On logarithmic integrals of the Riemann zeta-function and an approach to the Riemann Hypothesis by a geometric mean with respect to an ergodic transformation

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Received: 20 April 2015 / Revised: 19 August 2015 / Accepted: 28 August 2015 /
Published online: 18 September 2015
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Abstract We study the distribution of certain improper integrals associated with the Riemann zeta-function ζ , namely,

$$\frac{a}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma + it)|}{a^2 + t^2} dt, \quad \sigma \in \mathbb{R},$$

where $a > 0$ is real. For instance, for $a = \sigma \in (0, 1]$, we shall show that

$$\frac{1}{2\pi} \int_{\operatorname{Re} s = \sigma} \frac{\log |\zeta(s)|}{|s|^2} |ds| = \frac{\log |(2\sigma - 1)\zeta(2\sigma)|}{2\sigma} + \frac{1}{2\sigma} \sum_{\operatorname{Re} \varrho > \sigma} \log \left| \frac{\varrho}{2\sigma - \varrho} \right|.$$

In particular, for $\sigma = 1/2$, we get a well-known result proved by Balazard, Saias and Yor:

$$\frac{1}{2\pi} \int_{\operatorname{Re} s = 1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = \sum_{\operatorname{Re} \varrho > 1/2} \log \left| \frac{\varrho}{1 - \varrho} \right|.$$

In the final section, we study Boole's transformation, $T_a: x \mapsto (x - a^2/x)/2$ for $x \neq 0$ and $T_a 0 = 0$, and show by its ergodicity that, for $\sigma \in \mathbb{R}$, the geometric mean-value of $|\zeta(\sigma + iT_a^n x)|$ exists for almost all $x \in \mathbb{R}$ as $n \rightarrow +\infty$, and is independent of x . In particular, for $\sigma = 1/2$, we obtain a criterion for the Riemann Hypothesis:

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let $z_n(\mu) = \exp(2\pi i \{2^n \mu / \pi\})$ for fixed $n \in \mathbb{N}$ and $\mu \in [0, \pi]$, then a necessary condition for the truth of the Riemann Hypothesis is

$$\prod_{n=0}^{N-1} \left| \zeta \left(\frac{1}{1 - z_n(\mu)} \right) \right| = O(N^\alpha) \quad \text{for almost all } \mu \in [0, \pi] \text{ as } N \rightarrow +\infty$$

with arbitrary $\alpha \in \mathbb{R}$.

Keywords Riemann zeta-function · Riemann Hypothesis · Ergodic theory · Logarithmic integrals · Cauchy distributed function · Cauchy random walk · Entire functions

Mathematics Subject Classification 11M26 · 11K06 · 11K31 · 37A45

1 Introduction

Let $s = \sigma + it$. The Riemann zeta-function ζ is defined by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad \sigma > 1.$$

Euler found a relation between the ζ -function and the prime numbers by the product representation

$$\zeta(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p^s} \right)^{-1}, \quad \sigma > 1, \quad (1)$$

where $\mathcal{P}$ denotes the set of primes. Riemann discovered that the zeta-function has an analytic continuation to the whole complex plane except for a simple pole at $s = 1$ with residue 1, and satisfies the following functional equation [20, Chapter II]:

$$\xi(s) = \xi(1-s) \quad \text{with} \quad \xi(s) = (s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}+1\right)\zeta(s).$$

Notice that ξ is a regular function of order 1, hence, by Hadamard's product formula (see [10, Chapter I, p. 12] or [21, 8.24, p. 250]) one can write

$$\xi(s) = e^{As+B} \prod_{\varrho} \left(1 - \frac{s}{\varrho} \right) e^{s/\varrho} = \xi(0) \prod_{\varrho} \left(1 - \frac{s}{\varrho} \right). \quad (2)$$

Here ϱ ranges over the zeros of ξ according to their multiplicities. In the latter product zeros $\beta + i\gamma$ and $\beta - i\gamma$ are grouped together for convergence. The values ϱ are the complex zeros of the Riemann zeta-function in $\sigma > 0$ due to the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \text{with} \quad \chi(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}. \quad (3)$$

Notice that $\chi(s)$ is holomorphic for $\sigma \leq 1/2$ with only real zeros at $s = 0, -2, -4, \dots$. By Stirling's formula, for any bounded σ and $|t| \geq 2$,

$$\chi(s) = \left| \frac{t}{2\pi} \right|^{1/2-\sigma} \exp \left(i \left(\frac{\pi}{4} - t \log \frac{t}{2\pi e} \right) \right) \left\{ 1 + O \left(\frac{1}{|t|} \right) \right\}.$$

It follows that the function given by

$$H(s) = (s-1)\zeta(s)$$

is an entire function with the same zeros as the Riemann zeta-function. We recall that the Riemann zeta-function has, thanks to Euler product (1), no zeros in the half-plane $\sigma > 1$; the functional equation (3) implies the existence of simple zeros $s = -2, -4, \dots$ in the half-plane $\sigma < 0$, called the trivial zeros; apart from those there are no zeros in the left half-plane. Riemann conjectured and von Mangoldt proved the existence of an infinitude of non real zeros of the Riemann zeta-function, denoted $\varrho = \beta + i\gamma$, in the “critical” strip $0 \leq \sigma \leq 1$ (called the nontrivial zeros). Moreover, by (3) and by the reflection principle, it follows that the nontrivial zeros lie symmetrically with respect to the real axis and the critical line $\sigma = 1/2$. For $T > 0$ let $N(T)$ count the number of nontrivial zeros ϱ in the rectangle $[0, 1] \times [0, T]$ (with multiplicities), then the Riemann–von Mangoldt formula states that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Hadamard and de La Vallée Poussin [20, 3.2, Chapter III] proved that $\zeta(s)$ does not vanish on the line $\sigma = 1$, and so, by (3), $\zeta(s)$ does not vanish on $\sigma = 0$. Moreover, Riemann stated the famous, yet unproved so-called Riemann hypothesis that $\zeta(s) \neq 0$ for $\sigma > 1/2$.

Balazard et al. [4] proved that the function $H(1/(1-z))$ belongs to the Hardy class $H^{1/3}(D)$, where D is the unit disk (see for example [2] for the definition of Hardy spaces). It follows from the factorization theorem that

$$\int_{\sigma=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = 2\pi \sum_{\beta>1/2} \log \left| \frac{\varrho}{1-\varrho} \right|. \quad (4)$$

On the other hand, Eroğlu and Ostrovskii [8] proved that $H(1/(1-z))$ belongs to the Hardy class $H^p(D)$ if and only if $0 < p < 1$. Additionally, they proved that the integral (4) can be computed regardless to the theory of the Hardy classes using the Poisson representation given by a theorem due to Levin [13, Lecture 14.2, Theorem 3, pp. 104–105] including the subsequent remark.

Theorem 1.1 *Let $F(z)$ be an analytic function in the half-plane $\operatorname{Im} z \geq 0$ and assume that $\log |F(z)|$ has a positive harmonic majorant in $\operatorname{Im} z > 0$. Then, for all z with $\operatorname{Im} z > 0$, we have*

$$\log |F(z)| = \frac{\operatorname{Im} z}{\pi} \int_{-\infty}^{+\infty} \frac{\log |F(u)|}{|z-u|^2} du + \sum_{\operatorname{Im} r > 0} \log \left| \frac{z-r}{z-\bar{r}} \right| + A \operatorname{Im} z.$$

Here $\{r\}$ is the set of zeros of the function $F(z)$ and A is given by

$$A = \limsup_{v \rightarrow +\infty} \frac{\log |F(iv)|}{v}.$$

Hence, the Riemann Hypothesis is true if and only if the left hand-side of (4) is zero.

In this paper, we shall use Theorem 1.1 in order to compute the integral

$$\frac{a}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma + it)|}{a^2 + t^2} dt \quad (5)$$

for all $a > 0$ and $\sigma \in \mathbb{R}$. The existence of the integral (5) depends on the existence of a positive harmonic majorant for the function $\log |(s-1)\zeta(s)|$ in every half-plane $\sigma > \sigma_0$ with $\sigma_0 \in \mathbb{R}$. In particular, if $a = g(\sigma)$ with a given function g , then the integral

$$\frac{g(\sigma)}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma + it)|}{g(\sigma)^2 + t^2} dt$$

exists for all $\sigma \in \mathcal{H}_g = g^{-1}((0, +\infty))$.

Theorem 1.2 *For real numbers σ_0 and $a > 0$, the integral*

$$\frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt$$

exists, and

$$\begin{aligned} \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt &= \log \left| \frac{a + \sigma_0 - 1}{a + |1 - \sigma_0|} \zeta(a + \sigma_0) \right| \\ &\quad + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{a - \sigma_0 + \rho}{a + \sigma_0 - \rho} \right|, \end{aligned}$$

where ρ denotes the (trivial and nontrivial) zeros of the Riemann zeta-function.

We remark that if $\sigma_0 \geq 1$, then for all $a > 0$ we have

$$\frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt = \log \zeta(a + \sigma_0).$$

One can prove this particular result by an elementary method as the proof of Theorem 2.2 shows. Moreover, one can extract further interesting applications from our theorem, for instance, for $a = 1 - \sigma_0$ with $\sigma_0 < 1$, we obtain the following result.

Corollary 1.3 For every fixed real number σ_0 in $(-\infty, 1)$,

- for $0 < \sigma_0 < 1$, we have

$$\frac{1 - \sigma_0}{\pi} \int_{\operatorname{Re} s = \sigma_0} \frac{\log |\zeta(s)|}{|1 - s|^2} |ds| = -\log(2(1 - \sigma_0)) + \sum_{\beta > \sigma_0} \log \left| \frac{1 - 2\sigma_0 + \varrho}{1 - \varrho} \right|;$$

- for $-2 < \sigma_0 \leq 0$, we have

$$\frac{1 - \sigma_0}{\pi} \int_{\operatorname{Re} s = \sigma_0} \frac{\log |\zeta(s)|}{|1 - s|^2} |ds| = \log \left| \frac{\xi(2(1 - \sigma_0))}{1 - \sigma_0} \right|;$$

- for $\sigma_0 \leq -2$, we have

$$\frac{1 - \sigma_0}{\pi} \int_{\operatorname{Re} s = \sigma_0} \frac{\log |\zeta(s)|}{|1 - s|^2} |ds| = \log \left| \frac{\xi(2(1 - \sigma_0))}{1 - \sigma_0} \right| + \sum_{n=1}^{\lfloor -\sigma_0/2 \rfloor} \log \left| \frac{1 - 2(\sigma_0 + n)}{2n + 1} \right|.$$

Proof Let $a = 1 - \sigma_0$. Then the integral (5) exists for every $\sigma_0 \in (-\infty, 1)$. In view of Theorem 1.2 and since $\lim_{a \rightarrow 1 - \sigma_0} (a + \sigma_0 - 1)\zeta(\sigma_0 + a) = H(1) = 1$, we have

$$\frac{1 - \sigma_0}{\pi} \int_{\operatorname{Re} s = \sigma_0} \frac{\log |\zeta(s)|}{|1 - s|^2} |ds| = -\log(2(1 - \sigma_0)) + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{1 - 2\sigma_0 + \rho}{1 - \rho} \right|.$$

If σ_0 is fixed in $(-2, 1)$, then the zeros of $H(s)$ in the half-plane $\sigma \geq \sigma_0$ are exactly the nontrivial zeros (i.e. $\rho = \varrho = \beta + i\gamma$), so we have

$$\frac{1 - \sigma_0}{\pi} \int_{\sigma = \sigma_0} \frac{\log |\zeta(s)|}{|1 - s|^2} |ds| = -\log(2(1 - \sigma_0)) + \sum_{\beta > \sigma_0} \log \left| \frac{1 - 2\sigma_0 + \varrho}{1 - \varrho} \right|.$$

This implies the first assertion of Corollary 1.3.

In particular, for $\sigma_0 \leq 0$,

$$\sum_{\beta > \sigma_0} \log \left| \frac{1 - 2\sigma_0 + \varrho}{1 - \varrho} \right| = \sum_{\beta > 0} \log \left| \frac{1 - 2\sigma_0 + \varrho}{1 - \varrho} \right| = \log \left(\prod_{\varrho} \left| 1 - \frac{2(1 - \sigma_0)}{1 - \varrho} \right| \right).$$

On the other hand, (2) can be rewritten as

$$2\xi(s) = \prod_{\varrho} \left(1 - \frac{s}{1 - \varrho} \right).$$

It follows that, for $\sigma_0 \leq 0$,

$$\sum_{\beta > \sigma_0} \log \left| \frac{1 - 2\sigma_0 + \varrho}{1 - \varrho} \right| = \log |2\xi(2(1 - \sigma_0))|.$$

Consequently, for $-2 < \sigma_0 \leq 0$,

$$\begin{aligned} \frac{1-\sigma_0}{\pi} \int_{\sigma=\sigma_0} \frac{\log |\zeta(s)|}{|1-s|^2} |ds| &= -\log(2(1-\sigma_0)) + \sum_{\beta>\sigma_0} \log \left| \frac{1-2\sigma_0+\varrho}{1-\varrho} \right| \\ &= -\log(2(1-\sigma_0)) + \log |2\xi(2(1-\sigma_0))| \\ &= \log \left| \frac{\xi(2(1-\sigma_0))}{1-\sigma_0} \right|, \end{aligned}$$

which proves the second assertion of Corollary 1.3.

For $\sigma_0 \leq -2$, there are trivial and nontrivial zeros in the half-plane $\sigma \geq \sigma_0$. Thus,

$$\begin{aligned} \frac{1-\sigma_0}{\pi} \int_{\sigma=\sigma_0} \frac{\log |\zeta(s)|}{|1-s|^2} |ds| &= -\log(2(1-\sigma_0)) + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{1-2\sigma_0+\rho}{1-\rho} \right| \\ &= -\log(2(1-\sigma_0)) + \sum_{\beta>\sigma_0} \log \left| \frac{1-2\sigma_0+\varrho}{1-\varrho} \right| + \sum_{-2n>\sigma_0} \log \left| \frac{1-2\sigma_0-2n}{1+2n} \right| \\ &= \log(2(1-\sigma_0)) + \sum_{\beta>0} \log \left| \frac{1-2\sigma_0+\varrho}{1-\varrho} \right| + \sum_{n=1}^{\lfloor \sigma_0/2 \rfloor} \log \left| \frac{1-2\sigma_0-2n}{1+2n} \right| \\ &= \log \left| \frac{\xi(2(1-\sigma_0))}{1-\sigma_0} \right| + \sum_{n=1}^{\lfloor \sigma_0/2 \rfloor} \log \left| \frac{1-2\sigma_0-2n}{1+2n} \right|. \end{aligned}$$

This proves the last assertion of Corollary 1.3; the proof is complete. $\square$

Actually, the second assertion of Corollary 1.3 would hold on the interval $(-2, \delta]$, if there exists $0 < \delta < 1/2$ such that $\zeta(s) \neq 0$ for σ satisfying $-2 < \sigma < \delta$. In this case all nontrivial zeros are located in the strip $\delta < \sigma < 1-\delta$, hence $\sum_{\beta>\sigma} \log |(1-2\sigma+\varrho)/(1-\varrho)| = \log |\xi(2(1-\sigma))/(1-\sigma)|$ for all $\sigma \in (-2, \delta)$. Thus, we have

Corollary 1.4 *Let $\sigma_0 \in (0, 1/2]$. Then $\zeta(s) \neq 0$ in the strip $-2 < \sigma < \sigma_0$ if and only if one of the following equivalent assertions holds:*

•

$$\frac{1-\sigma_0}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0+it)|}{(1-\sigma_0)^2+t^2} dt = \log \left| \frac{\xi(2(1-\sigma_0))}{1-\sigma_0} \right|,$$

•

$$\frac{\sigma_0}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(1-\sigma_0+it)|}{\sigma_0^2+t^2} dt = -\log(2\sigma_0).$$

In particular, if $\sigma_0 = 1/2$, then the Riemann Hypothesis is true if and only if

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(1/2 + it)|}{1/4 + t^2} dt = 0.$$

One can rephrase Corollary 1.4 as follows: the Riemann Hypothesis is false if and only if there exists a real number $\sigma_0 \in [1/2, 1)$ such that

$$\frac{1 - \sigma_0}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{(1 - \sigma_0)^2 + t^2} dt > -\log(2(1 - \sigma_0)).$$

Since $|1 - 2\sigma + \varrho| \geq |1 - \varrho|$ with $\beta > \sigma$ for every given $\sigma \in (0, 1)$, we have $\sum_{\beta > \sigma} \log |(1 - 2\sigma + \varrho)/(1 - \varrho)| \geq 0$. Thus, for all $\sigma \in (0, 1)$, the inequality

$$\frac{1 - \sigma}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma + it)|}{(1 - \sigma)^2 + t^2} dt \geq -\log(2(1 - \sigma)) \quad (6)$$

holds. Notice that inequality (6) is an equality if only $\zeta(x + it) \neq 0$ for all $x > \sigma$. Thus, Corollary 1.3 is not unrelated with a hypothetical vertical zero-free region for the Riemann zeta-function. Moreover, since by [16] there is only an infinitesimal proportion of the nontrivial zeros of ζ outside the strip $|\sigma - 1/2| < \delta$ for (small) $\delta > 0$, the integral $(1 - \sigma)/\pi \int_{\text{Re } s = \sigma} \log |\zeta(s)|/|1 - s|^2 |ds|$ equals approximately $-\log(2(1 - \sigma))$ for $\sigma \in (1/2, 1)$.

2 Proof of Theorem 1.2

Let σ_0 be a fixed real number. We put $z = z(s) = i(s - \sigma_0)$ and let F be the function defined by

$$F(z) = (s - 1)\zeta(s).$$

2.1 Function F satisfies conditions of Theorem 1.2

The transformation $z(s)$ maps the half-plane $\sigma \geq \sigma_0$ to the half-plane $\text{Im } z \geq 0$. Since F is analytic in the latter half-plane $\text{Im } z \geq 0$, the zeros of $F(z)$ are of the form $r = i(\rho - \sigma_0)$ where ρ stands for a zero of $\zeta(s)$ in the half-plane $\sigma \geq \sigma_0$. Please notice that here ρ denotes both, the trivial and the nontrivial zeros (ϱ) of ζ in the half-plane $\sigma \geq \sigma_0$.

It remains to prove that F has a positive harmonic majorant in $\text{Im } z \geq 0$. Hence, it is sufficient to prove that the function $H(s) = (s - 1)\zeta(s)$ has a positive harmonic majorant in the half-plane $\sigma \geq \sigma_0$.

First case: $\sigma_0 \in [1/2, +\infty)$. It is well-known that, for all $\sigma \geq 1/2$, $s \neq 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of the real number x [10, p. 3]. Thus, for all $\sigma > \sigma_0$,

$$|H(s)| = |s| \left| 1 - (s-1) \int_1^{+\infty} \frac{\{x\}}{x^{s+1}} dx \right| \leq |s|(1+2|s-1|) \leq 4|s+1|^2 \leq 4|s-\sigma_0+2|^2.$$

It is clear that $|s-\sigma_0+2| \geq 2$ for all $\sigma > \sigma_0$, hence $4 \leq |s-\sigma_0+2|^2$. Consequently, for all $\sigma > \sigma_0$, we have $|H(s)| \leq |s-\sigma_0+2|^4$. It follows that, for all $\sigma > \sigma_0$,

$$\log |H(s)| \leq 4 \log |s-\sigma_0+2|. \quad (7)$$

Since the positive function on the right-hand side is harmonic, $\log |H(s)|$ has a positive harmonic majorant in the half-plane $\sigma > \sigma_0$, where $\sigma_0 \geq 1/2$.

Second case: $\sigma_0 < 1/2$. Let $\sigma \in (\sigma_0, 1/2]$. It follows from the functional equation (3) that

$$H(s) = \Delta(s)H(1-s),$$

where

$$\Delta(s) = \frac{1-s}{s} \chi(s) = \pi^{s-1/2} \frac{\Gamma((3-s)/2)}{\Gamma((2+s)/2)}.$$

Notice that $\Delta(s)$ is holomorphic in the half-plane $\sigma \leq 1/2$ with only real zeros at $s = -2n$ for $n = 1, 2, \dots$, and satisfies the functional equation $\Delta(s)\Delta(1-s) = 1$. By Stirling's formula, we have, for all $\sigma_0 < \sigma \leq 1/2$ and $|t| > 1$,

$$\Delta(s) = \left(\frac{|t|}{2\pi e} \right)^{1/2-\sigma} \exp \left(-i \left(\frac{3\pi}{4} + t \log \left(\frac{t}{2\pi e} \right) \right) \right) \left\{ 1 + O \left(\frac{1}{|t|} \right) \right\}.$$

Hence, for $\sigma > \sigma_0$ (with $\sigma_0 < 1/2$),

$$\log |\Delta(s)| \leq k \log |s-\sigma_0+2|,$$

where $k = k(\sigma_0)$ is a positive constant. Also, in a similar way as (7), the inequality

$$\log |H(1-s)| \leq k' \log |s-\sigma_0+2|$$

holds for some positive constant k' depending on σ_0 . Consequently, for a fixed real $\sigma_0 \in \mathbb{R}$, we have

$$\log |H(s)| \leq K \log |s-\sigma_0+2|, \quad \sigma > \sigma_0,$$

with a suitable positive constant $K = K(\sigma_0)$. Hence, $\log |H(s)|$ has a positive harmonic majorant in the half-plane $\sigma > \sigma_0$. Moreover, it follows that $\log |F(z)|$ has a

positive harmonic majorant in the half-plane $\operatorname{Im} z > 0$. Thus, F satisfies the conditions of Theorem 1.1.

2.2 Application of Theorem 1.1 to the function F

Since $F(z)$ satisfies the conditions of Theorem 1.1, for all w satisfying $\operatorname{Im} w > 0$,

$$\log |F(w)| = \frac{\operatorname{Im} w}{\pi} \int_{-\infty}^{+\infty} \frac{\log |F(u)|}{|w - u|^2} du + \sum_{\operatorname{Im} r > 0} \log \left| \frac{w - r}{w - \bar{r}} \right| + A \operatorname{Im} w, \quad (8)$$

where $\{r = i(\rho - \sigma_0)\}$ is the set of zeros of $F(z)$ and

$$\begin{aligned} A &= \limsup_{v \rightarrow +\infty} \frac{\log |F(iv)|}{v} = \limsup_{v \rightarrow +\infty} \frac{\log |H(v + \sigma_0)|}{v} \\ &= \limsup_{v \rightarrow +\infty} \frac{\log(v)}{v} + \frac{\log |\zeta(v)|}{v} = 0 \end{aligned}$$

(since $\zeta(v) \rightarrow 1$ for $v \rightarrow +\infty$).

Next we take, for every $a > 0$, $w = ia$ in formula (8), and obtain $F(w) = H(a + \sigma_0) = (a + \sigma_0 - 1)\zeta(a + \sigma_0)$ as well as

$$\frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |H(\sigma_0 - it)|}{|ia - t|^2} dt + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{a + \sigma_0 - \rho}{a - \sigma_0 + \rho} \right| = \log |H(a + \sigma_0)|,$$

i.e.,

$$\frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |H(\sigma_0 + it)|}{|a + it|^2} dt = \log |H(a + \sigma_0)| + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{a - \sigma_0 + \rho}{a + \sigma_0 - \rho} \right|.$$

Since

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\log |H(\sigma_0 + it)|}{|a + it|^2} dt &= \int_{-\infty}^{+\infty} \frac{\log |\sigma_0 - 1 + it|}{|a + it|^2} dt + \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{|a + it|^2} dt \\ &= \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{|a + it|^2} dt + \int_{\sigma=\sigma_0} \frac{\log |s - 1|}{|s - \sigma_0 + a|^2} |ds|, \end{aligned}$$

and

$$\int_{\sigma=\sigma_0} \frac{\log |s - 1|}{|s - \sigma_0 + a|^2} |ds| = \frac{\pi}{a} \log ||1 - \sigma_0| + a|,$$

this shows

$$\begin{aligned} \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt \\ = \log \left| \frac{a + \sigma_0 - 1}{a + |1 - \sigma_0|} \zeta(a + \sigma_0) \right| + \sum_{\operatorname{Re} \rho > \sigma_0} \log \left| \frac{a - \sigma_0 + \rho}{a + \sigma_0 - \rho} \right|. \end{aligned}$$

This proves Theorem 1.2.

Now we give an additional important application of Theorem 1.2, for $a = \sigma \in (0, 1]$.

Corollary 2.1 *For any $\sigma \in (0, 1]$,*

$$\frac{1}{2\pi} \int_{\operatorname{Re} s = \sigma} \frac{\log |\zeta(s)|}{|s|^2} |ds| = \frac{1}{2\sigma} \log |(2\sigma - 1)\zeta(2\sigma)| + \frac{1}{2\sigma} \sum_{\beta > \sigma} \log \left| \frac{\varrho}{2\sigma - \varrho} \right|.$$

One can interpret this corollary in terms of a Brownian motion: if B_t denotes the complex Brownian motion starting at the origin and $\tau_\sigma = \inf \{t > 0, \operatorname{Re} B_t = \sigma\}$ is the first passage time to the vertical line $\operatorname{Re} s = \sigma$ with $\sigma \in (0, 1]$, then τ_σ follows a Lévy distribution having a scale σ^2 and a location parameter 0 [3]; thus the imaginary part of B_{τ_σ} has a symmetric Cauchy distribution with scale σ and so the integral represented in Corollary 2.1 is the expectation $\mathbb{E}[\log |\zeta(B_{\tau_\sigma})|]$.

Next we shall prove Theorem 1.2 by an easy method for $\sigma > 1$.

Theorem 2.2 *For every $\sigma > 1$ and $a > 0$, we have*

$$\frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma + it)|}{a^2 + t^2} dt = \log \zeta(a + \sigma).$$

Proof Let $\sigma > 1$. Then, by the Euler product representation, $\zeta(s)$ does not vanish in the half-plane $\sigma > 1$, and the principal value $\log \zeta(s)$ is analytic in this half-plane with $\log |\zeta(s)| = \operatorname{Re} \log \zeta(s)$. Thus, for all $\sigma > 1$,

$$\log \zeta(s) = - \sum_{p \in \mathcal{P}} \log \left(1 - \frac{1}{p^s} \right).$$

Since for all $|z| < 1$ we have $-\log(1 - z) = \sum_{k \geq 1} z^k / k$,

$$\log \zeta(s) = \sum_{p \in \mathcal{P}} \sum_{k \geq 1} \frac{1}{kp^{ks}}, \quad \sigma > 1.$$

Now let $\lambda(n)$ be the function defined for all integers $n \geq 2$ by

$$\lambda(n) = \begin{cases} \frac{1}{k}, & \text{if } n = p^k, \quad k = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\log \zeta(s) = \sum_{n \geq 2} \frac{\lambda(n)}{n^s}. \quad (9)$$

Taking the real part, we obtain

$$\log |\zeta(s)| = \sum_{n \geq 2} \frac{\lambda(n)}{n^\sigma} \cos(t \log n).$$

Since $|\lambda(n)| < 1$ for all integers $n \geq 2$, we find $|\log |\zeta(s)|| < \zeta(\sigma)$ for $\sigma > 1$. So by the dominated convergence theorem we have, for $\sigma_0 > 1$,

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt &= \int_{-\infty}^{+\infty} \sum_{n \geq 2} \frac{\lambda(n)}{n^{\sigma_0}} \frac{\cos(t \log n)}{a^2 + t^2} dt \\ &= \sum_{n \geq 2} \frac{\lambda(n)}{n^{\sigma_0}} \int_{-\infty}^{+\infty} \frac{\cos(t \log n)}{a^2 + t^2} dt. \end{aligned}$$

Integrating the function $e^{iz \log n} / (a^2 + z^2)$ over the contour consisting of $[-R, R]$ and the upper semicircle $|z| = R$ and letting $R \rightarrow +\infty$, we get by the residue theorem

$$\int_{-\infty}^{+\infty} \frac{\cos(t \log n)}{a^2 + t^2} dt = \frac{\pi}{a} e^{-a \log n} = \frac{\pi}{a} \frac{1}{n^a}.$$

Thus,

$$\int_{\mathbb{R}} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt = \frac{\pi}{a} \sum_{n \geq 2} \frac{\lambda(n)}{n^{\sigma_0 + a}}.$$

In combination with (9) it follows that

$$\frac{a}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma_0 + it)|}{a^2 + t^2} dt = \log \zeta(\sigma_0 + a), \quad \sigma_0 > 1.$$

This finishes the proof. $\square$

There is another interpretation of Theorem 1.2 in terms of probability theory: Notice that the standard density function of a Cauchy distributed random variable X is given by $v \mapsto 1/\pi(1+v^2)$; hence, the expectation of $\log |\zeta(\sigma + iX)|$, for any fixed $\sigma \in \mathbb{R}$, equals

$$\mathbb{E}[\log |\zeta(\sigma + iX)|] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma + iv)|}{1+v^2} dv.$$

In particular, for $\sigma = 1/2$,

$$\mathbb{E} \left[\log \left| \zeta \left(\frac{1}{2} + iX \right) \right| \right] = \log \left(\frac{1}{3} \zeta \left(\frac{3}{2} \right) \right) + \sum_{\beta > 1/2} \log \left| \frac{1/2 + \varrho}{3/2 - \varrho} \right|.$$

Assuming the Riemann Hypothesis, the expectation of $\log |\zeta(1/2 + iX)|$ is equal to $\log \zeta(3/2) - \log 3 \approx -0.138352$.

Thereby, since the Cauchy random walk $S_n = \sum_{k=1}^n X_k$ has distribution density $v \mapsto n/\pi \cdot 1/(n^2 + v^2)$, the expectation of $\log |\zeta(1/2 + iS_n)|$ is

$$\begin{aligned} \mathbb{E} \left[\log \left| \zeta \left(\frac{1}{2} + iS_n \right) \right| \right] \\ = \log \left| \frac{n-1/2}{n+1/2} \zeta \left(n + \frac{1}{2} \right) \right| + \sum_{\beta > 1/2} \log \left| \frac{n-1/2 + \varrho}{n+1/2 - \varrho} \right|. \end{aligned} \quad (10)$$

Thus, if the Riemann Hypothesis is true, then the limit of the expectation above tends to 0 as $n \rightarrow +\infty$, i.e. the random variable $\log |\zeta(1/2 + iS_n)|$ converges in mean to zero. Without assuming any unproved hypothesis we have the following

Theorem 2.3 For $n \in \mathbb{N}^*$ sufficiently large,

$$\mathbb{E} \left[\log \left| \zeta \left(\frac{1}{2} + iS_n \right) \right| \right] = O(\log n).$$

Proof Let n be a positive integer. Then

$$\begin{aligned} \sum_{\beta > 1/2} \log \left| \frac{n-1/2 + \varrho}{n+1/2 - \varrho} \right| &= \frac{1}{2} \sum_{\beta > 1/2} \log \left(\frac{(n-1/2 + \beta)^2 + \gamma^2}{(n+1/2 - \beta)^2 + \gamma^2} \right) \\ &\leq \frac{1}{2} \sum_{\beta > 1/2} \log \left(\frac{(n+1/2)^2 + \gamma^2}{(n-1/2)^2 + \gamma^2} \right) \\ &= \frac{1}{2} \sum_{\beta > 1/2} \log \left(1 + \frac{2n}{(n-1/2)^2 + \gamma^2} \right) \\ &\leq n \sum_{\beta > 1/2} \frac{1}{(n-1/2)^2 + \gamma^2}. \end{aligned}$$

Let $N^+(t)$ count the nontrivial zeros $\varrho = \beta + i\gamma$ of the zeta-function inside the strip $1/2 < \sigma \leq 1$ with $\gamma \in [0, t]$. Then

$$\sum_{\beta > 1/2} \frac{1}{(n - 1/2)^2 + \gamma^2} = 2 \int_0^{+\infty} \frac{dN^+(t)}{(n - 1/2)^2 + t^2} = 4 \int_0^{+\infty} \frac{t N^+(t)}{((n - 1/2)^2 + t^2)^2} dt.$$

Since $N^+(T) = O(T \log T / 2\pi e)$ for $T > 0$, there exists a positive number B such that

$$4 \int_0^{+\infty} \frac{t N^+(t)}{((n - 1/2)^2 + t^2)^2} dt \leq B \frac{\log(n - 1/2)}{n - 1/2}$$

for all $n \geq 2$. Hence,

$$\sum_{\beta > 1/2} \log \left| \frac{n - 1/2 + \varrho}{n + 1/2 - \varrho} \right| \leq B \frac{n}{n - 1/2} \log \left(n - \frac{1}{2} \right), \quad n \geq 2,$$

holds for some positive constant B . In view of (10) we deduce that

$$\mathbb{E} \left[\log \left| \zeta \left(\frac{1}{2} + iS_n \right) \right| \right] = O(\log n)$$

for sufficiently large n . The proof is complete. $\square$

One can extend Theorem 2.3 by replacing the Cauchy random walk with other stochastic processes; the closest one is the symmetric α -stable process with $\alpha \in [1, 2]$ (see [17]) which includes the Cauchy random walk as a special case ($\alpha = 1$) and the Brownian motion (when $\alpha = 2$).

Next we are giving another interpretation of the integral represented in Theorem 1.2 in terms of Cesàro means under application of the pointwise ergodic theorem and some related results.

3 Ergodic Cesàro means of $\log |\zeta(s)|$

First, we want to give a brief historical background of our work. In May 7, 1857, Boole wrote a paper in which he used transformations of the form $x \mapsto x - a/x$, $a > 0$, in order to calculate some definite integrals using the following remarkable identity [6, Equation (3), p.37]: for all continuous functions f on $\mathbb{R}$, the equality $\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f(x - a/x) dx$ holds. In 1973, Adler and Weiss [1] proved that the Boole-transformation $x \mapsto x - 1/x$ preserves Lebesgue measure, and, in addition, that it is ergodic. In 2009, Lifshits and Weber [15] published a paper in which they proved that almost surely

$$\frac{1}{N} \sum_{n=1}^N \zeta \left(\frac{1}{2} + iS_n \right) = 1 + O \left(\frac{(\log N)^b}{\sqrt{N}} \right), \quad b > 2,$$

where $S_n = \sum_{k \leq n} X_k$ is a Cauchy random walk (and each X_j is a Cauchy distributed random variable). Hence, the Cesàro mean of $\zeta(s)$ with respect to the Cauchy random walk $s = 1/2 + iS_n$ converges almost surely to 1, which indicates that the values of the Riemann zeta-function are small on average. Recently, in 2012, Steuding [19] published a paper which contains beautiful results concerning the Cesàro means of the Riemann zeta-function on vertical lines $s = \sigma + i\mathbb{R}$ with respect to Boole's ergodic transformation $T: x \mapsto (x - 1/x)/2$ (with $T0 = 0$). Steuding proved that, for $\sigma > -1/2$, the Cesàro mean of $\zeta(s + iT^n x)$ exists for almost all real values x , where the notation T^n means $T^n = T \circ T^{n-1}$. In addition,

$$\begin{aligned} \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta(s + iT^n x) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\zeta(s + i\tau)}{1 + \tau^2} d\tau \\ &= \begin{cases} \zeta(s+1) - \frac{2}{s(2-s)}, & -\frac{1}{2} < \sigma < 1, \\ \zeta(2+it) - \frac{1}{1+t^2}, & s = 1+it, \quad t \in \mathbb{R}, \\ \zeta(s+1), & \sigma > 1. \end{cases} \end{aligned}$$

The convergence holds for almost all $x \in \mathbb{R}$. In particular, for almost all $x \in \mathbb{R}$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n \leq N} \zeta\left(\frac{1}{2} + iT^n x\right) = \zeta\left(\frac{3}{2}\right) - \frac{8}{3} \approx -0.05429.$$

In this paper, we are interested in a generalized Boole-transformation $T_a: \mathbb{R} \rightarrow \mathbb{R}$ defined, for a fixed real number $a \in (0, +\infty)$, by

$$T_a x = \begin{cases} \frac{1}{2} \left(x - \frac{a^2}{x} \right) & \text{if } x \neq 0, \\ T_a 0 = 0 & \text{otherwise.} \end{cases} \quad (11)$$

Let $\mathcal{B}$ denote the Borel sigma-algebra associated with $\mathbb{R}$. Hence, $(\mathbb{R}, \mathcal{B})$ is a measurable space. It is easy to see that T_a is measurable and, if we put $t = T_a x$, then $dt/(a^2 + t^2) = 2dx/(a^2 + x^2)$. It turns out that, for any Lebesgue integrable function f ,

$$\int_{\mathbb{R}} f \circ T_a d\mathbf{P}_a = \frac{a}{\pi} \int_{\mathbb{R}} \frac{f(T_a x)}{a^2 + x^2} dx = \frac{a}{\pi} \int_{\mathbb{R}} \frac{f(t)}{a^2 + t^2} dt = \int_{\mathbb{R}} f d\mathbf{P}_a, \quad (12)$$

where $\mathbf{P}_a$ is the probability measure, for $A \in \mathcal{B}$, defined by

$$\mathbf{P}_a(A) = \frac{a}{\pi} \int_A \frac{d\tau}{a^2 + \tau^2}.$$

Hence, it follows from (12) that T_a is a measure-preserving transformation with respect to $\mathbf{P}_a$. Obviously, the only T_a -invariant sets are $A = \{0\}$ and $B = \mathbb{R}$ for which $\mathbf{P}_a(A) = 0$ and $\mathbf{P}_a(B) = 1$. Consequently, for any fixed real number $a > 0$, $(\mathbb{R}, \mathcal{B}, \mathbf{P}_a, T_a)$ is

an ergodic system. Then, by the pointwise ergodic theorem (or Birkhoff–Khinchin theorem [5, 12], resp. [7, Section 3.2]), we have, for any $f \in L^1(\mathbf{P}_a)$ and for almost all $x \in \mathbb{R}$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T_a^n x) = \int_{\mathbb{R}} f \, d\mathbf{P}_a. \quad (13)$$

One may notice that for every $a > 0$ there exists a functional relation between T_a and T , namely, $T_a x = aTx/a$ for all $x \in \mathbb{R}$, so that $T_1 \equiv T$. Thus, (13) is equivalent to

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^{N-1} f(aT^n x) = \int_{\mathbb{R}} f_a \, d\mathbf{P}_1,$$

where $f_a(\tau) = f(a\tau)$, being valid for almost all $x \in \mathbb{R}$. Notice that the exceptional values of x constitute the set

$$\mathcal{E} = \left\{ \alpha_n \in \mathbb{R} : \alpha_n = \pm \alpha_{n-1} \pm \sqrt{\alpha_{n-1}^2 + a^2} \text{ with } \alpha_0 = 0 \right\}_{n \geq 1}, \quad (14)$$

that is the set of preimages of 0, i.e. $\{T_a^{-n}0\}_{n \geq 0}$.

Let $a > 0$ and σ be a fixed number from $\mathbb{R}$, then by Theorems 2.2 and 1.2, the function $t \rightarrow \log |\zeta(\sigma + it)|$ is $\mathbf{P}_a$ -integrable. Consequently, the pointwise ergodic theorem implies the following result.

Theorem 3.1 *Let $\sigma \in \mathbb{R}$ be a fixed real number and $a > 0$. Then, for almost all $x \in \mathbb{R}$,*

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log |\zeta(\sigma + iT_a^n x)| = \frac{a}{\pi} \int_{\mathbb{R}} \frac{\log |\zeta(\sigma + it)|}{a^2 + t^2} \, dt.$$

Since the real logarithm is continuous, the left-hand side of the formula in Theorem 3.1 can be interpreted as the limit of the geometric mean of $|\zeta(\sigma + iT_a^n x)|$ with respect to the ergodic transformation T_a . For the standard Boole transformation T , $a = 1$, we thus have

Corollary 3.2 *For any fixed $\sigma \in \mathbb{R}$ and almost all $x \in \mathbb{R}$,*

$$\lim_{N \rightarrow +\infty} \left| \prod_{n=0}^{N-1} \zeta(\sigma + iT^n x) \right|^{1/N} = \frac{|\sigma \zeta(\sigma + 1)|}{1 + |1 - \sigma|} \prod_{\beta > \sigma} \left| \frac{1 - \sigma + \varrho}{1 + \sigma - \varrho} \right|.$$

In particular, for $\sigma = 1/2$, we get under assumption of the Riemann Hypothesis

$$\lim_{N \rightarrow +\infty} \left| \prod_{n=0}^{N-1} \zeta\left(\frac{1}{2} + iT^n x\right) \right|^{1/N} = \frac{1}{3} \zeta\left(\frac{3}{2}\right). \quad (15)$$

We would like to mention that Srichan [18] has obtained a similar result in his thesis by a different method. He proved, using the Tsang Lemma [22, Lemma 5, p. 378], that for almost all $x \in \mathbb{R}$ and $\sigma \in [1/2, 2]$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{0 \leq n < N} \log \zeta(s + iT^n x) = \log \zeta(s + 1) + 2 \sum_{\beta > \sigma} \int_0^{\beta - \sigma} \frac{d\alpha}{1 + (\gamma - t - i\alpha)^2} - 2 \int_{\min(1-\sigma, 0)}^{1-\sigma} \frac{d\alpha}{1 + (t + i\alpha)^2}.$$

Thus, under assumption of the truth of the Riemann Hypothesis,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{0 \leq n < N} \log \left| \zeta \left(\frac{1}{2} + iT^n x \right) \right| = \log \left(\frac{1}{3} \zeta \left(\frac{3}{2} \right) \right)$$

for almost all $x \in \mathbb{R}$, which is the same result as in (15).

Moreover, with $a = 1/2$ in (11) one can find the following powerful result as Theorem 4.2 from Steuding in [19]. Notice that in [19] the same notation is used for the transformation T with $T \equiv T_1$ and the special parameter $a = 1$.

Corollary 3.3 *For almost all $x \in \mathbb{R}$,*

$$\lim_{N \rightarrow +\infty} \left| \prod_{n=0}^{N-1} \zeta \left(\frac{1}{2} + iT_{1/2}^n x \right) \right|^{1/N} = \prod_{\beta > 1/2} \left| \frac{\varrho}{1 - \varrho} \right|. \quad (16)$$

In particular, the Riemann hypothesis is true if and only if, for almost all $x \in \mathbb{R}$,

$$\lim_{N \rightarrow +\infty} \left| \prod_{n=0}^{N-1} \zeta \left(\frac{1}{2} + iT_{1/2}^n x \right) \right|^{1/N} = 1.$$

3.1 Some numerical experiments

Let $a > 0$ be a fixed real number. Since $(\mathbb{R}, \mathcal{B}, \mathbf{P}_a, T_a)$ is an ergodic system, for almost every initial value $x \in \mathbb{R}$ the orbit $\{T_a^n x\}$ lies dense in $\mathbb{R}$, and by the pointwise ergodic theorem the sojourn time of this orbit in any given interval $A \in \mathcal{B}$ is given by the positive quantity

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^{N-1} \mathbb{1}_A(T_a^n x) = \mathbf{P}_a(A),$$

where $\mathbb{1}_A$ is the indicator function of A , i.e.

$$\mathbb{1}_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Poincaré's recurrence theorem affirms that almost every point of any interval $A \in \mathcal{B}$ returns to A infinitely often. If we denote $\tau_A(x)$ for the first return time of the orbit of x to A , i.e. $\tau_A(x) = \min \{n \in \mathbb{N}^* : T_a^n x \in A\}$, then by Kac's lemma [11] the mean return time is given by the positive quantity

$$\int_A \tau_A d\mathbf{P}_a = \mathbf{P}_a(\{\tau_A < +\infty\}).$$

Since the system is ergodic, the integral above equals $1/\mathbf{P}_a(A)$. For the sake of completeness, we consider the most interesting case, that is $a = 1/2$. Then the first element of the orbit for $x = 7$ in the interval $A = (100, 1000)$ appears after $n = 498$ iterations, (that is $T_{1/2}^{499} 7 \approx 122.194$) whereas the first return of the orbit of $x = 7$ to the interval A is expected at time $n = 498 + (\mathbf{P}_{1/2}(A))^{-1} \approx 1196$. The numerical calculations show that large values of $T_{1/2}^n x$ are rare and that small values are more frequent.

One can calculate the n -th iteration of T_a easily using recursive sequences. Indeed, let x be a fixed real number and let $(u_n)_{n \in \mathbb{N}}$ be the sequence in $\mathbb{R} \setminus \pi\mathbb{Z}$ defined by $a \cot(u_n) = T_a^n x$, where $a > 0$ is fixed. One can observe that $a \cot(\pm \pi/2^m)$ is for every $m \in \mathbb{N}^*$ an exceptional value of x from the set $\mathcal{E}$ defined in (14), i.e. $\{a \cot(\pm \pi/2^m)\}_{m \geq 0} = \{T_a^{-n} 0\}_{n \geq 1}$. Then, for almost all real values x , we have

$$\begin{aligned} T_a^{n+1} x &= T_a(T_a^n x) = \frac{1}{2} \left(T_a^n x - \frac{a^2}{T_a^n x} \right) \\ &= \frac{a}{2} \left(\cot(u_n) - \frac{1}{\cot(u_n)} \right) = a \cot(2u_n). \end{aligned}$$

Thus, $\cot(u_{n+1}) = \cot(2u_n)$, which is equivalent to $u_{n+1} = 2u_n \bmod \pi$ and $u_n = 2^n u_0 \bmod \pi$ for every $n \in \mathbb{N}$. Then, finally, for every positive integer n and almost all $x \in \mathbb{R}$,

$$T_a^n x = a \cot \left(2^n \cot^{-1} \left(\frac{x}{a} \right) \right).$$

In particular, for almost all $x \in \mathbb{R}$,

$$T_{1/2}^n x = \frac{1}{2} \cot(2^n \cot^{-1}(2x)).$$

By the well-known formula $1/2 + i/2 \cot(\theta) = 1/(1 - e^{2i\theta})$, the product (16) can be rewritten as follows: for almost all real x ,

$$\lim_{N \rightarrow +\infty} \left| \prod_{n=0}^{N-1} \zeta \left(\frac{1}{1 - z_n(x)} \right) \right|^{1/N} = \prod_{\beta < 1/2} \left| 1 - \frac{1}{\varrho} \right|, \quad (17)$$

where $z_n(x) = e^{i2^{n+1} \cot^{-1}(2x)}$. In fact, since the function $e^{i\theta}$ is 2π -periodic, for every $\theta \in \mathbb{R}$ the equality $e^{i\theta} = e^{i2\pi\{\theta/2\pi\}}$ holds, where $\{\cdot\}$ is the fractional part function.

Table 1 Values of the geometric mean $|\prod_{n=0}^{N-1} \zeta(1/(1 - z_n(x)))|^{1/N}$ for $N = 100$ and $N = 1000$ with the initial values $x = 7$ and $x = \sqrt{7}$

	$N = 100$	$N = 1000$
$x = 7$	0.963292...	0.994179...
$x = \sqrt{7}$	0.974527...	1.00604...

It seems to be better to apply this equality to $z_n(x)$ (since $2^{n+1} \cot^{-1}(2x)$ takes large values for large n). Thus, for almost all $x \in \mathbb{R}$, let

$$z_n(x) = \exp\left(2i\pi \left\{ \frac{2^n \cot^{-1}(2x)}{\pi} \right\}\right), \quad n \geq 0.$$

Some numerical evaluations of the geometric mean $|\prod_{n=0}^{N-1} \zeta(1/(1 - z_n(x)))|^{1/N}$ are given in Table 1.

Of course, the numerical computation of the geometric mean $(\prod_{n=0}^{N-1} |\zeta(1/(1 - z_n(x)))|)^{1/N}$ for large values of N and arbitrary $x \in \mathbb{R}$ does not provide any rigorous and reasonable approach to the Riemann Hypothesis because “almost all nontrivial zeros of the Riemann zeta function are close to the critical line” (proved first by Selberg [16] and later by Levinson and Montgomery [14]), that means that the product on the right-hand side of (17) is close to 1. As a matter of fact, Gourdon [9] stated, using the Odlyzko–Schönhage algorithm, that the first 10^{13} nontrivial zeros of the Riemann zeta-function are on the critical line. Thus, since $|\varrho_n| \sim 2\pi n/\log n$ as $n \rightarrow \infty$, where ϱ_n denotes the n -th nontrivial zero of ζ , the first “exceptional” nontrivial zeros, denoted as ϱ_{ex} ($\text{Re } \varrho_{\text{ex}} \neq 1/2$), must be greater than 10^{10} . It follows that $|1 - \varrho_{\text{ex}}| \approx |\varrho_{\text{ex}}|$ and $|(1 - \varrho_{\text{ex}})/\varrho_{\text{ex}}| = 1 + o(1)$, hence $\prod_{\beta < 1/2} |1 - 1/\varrho|$ could still be $\approx \sqrt{\xi(1)/\xi(0)} + o(1) = 1 + o(1)$. But, a quantitative approximate formula for our geometric mean with respect to N could provide a reasonable approach to the Riemann Hypothesis, for example, if for almost all real values x ,

$$\prod_{n=0}^{N-1} \left| \zeta\left(\frac{1}{1 - z_n(x)}\right) \right| = O(N^\alpha), \quad \text{as } N \rightarrow \infty,$$

where $\alpha \in \mathbb{R}$, then the Riemann Hypothesis would be true thanks to Corollary 3.3.

Acknowledgments Lahoucine Elaissoui would like to thank Jörn Steuding for his support and valuable remarks. Furthermore, the authors would like to express their gratitude to the anonymous referees.

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