigcup_{k=1}^{\infty} \overline{G_{k,V}}$. Observe that

$$f^{-1}(V) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} f_n^{-1}(\overline{G_{k,V}}).$$

Denote $F_{k,V} = \bigcap_{n=k}^{\infty} f_n^{-1}(\overline{G_{k,V}})$ and notice that every set $F_{k,V}$ is functionally closed in X . For all $k, m \in \mathbb{N}$ we put $\mathcal{B}_{k,m} = (F_{k,V} : V \in \mathcal{V}_m)$ and $\mathcal{B} = \bigcup_{k,m=1}^{\infty} \mathcal{B}_{k,m}$. Then $\mathcal{B}$ is a base for f . Moreover, every family $\mathcal{B}_{k,m}$ is strongly functionally discrete, since $F_{k,V} \subseteq f_k^{-1}(V)$ and the family $(f_k^{-1}(V) : V \in \mathcal{V}_m)$ is discrete and consists of functionally open sets. Hence, $\mathcal{B}$ is a σ -sfd base for f . Thus, $f \in \Sigma_0^{f*}(X, Y)$. $\square$

Proposition 2.6 *Let X and Y be topological spaces. Then*

- (i) $K_1(X, Y) \subseteq K_1^w(X, Y)$;
- (ii) $K_1(X, Y) = H_1(X, Y)$ if X is perfectly normal;
- (iii) $K_1(X, Y) = K_1^w(X, Y) \subseteq H_1(X, Y)$ if Y is perfectly normal;
- (iv) $H_1(X, Y) \subseteq K_1^w(X, Y)$ if X is normal;
- (v) $H_1(\nu X, Y) \subseteq K_1^w(\nu X, Y)$ if X is normal.

Proof Since (i), (ii) and (iii) are obvious, we prove (iv) and (v). Let F be a functionally closed subset of Y and $\varphi : Y \rightarrow [0, 1]$ be a continuous function with $F = \varphi^{-1}(0)$. We put $G_n = \varphi^{-1}([0, 1/n])$ for every $n \in \mathbb{N}$.

(iv) Let $f \in H_1(X, Y)$. For every n the sets $A_n = f^{-1}(G_n)$ and $B_n = X \setminus f^{-1}(\overline{G_n})$ are disjoint and F_σ in X . Then, by [17, Lemma 1.7], for every n there exists a functionally G_δ -set C_n in X such that $A_n \subseteq C_n$ and $C_n \cap B_n = \emptyset$. Notice that $f^{-1}(F) = \bigcap_{n=1}^{\infty} C_n$. Hence, $f^{-1}(F)$ is a functionally G_δ -set in X . Therefore, $f \in K_1^w(X, Y)$.

(v) Let $f \in H_1(\nu X, Y)$. Put

$$\begin{aligned} A &= f^{-1}(F), & A_n &= f^{-1}(G_{n+2}), \\ B_n &= \nu X \setminus f^{-1}(G_n), & C_n &= \nu X \setminus f^{-1}(\overline{G_{n+1}}). \end{aligned}$$

Then $A \subseteq A_n$, $B_n \subseteq C_n$, $A_n \cap C_n = \emptyset$, A and B_n are G_δ -sets in νX , A_n and C_n are F_σ -sets in νX . If $A = \emptyset$ or $\nu X \setminus A = \emptyset$, then A is a functionally G_δ -set. Thus, it is sufficient to consider the case of $A \neq \emptyset$ and $\nu X \setminus A \neq \emptyset$. Moreover, we may assume that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$. Since X intersects every nonempty G_δ -subset of νX (and, consequently, every nonempty $G_{\delta\sigma}$ -set) by [1, Corollary 2.6], $A_X = A \cap X \neq \emptyset$ and $B_{n,X} = B_n \cap X \neq \emptyset$. Then $A_{n,X} = A_n \cap X$ and $C_{n,X} = C_n \cap X$ are nonempty disjoint F_σ -subsets of the normal space X . Applying [17, Lemma 1.7], for every n we choose a functionally G_δ -set $D_{n,X}$ in X such that $A_{n,X} \subseteq D_{n,X}$ and $D_{n,X} \cap C_{n,X} = \emptyset$. Since X is a C -embedded subspace of νX , there exists a functionally G_δ -set D_n in νX such that $D_n \cap X = D_{n,X}$.

Notice that $D_n \cap B_n = \emptyset$, since otherwise $D_n \cap B_n$ is a nonempty G_δ -subset of νX which does not intersect X , what contradicts to the G_δ -density of X in νX . Notice that $A \setminus D_n$ is a $G_{\delta\sigma}$ -set in νX and $(A \setminus D_n) \cap X \subseteq A_{n,X} \setminus D_{n,X} = \emptyset$. Therefore, $A \subseteq D_n$.

Hence, $(D_n)_{n=1}^\infty$ is a sequence of functionally G_δ -subsets of νX such that $A \subseteq D_n \subseteq f^{-1}(G_n)$ for every n . Then the set $f^{-1}(F) = \bigcap_{n=1}^\infty D_n$ is functionally G_δ in νX . Therefore, $f \in K_1^w(\nu X, Y)$. $\square$

Let us observe that the space

$$X = \{(x_t)_{t \in [0,1]} \in \mathbb{R}^{[0,1]} : |\{t \in [0,1] : x_t \neq 0\}| \leq \aleph_0\}$$

is normal [2], but its Hewitt–Nachbin completion $\nu X = \mathbb{R}^{[0,1]}$ is not normal [16].

The following examples show that the conditions on X and Y in Proposition 2.6 are essential.

Example 2.7 There exists a Hausdorff compact space Y such that

$$K_1^w(\mathbb{R}, Y) \setminus H_1(\mathbb{R}, Y) \neq \emptyset.$$

Proof Let $Y = \mathbb{R} \cup \{\infty\}$ be the Alexandroff one-point compactification of the real line $\mathbb{R}$ with the discrete topology and let $f: \mathbb{R} \rightarrow Y$ be a mapping such that $f(0) = \infty$ and $f(x) = x$ for all $x \in \mathbb{R} \setminus \{0\}$. Then the preimage $f^{-1}(F)$ of any functionally closed subset $F \subseteq Y$ is either finite, or has a countable complement, i.e. $f^{-1}(F)$ is G_δ in $\mathbb{R}$. Hence, $f \in K_1^w(\mathbb{R}, Y)$. On the other hand, if V is any non-Borel subset of $\mathbb{R}$, then V is open in Y , but $f^{-1}(V)$ is not F_σ . Therefore, $f \notin H_1(\mathbb{R}, Y)$. $\square$

Example 2.8 There exists a completely regular space X such that

$$H_1(X, \mathbb{R}) \setminus K_1^w(X, \mathbb{R}) \neq \emptyset.$$

Proof Let X be the Niemytski plane, i.e. $X = \mathbb{R} \times [0, +\infty)$, where a base of neighborhoods of $(x, y) \in X$ with $y > 0$ consists of open balls with the center in (x, y) , and a base of neighborhoods of $(x, 0)$ consists of sets $U \cup \{(x, 0)\}$, where U is an open ball tangent to $\mathbb{R} \times \{0\}$ in the point $(x, 0)$. It is well known that the space X is perfect and completely regular, but is not normal.

Let E be a set which is not of the $G_{\delta\sigma}$ -type in $\mathbb{R}$ and consider the subspace $A = E \times \{0\}$ of X . Then A is F_σ - and G_δ -subset of X . We show that A is not functionally F_σ in X . Assume the contrary and write $A = \bigcup_{n=1}^\infty A_n$, where each A_n is a functionally closed subset of X . For every $n \in \mathbb{N}$ we take a continuous function $f_n: X \rightarrow [0, 1]$ such that $A_n = f_n^{-1}(0)$ and for all $(x, y) \in X$ and $m \in \mathbb{N}$ we put

$$g_{n,m}(x, y) = \begin{cases} f_n(x, y), & y \geq \frac{1}{m}, \\ f_n\left(x, \frac{1}{m}\right), & 0 \leq y < \frac{1}{m}. \end{cases}$$

Then $g_{n,m}: \mathbb{R} \times [0, +\infty) \rightarrow [0, 1]$ is a continuous function and $\lim_{m \rightarrow \infty} g_{n,m}(x, y) = f_n(x, y)$ for every $(x, y) \in X$. Therefore, $f_n \in B_1(\mathbb{R} \times [0, +\infty), [0, 1])$ for every $n \in \mathbb{N}$, which implies that every set A_n is a G_δ -subset of $\mathbb{R} \times \{0\}$, a contradiction.

It is easy to see that $\chi_A \in H_1(X, \mathbb{R}) \setminus K_1^w(X, \mathbb{R})$. $\square$

3 Approximation lemma

Definition 3.1 A family $\mathcal{A}$ of subsets of topological space X is called *strictly functionally discrete* if for every $A \in \mathcal{A}$ there exists a continuous function $f_A: X \rightarrow [0, 1]$ such that $A \subseteq f_A^{-1}(0)$ and the family $(f_A^{-1}([0, 1])) : A \in \mathcal{A}$ is discrete.

Evidently, each strictly functionally discrete family is strongly functionally discrete and these two concepts coincide in the case when $\mathcal{A}$ consists of functionally closed sets.

For a family $\mathcal{A} = (A_i : i \in I)$ the set $\bigcup_{i \in I} A_i$ is denoted by $\bigcup \mathcal{A}$.

Proposition 3.2 Let X be a topological space, (Y, d) a metric space and $f: X \rightarrow Y$ a mapping. Assume that for some integer $n \geq 2$ there are strictly functionally discrete families $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_n$ of subsets of X and mappings $p_k: \mathcal{F}_k \rightarrow Y$ for $k \in \{0, \dots, n\}$ such that

- (A) the family $\mathcal{F}_0$ consists of a single set X ;
- (B) for every $k \leq n$ and $F \in \mathcal{F}_k$ we get $\sup_{x \in F} d(f(x), p_k(F)) < 1/2^{k+2}$;
- (C) for every $k < n$ each set of $\mathcal{F}_{k+1}$ is contained in some set of $\mathcal{F}_k$;
- (D) for every $F \in \mathcal{F}_1$ the points $p_0(X)$ and $p_1(F)$ can be joined by an arc in Y ;
- (E) for any $k \in \{1, \dots, n-1\}$ and sets $F \in \mathcal{F}_{k+1}$ and $F' \in \mathcal{F}_k$ with $F \subseteq F'$ the points $p_{k+1}(F)$ and $p_k(F')$ can be joined by an arc of diameter less than $1/2^{k+2}$ in Y .

Then there exists a continuous mapping $g: X \rightarrow Y$ such that for any $k \in \{1, \dots, n\}$ and $x \in \bigcup \mathcal{F}_k$ we get

$$d(f(x), g(x)) < \frac{1}{2^k}.$$

Proof For every $k \in \{1, \dots, n\}$ and $F \in \mathcal{F}_k$ we take a continuous function $\psi_{F,k}: X \rightarrow [0, 1]$ such that $F \subseteq \psi_{F,k}^{-1}(0)$ and the family $(\psi_{F,k}^{-1}([0, 1])) : F \in \mathcal{F}_k$ is discrete. Put $\varphi_{F,1} = \psi_{F,1}$ for every $F \in \mathcal{F}_1$. Now for every $F \in \mathcal{F}_2$ we choose $F' \in \mathcal{F}_1$ such that $F \subseteq F'$ and set $U_{F,2} = \psi_{F',1}^{-1}([0, 1/2])$. There exists a continuous function $\varphi_{F,2}: X \rightarrow [0, 1]$ such that $\varphi_{F,2}^{-1}(0) = \psi_{F,2}^{-1}(0)$ and $\varphi_{F,2}^{-1}([0, 1]) = U_{F,2}$. Proceeding inductively in this way we obtain a sequence of families $(\varphi_{F,k} : F \in \mathcal{F}_k)_{1 \leq k \leq n}$ of continuous functions $\varphi_{F,k}: X \rightarrow [0, 1]$ such that for any $k \in \{2, \dots, n\}$ and $F \subseteq F_k$ there is $F' \in \mathcal{F}_{k-1}$ with $F \subseteq F'$ and

$$F \subseteq \varphi_{F,k}^{-1}(0) \subseteq \varphi_{F,k}^{-1}([0, 1]) \subseteq \varphi_{F',k-1}^{-1}\left(\left[0, \frac{1}{2}\right]\right),$$

and the family $(U_{F,k} : F \in \mathcal{F}_k)$ is discrete, where $U_{F,k} = \varphi_{F,k}^{-1}([0, 1])$. For every k we put $U_k = \bigcup_{F \in \mathcal{F}_k} U_{F,k}$ and notice that the sets

$$H_k = \bigcup_{F \in \mathcal{F}_k} \varphi_{F,k}^{-1}\left(\left[0, \frac{1}{2}\right]\right), \quad E_k = X \setminus U_k$$

are disjoint and functionally closed in X . Hence, there exists a continuous function $h_k: X \rightarrow [0, 1]$ such that $H_k = h_k^{-1}(1)$ and $E_k = h_k^{-1}(0)$.

For all $x \in X$ let

$$g_0(x) = p_0(X).$$

For every $F \in \mathcal{F}_1$ we take a continuous function $\gamma_{F,1}: [0, 1] \rightarrow Y$ with $\gamma_{F,1}(0) = p_0(X)$ and $\gamma_{F,1}(1) = p_1(F)$. Define $g_1: X \rightarrow Y$ by the formula

$$g_1(x) = \begin{cases} g_0(x) & \text{if } x \in E_1, \\ \gamma_{F,1}(h_1(x)) & \text{if } x \in \overline{U_{F,1}} \text{ for some } F \in \mathcal{F}_1. \end{cases}$$

Let us observe that the family $(U_{F,1} : F \in \mathcal{F}_1)$ is discrete and for every $x \in \overline{U_{F,1}} \setminus U_{F,1}$ with $F \in \mathcal{F}_1$ we have $h_1(x) = 0$ and $g_1(x) = \gamma_{F,1}(0) = g_0(x)$. Moreover, $g_1(x) = p_1(F)$ for $x \in \varphi_{F,1}^{-1}([0, 1/2])$, $F \in \mathcal{F}_1$. Therefore, g_1 is a well-defined continuous mapping.

Assume that we have already defined continuous mappings $g_1, \dots, g_k: X \rightarrow Y$ for $1 \leq k < n$ such that $g_k(x) = g_{k-1}(x)$ if $x \in E_k$ and $g_k(x) = p_k(F)$ if $x \in \varphi_{F,k}^{-1}([0, 1/2])$ for some $F \in \mathcal{F}_k$. Using (E), for every $F \in \mathcal{F}_{k+1}$ we choose a set $F' \in \mathcal{F}_k$ with $F \subseteq F'$ and a continuous arc $\gamma_{F,k+1}: [0, 1] \rightarrow Y$ of diameter less than $1/2^{k+2}$ such that $\gamma_{F,k+1}(0) = p_k(F')$ and $\gamma_{F,k+1}(1) = p_{k+1}(F)$. Define a continuous mapping $g_{k+1}: X \rightarrow Y$ by the formula

$$g_{k+1}(x) = \begin{cases} g_k(x) & \text{if } x \in E_{k+1}, \\ \gamma_{F,k+1}(h_{k+1}(x)) & \text{if } x \in \overline{U_{F,k+1}} \text{ for some } F \in \mathcal{F}_{k+1}. \end{cases}$$

We prove that the inequality

$$d(g_{k+1}(x), g_k(x)) < \frac{1}{2^{k+2}} \quad (1)$$

holds for every $x \in X$. Clearly, (1) is valid for $x \in E_{k+1}$. If $x \in U_{F,k+1}$ for $F \in \mathcal{F}_{k+1}$, then $x \in \varphi_{F',k}^{-1}([0, 1/2])$ and

$$g_{k+1}(x) = \gamma_{F,k+1}(h_{k+1}(x)), \quad g_k(x) = p_k(F') = \gamma_{F,k+1}(0),$$

which implies (1).

Proceeding inductively we define continuous mappings $g_1, \dots, g_n: X \rightarrow Y$ which satisfy (1) for all $k \in \{1, \dots, n-1\}$ and $x \in X$. Moreover, $g_k|_F = p_k(F)$ for every $k \in \{1, \dots, n\}$ and $F \in \mathcal{F}_k$.

We denote $g = g_n$. Let $1 \leq k \leq n$ and $x \in \bigcup \mathcal{F}_k$. Then $x \in F$ for some $F \in \mathcal{F}_k$ and $g_k(x) = p_k(F)$. Taking into consideration (B), we get

$$d(f(x), g_k(x)) \leq \frac{1}{2^{k+2}}.$$

Then (1) implies

$$d(f(x), g(x)) \leq d(f(x), g_k(x)) + \sum_{i=k}^{n-1} d(g_i(x), g_{i+1}(x)) < \frac{1}{2^{k+2}} + \frac{1}{2^{k+1}} < \frac{1}{2^k},$$

which completes the proof. $\square$

4 A generalization of the Lebesgue–Hausdorff theorem

Theorem 4.1 *Let X be a topological space and Y a metrizable connected and locally arcwise connected space. Then $\Sigma_0^{f*}(X, Y) \subseteq B_1(X, Y)$.*

Proof Let d be a metric on Y which generates its topological structure. Using locally arcwise connectedness of Y , for every $k \in \mathbb{N}$ we choose a covering $\mathcal{U}_k$ of Y by open balls of diameters less than $1/2^{k+2}$ such that any two points in any set $U \in \mathcal{U}_k$ can be joined by an arc of diameter less than $1/2^{k+2}$.

Let $f \in \Sigma_0^{f*}(X, Y)$ and let $\mathcal{B}$ be a σ -sfd base for f which consists of functionally closed sets in X . For every $k \in \mathbb{N}$ we put

$$\mathcal{B}_k = \{B \in \mathcal{B} : B \subseteq f^{-1}(U) \text{ for some } U \in \mathcal{U}_k\}.$$

Then $\mathcal{B}_k$ is a σ -sfd family and $X = \bigcup \mathcal{B}_k$ for every k . According to [8, Lemma 13], for every $k \in \mathbb{N}$ there exists a sequence $(\mathcal{B}_{k,n})_{n=1}^\infty$ of sfd families $\mathcal{B}_{k,n} = (B_{k,n,i} : i \in I_{k,n})$ of functionally closed subsets of X such that $\mathcal{B}_{k,n} < \mathcal{B}_k$, $\mathcal{B}_{k,n} < \mathcal{B}_{k,n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty \bigcup \mathcal{B}_{k,n} = X$. For all $k, n \in \mathbb{N}$ we put

$$\mathcal{F}_{k,n} = (B_{1,n,i_1} \cap \cdots \cap B_{k,n,i_k} : i_m \in I_{m,n}, 1 \leq m \leq k)$$

and notice that every $\mathcal{F}_{k,n}$ is a strongly functionally discrete family of functionally closed sets with

- (a) $\mathcal{F}_{k+1,n} < \mathcal{F}_{k,n}$,
- (b) $\mathcal{F}_{k,n} < \mathcal{F}_{k,n+1}$,
- (c) $\bigcup_{n=1}^\infty \bigcup \mathcal{F}_{k,n} = X$.

Fix $n \geq 2$ and for every $k = 1, \dots, n$ we denote $\mathcal{F}_k = \mathcal{F}_{k,n}$. Take any $y_0 \in Y$ and put $p_0(X) = y_0$. For every $F \in \mathcal{F}_1$ we take $U_F \in \mathcal{U}_1$ with $f(F) \subseteq U_F$ and any $y_F \in U_F$. Put $p_1(F) = y_F$. Notice that Y is arcwise connected, therefore, condition (D) of Proposition 3.2 holds.

Now for every $F \in \mathcal{F}_2$ there exist $F' \in \mathcal{F}_1$ with $f(F) \subseteq f(F')$ and $U_F \in \mathcal{U}_2$ with $f(F) \subseteq U_F$. Take any point $y_F \in U_F \cap U_{F'}$ and put $p_2(F) = y_F$. Notice that we can join $p_2(F)$ and $p_1(F')$ by an arc of diameter less than $1/2^4$. Proceeding inductively we obtain mappings $p_k : \mathcal{F}_k \rightarrow Y$ which for every $k \in \{0, \dots, n\}$ satisfy the conditions of Proposition 3.2. Hence, there exists a continuous mapping $g_n : X \rightarrow Y$ such that $d(f(x), g_n(x)) < 1/2^k$ if $x \in \bigcup \mathcal{F}_{k,n}$ for $k \leq n$.

We show that the sequence $(g_n)_{n=2}^\infty$ converges pointwise to f on X . Let $x \in X$, $\varepsilon > 0$ and $k \in \mathbb{N}$ be such that $1/2^k < \varepsilon$. Conditions (b) and (c) imply that there exists $n_0 \geq k$ such that $x \in \mathcal{F}_{k,n}$ for all $n \geq n_0$. Then we have

$$d(f(x), g_n(x)) < \frac{1}{2^k} < \varepsilon$$

for all $n \geq n_0$. Hence, $f \in B_1(X, Y)$. $\square$

Combining Theorems 2.5 and 4.1 we get the following result.

Theorem 4.2 *Let X be a topological space and Y a metrizable connected and locally arcwise connected space. Then $K_1(X, Y) \cap \Sigma^f(X, Y) = B_1(X, Y)$.*

Remark 4.3 If X is a normal space, then Proposition 2.6 and Theorem 4.2 imply Theorem C.

For a product $Y = \prod_{n=1}^\infty Y_n$ of topological spaces we denote by π_n a projection mapping $\pi_n: Y \rightarrow Y_n$, $\pi_n(y) = y_n$ for every $y = (y_n)_{n=1}^\infty \in Y$.

Corollary 4.4 *Let X be a topological space, $(Y_n)_{n=1}^\infty$ a sequence of metrizable connected and locally arcwise connected spaces and let $Y \subseteq \prod_{n=1}^\infty Y_n$ be a subspace such that for every $n \in \mathbb{N}$ there exists a continuous mapping $h_n: \prod_{k=1}^n Y_k \rightarrow Y$ with $\pi_n(h_n(y)) = y$ for every $y \in \prod_{k=1}^n Y_k$. Then*

$$K_1(X, Y) \cap \Sigma^f(X, Y) = B_1(X, Y).$$

Proof We only need to prove the inclusion $\Sigma_0^{f*}(X, Y) \subseteq B_1(X, Y)$. Let $f \in \Sigma_0^{f*}(X, Y)$. Then $\pi_n \circ f \in B_1(X, Y_n)$, by Theorem 4.2, for every n . Therefore, there exists a sequence of continuous mappings $f_{nm}: X \rightarrow Y_n$ such that $\lim_{m \rightarrow \infty} f_{nm}(x) = \pi_n(f(x))$ for all $x \in X$ and $n \in \mathbb{N}$. We put $f_n = h_n \circ f_{nn}$. Then $(f_n)_{n=1}^\infty$ is a sequence of continuous mappings $f_n: X \rightarrow Y$ which converges to f pointwise on X . $\square$

Remark 4.5 Let $Y = \bigcup_{n=1}^\infty \{(y_n)_{n \in \mathbb{N}} \in [0, 1]^\omega : y_n = 1, y_k = 0, k > n\}$. Then Y is neither arcwise connected nor locally arcwise connected, but the equality $K_1(X, Y) = B_1(X, Y)$ holds for any space X , by Corollary 4.4.

Definition 4.6 A topological space X is called

- *almost arcwise connected* if each pair of nonempty open subsets of X can be joined by an arc in X ;
- *locally almost arcwise connected at $x \in X$* if for any neighborhood V of x there exists a neighborhood $U \subseteq V$ of x such that each pair of nonempty open subsets of U can be joined by an arc in $\bar{V}$;
- *locally almost arcwise connected* if it is locally almost arcwise connected at every point.

Lemma 4.7 *Let Y be a topological space such that $H_1([0, 1], Y) \subseteq B_1([0, 1], Y)$. Then Y is almost arcwise connected.*

Proof Consider nonempty open sets U and V in Y and take different points $u \in U$ and $v \in V$. Let $f(x) = u$ if $x \in [0, 1)$ and $f(1) = v$. Then $f \in H_1([0, 1], Y)$. By the assumption of the theorem, there exists a sequence of continuous mappings $f_n: [0, 1] \rightarrow Y$ which is pointwise convergent to f . Then there exists a number n such that $f_n(0) \in U$ and $f_n(1) \in V$, which implies almost arcwise connectedness of Y . $\square$

Lemma 4.8 *Let Y be a first-countable topological space such that $H_1([0, 1], Y) \subseteq B_1([0, 1], Y)$. Then Y is locally almost arcwise connected.*

Proof Suppose to the contrary that there exists a point $y_0 \in Y$ and its neighborhood V such that the condition of locally almost arcwise connectedness at y_0 is not satisfied. Let $(V_n : n \in \mathbb{N})$ be a family of all basic neighborhoods of y_0 which are contained in V . For every $n \in \mathbb{N}$ we take nonempty open sets $U_{n,0}$ and $U_{n,1}$ in V_n such that any arc starting in $U_{n,0}$ and ending in $U_{n,1}$ necessarily intersects $Y \setminus \overline{V}$. For all $n \in \mathbb{N}$ and $i = 0, 1$ fix any point $y_{n,i} \in U_{n,i}$. Let $Q = \{[r_{n,0}, r_{n,1}] : n \in \mathbb{N}\}$ be a dense subset of $\mathbb{Q} \cap [0, 1]^2$, where all $r_{n,i}$ are distinct. Define a mapping $f: [0, 1] \rightarrow Y$ by the formula

$$f(x) = \begin{cases} y_0 & \text{if } x \in [0, 1] \setminus Q, \\ y_{n,i} & \text{if } x = r_{n,i} \text{ for some } n \in \mathbb{N} \text{ and } i \in \{0, 1\}. \end{cases}$$

Notice that $f([0, 1]) \subseteq V$ and $f \in H_1([0, 1], Y)$, since every sequence $(y_{n,i})_{n=1}^\infty$ converges to y_0 in Y . Then there exists a sequence $(f_k)_{k=1}^\infty$ of continuous mappings $f_k: [0, 1] \rightarrow Y$ which is pointwise convergent to f .

Fix $m \in \mathbb{N}$. Consider an open set

$$G_m = \bigcup_{k \geq m} f_k^{-1}(Y \setminus \overline{V})$$

and prove that it is dense in $[0, 1]$. Let $n \in \mathbb{N}$ and $I = [r_{n,0}, r_{n,1}]$. Since $f_k(r_{n,i}) \rightarrow f(r_{n,i}) = y_{n,i}$, there exists $k \geq m$ such that $f_k(r_{n,i}) \in U_{n,i}$ for $i = 0, 1$. Then $f_k(I) \setminus \overline{V} \neq \emptyset$. It follows that $I \cap G_m \neq \emptyset$.

Notice that $\bigcap_{m=1}^\infty G_m \subseteq f^{-1}(Y \setminus V) = \emptyset$, which contradicts to the Baireness of $[0, 1]$. $\square$

The last two lemmas immediately imply the following result.

Corollary 4.9 *Let Y be a first-countable space such that $H_1([0, 1], Y) \subseteq B_1([0, 1], Y)$. Then Y is almost arcwise connected and locally almost arcwise connected.*

We observe that the converse statement is not true even for metrizable separable spaces Y as [3, Example 2] shows.

Lemma 4.10 *Every regular locally almost arcwise connected space X is locally connected.*

Proof Fix a point $x \in X$ and an open neighborhood W of x . Let U and V be open neighborhoods of x such that $U \subseteq \overline{V} \subseteq W$ and each two open nonempty subsets of U can be joined by an arc in $\overline{V}$. Consider a family $\mathcal{C}$ which consists of all arcs intersecting U and lying in W . Put $G = \bigcup \mathcal{C} \cup U$ and show that G is connected. Indeed, assume that $G = G_0 \cup G_1$, where G_0 and G_1 are nonempty disjoint relatively open subsets of G . By definition of G , we have $U \cap G_0 \neq \emptyset$ and $U \cap G_1 \neq \emptyset$. Then the sets $U \cap G_0$ and $U \cap G_1$ can be joined by an arc γ lying in W . Since $\gamma \in \mathcal{C}$ and the set γ is connected, it follows that $\gamma \subseteq G_0$ or $\gamma \subseteq G_1$, a contradiction. Hence, $G \subseteq W$ is a connected neighborhood of x . Therefore, X is connected in the small at x . It remains to apply [12, Theorem 10, p.90]. $\square$

Lemma 4.11 *Any completely metrizable almost arcwise connected and locally almost arcwise connected space X is arcwise connected and locally arcwise connected.*

Proof It is easy to see that X is connected. Lemma 4.10 implies that X is locally connected. Then X is arcwise connected and locally arcwise connected by [9, Theorem 1, p. 254]. $\square$

Theorem 4.12 *Let Y be a completely metrizable space. Then the following statements are equivalent:*

- (i) Y is connected and locally arcwise connected;
- (ii) $K_1(X, Y) \cap \Sigma^f(X, Y) = B_1(X, Y)$ for any topological space X ;
- (iii) $H_1([0, 1], Y) = B_1([0, 1], Y)$.

Proof The implication (i) $\Rightarrow$ (ii) follows from Theorem 4.2. The implication (ii) $\Rightarrow$ (iii) follows from the completeness of $[0, 1]$ and [4, Theorem 3]. Finally, Lemmas 4.7, 4.8 and 4.11 imply (iii) $\Rightarrow$ (i). $\square$

According to [4, Theorem 3], we have $H_1(X, Y) \subseteq \Sigma^f(X, Y)$ for any metrizable space Y . Therefore, Theorem 4.2 implies the following fact.

Corollary 4.13 *Let X be a completely metrizable space and Y a metrizable connected and locally arcwise connected space. Then*

$$H_1(X, Y) = K_1(X, Y) = B_1(X, Y).$$

The following example shows that Corollary 4.13 is not valid if Y is not metrizable.

Example 4.14 There exists a connected and locally arcwise connected first-countable Lindelöf space Y such that

$$H_1(\mathbb{R}, Y) \setminus B_1(\mathbb{R}, Y) \neq \emptyset.$$

Proof Let Y be the set $(\mathbb{R} \times [0, +\infty), \tau)$ equipped with a topology τ such that the sets $(x - \varepsilon, x + \varepsilon) \times (y - \varepsilon, y + \varepsilon)$, $\varepsilon > 0$, are a neighborhood base of a point (x, y) if $y > 0$, and the sets $\{(x, 0)\} \cup ((x, x + \varepsilon) \times [0, \varepsilon))$, $\varepsilon > 0$, are a neighborhood base of (x, y) if $y = 0$. Notice that Y is a Lindelöf first-countable connected and locally

arcwise connected space. We observe that Y is not regular, since each point $(x, 0)$ has not a closed neighborhood base. Evidently, the restriction of τ to the upper half-plane coincides with the Euclidean topology, and the restriction of τ to $\mathbb{R} \times \{0\}$ coincides with the topology of the Sorgenfrey line.

Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$ and $T = \mathbb{R} \setminus \mathbb{Q}$. Define a function $f : \mathbb{R} \rightarrow Y$ by

$$f(x) = \begin{cases} \left(x - \frac{1}{n}, 0\right) & \text{if } x = r_n, \\ (x, 0) & \text{if } x \in T. \end{cases}$$

In order to show that $f \in H_1(\mathbb{R}, Y)$ it is sufficient to prove that the preimage of a set $V = [a, b] \times \{0\}$, $a < b$, is F_σ in $\mathbb{R}$. Denote $E = f^{-1}(V)$. Notice that $A = [a, b] \setminus E$ and $B = E \setminus (a, b)$ are subsets of $\mathbb{Q}$ and

$$E = ([a, b] \setminus A) \cup B. \quad (2)$$

Let $A = \{r_{n_k} : k \in \mathbb{N}\}$, where $(n_k)_{k=1}^\infty$ is an increasing sequence. Then for every k we have

$$a \leq r_{n_k} \leq a + \frac{1}{n_k}.$$

Hence, $r_{n_k} \rightarrow a$ in $\mathbb{R}$. It follows that A is G_δ in $\mathbb{R}$. Then E is F_σ in $\mathbb{R}$ by (2).

Now we prove that $f \notin B_1(\mathbb{R}, Y)$. Assume to the contrary that there exists a sequence of continuous mappings $f_n : \mathbb{R} \rightarrow Y$ which is pointwise convergent to f on $\mathbb{R}$. For $x \in \mathbb{R}$ we denote

$$B(x) = \{(x, 0)\} \cup ((x, +\infty) \times [0, +\infty))$$

and define

$$A_n = \bigcap_{k=n}^\infty \{x \in \mathbb{R} : f_k(x) \in B(f(x))\}$$

for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^\infty A_n = \mathbb{R}$. Since T is a Baire space, there exist $n \in \mathbb{N}$ and $(a, b) \subseteq \mathbb{R}$ with

$$(a, b) \subseteq \overline{A_n \cap T}.$$

Notice that $A_n \cap T \subseteq F$, where $F = \bigcap_{k=n}^\infty \{x \in \mathbb{R} : \pi_X(f_k(x)) \geq x\}$. Since the projection $\pi_X : Y \rightarrow \mathbb{R}$, $\pi_X(x, y) = x$, is upper semi-continuous, the set F is closed in $\mathbb{R}$. Then $(a, b) \subseteq F$. It follows that $\pi_X(f(x)) \geq x$ for all $x \in (a, b)$, a contradiction. $\square$

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