Reflections in small continuous images of ordered spaces

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Abstract We prove that countable i -weight reflects in continuous images of weight $\leq \omega_1$ for all Tychonoff spaces while separability reflects in continuous images of weight $\leq \omega_1$ for GO spaces. If X is a GO space and all continuous images of X of weight $\leq \kappa^+$ have tightness at most κ , then $t(X) \leq \kappa$. All continuous images of weight $\leq \omega_1$ of a GO space X have countable pseudocharacter if and only if X is hereditarily Lindelöf. Besides, all continuous images of weight $\leq \omega_1$ of a linearly ordered space X have G_δ -diagonal if and only if X second countable.

Keywords Small continuous images · Weight · Reflection · Continuous map · Linearly ordered space · GO space · i -weight · G_δ -diagonal · Tightness · Pseudocharacter

Mathematics Subject Classification 54A25 · 54C05 · 54C25 · 54D55 · 54F05

1 Introduction

Say that a property $\mathcal{P}$ is reflectable in continuous images of weight κ if a space X has $\mathcal{P}$ whenever all continuous images of X of weight $\leq \kappa$ have $\mathcal{P}$. Studying reflections of a property $\mathcal{P}$ helps to determine whether a space X has $\mathcal{P}$ by analyzing relatively small objects associated with X . First results on reflectability in small continuous images were obtained by Tkachenko in [7] where he established that weight and spread are

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reflectable in small continuous images. In [11], Tkachuk systematically studied this kind of reflection. He proved, among other things, that character and pseudocharacter reflect in small continuous images for all Tychonoff spaces. In compact spaces, tightness $\leq \kappa$ is reflectable in continuous images of weight at most κ^+ while the Fréchet–Urysohn property fails to reflect in continuous images of weight $\leq \omega_1$ under $\text{MA} + \neg\text{CH}$. Also, the property of having a G_δ -point reflects in small continuous images if Jensen’s Axiom $\diamond$ holds.

The work presented in the paper [11] was continued in [9] where it was established that π -weight and π -character reflect in small continuous images for all Tychonoff spaces. It was also proved in [9] that it is independent of ZFC whether the countable network weight or Fréchet–Urysohn property reflect in continuous images of weight $\leq \omega_1$.

If a topological property $\mathcal{P}$ is invariant under continuous images and reflects in continuous images of weight λ , then a space X has $\mathcal{P}$ if and only if all continuous images of X of weight $\leq \lambda$ have $\mathcal{P}$. If we do not have invariance of $\mathcal{P}$ under continuous images, then the “only if” part often turns out to be false because having $\mathcal{P}$ in all continuous images of weight $\leq \lambda$ usually implies very strong additional properties of the space X .

With this idea in mind we analyze spaces that have pseudocharacter $\leq \kappa$ in all continuous images of weight $\leq \kappa^+$. This helps to prove that i-weight and diagonal number in GO spaces are determined by their values in small continuous images. We establish, for example, that all continuous images of a GO space X have countable pseudocharacter if and only if X is hereditarily Lindelöf. This is, of course, much stronger, than concluding that $\psi(X) = \chi(X) \leq \omega$; besides, all continuous images of weight $\leq \omega_1$ of a GO space X have a G_δ -diagonal if and only if $\text{hl}(X) = \text{iw}(X) = \omega$. Therefore the G_δ -diagonal property reflects in continuous images of weight $\leq \omega_1$ for GO spaces.

We also prove that a GO space X is separable if and only if all continuous images of X of weight $\leq \omega_1$ are separable; this is an example of improving the behavior of separability in presence of an order structure because separability does not reflect in continuous images of weight ω_1 even for compact spaces (see [11]).

The main result of this paper is a solution of Problem 3.10 from the paper [11]; we show that countable i-weight reflects in continuous images of weight $\leq \omega_1$ for all Tychonoff spaces. It is still an open problem whether countable tightness reflects in continuous images of weight $\leq \omega_1$ (see [11, Problem 3.1]). In this paper we establish that a GO space X has tightness at most κ whenever all continuous images of X of weight $\leq \kappa^+$ have tightness $\leq \kappa$.

2 Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X , the family $\tau(X)$ is its topology, $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$ and $\tau(x, X) = \{U \in \tau(X) : x \in U\}$ for any $x \in X$. The set $\mathbb{R}$ is the real line with its usual topology.

A map $f: X \rightarrow Y$ is called a *condensation* if it is a continuous bijection; in this case we say that X *condenses onto* Y . A space X is *Fréchet–Urysohn* if for any $A \subset X$ and $x \in \overline{A}$ there exists a sequence $\{a_n : n \in \omega\} \subset A$ such that $a_n \rightarrow x$. The space X

is *sequential* if for any non-closed $A \subset X$ we can find a sequence $\{a_n : n \in \omega\} \subset A$ with $a_n \rightarrow x$ for some point $x \in X \setminus A$. Given an infinite cardinal κ , we say that $t(x, X) \leq \kappa$ if for any set $A \subset X$ with $x \in \overline{A}$ there exists a set $B \subset A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$. The cardinal $t(X) = \sup\{t(x, X) : x \in X\}$ is called the *tightness* of the space X .

The *weight* $w(X)$ of a space X is the minimal cardinality of a base in X . The cardinal $iw(X) = \min\{\kappa : \text{the space } X \text{ has a weaker Tychonoff topology of weight } \leq \kappa\}$ is called the *i-weight* of X . A family $\mathcal{N}$ of subsets of a space X is a *network* in X if for any $U \in \tau(X)$, there exists a subfamily $\mathcal{N}' \subset \mathcal{N}$ with $U = \bigcup \mathcal{N}'$. The cardinal $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network of } X\}$ is called the *network weight* of X .

Let $s(X) = \sup\{|D| : D \text{ is a discrete subset of } X\}$; the cardinal $s(X)$ is called the *spread* of the space X . We denote by $\text{ext}(X)$ the cardinal $\sup\{|D| : D \text{ is a closed discrete subset of } X\}$ which is called the *extent* of X . Say that $l(X) \leq \kappa$ if every open cover of X has a subcover of cardinality $\leq \kappa$; then $hl(X) \leq \kappa$ if $l(Y) \leq \kappa$ for any $Y \subset X$.

Given a space X , the minimal cardinality of a local base at a point $x \in X$ is denoted by $\chi(x, X)$ and called the *character of X at x* . The *character* $\chi(X)$ of the space X is the cardinal $\sup\{\chi(x, X) : x \in X\}$. If κ is an infinite cardinal then a G_κ -set in X is the intersection of at most κ -many open subsets of X ; the expression $\psi(F, X) \leq \kappa$ says that F is a G_κ -subset of X . The cardinal $\psi(x, X) = \min\{\kappa : \{x\} \text{ is a } G_\kappa\text{-subset of } X\}$ is called the *pseudocharacter of X at x* and $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ is the *pseudocharacter of X* . We say that the *diagonal number* of a space X does not exceed κ if the diagonal $\{(x, x) : x \in X\}$ of the space X is a G_κ -set in $X \times X$. The space X has a G_δ -diagonal if its diagonal number is countable.

A family $\mathcal{B} \subset \tau^*(X)$ is a π -base of X (at a point $x \in X$) if for every $U \in \tau^*(X)$ ($U \in \tau(x, X)$ respectively) there is $B \in \mathcal{B}$ such that $B \subset U$. The cardinal $\pi w(X)$ (called the π -weight of X) is the minimal cardinality of a π -base of the space X . The minimal cardinality of a π -base at a point $x \in X$ is called the π -character of X at x and denoted by $\pi \chi(x, X)$; besides, $\pi \chi(X) = \sup\{\pi \chi(x, X) : x \in X\}$. If $f_t : X \rightarrow Y_t$ is a map for any $t \in T$ and $A = \{f_t : t \in T\}$ then the *diagonal product* $\varphi = \Delta(A) : X \rightarrow \prod_{t \in T} Y_t$ of the family A is defined by $\varphi(x)(t) = f_t(x)$ for any $x \in X$ and $t \in T$. For any spaces X and Y , the set $C(X, Y)$ consists of continuous functions from X to Y ; we write $C(X)$ instead of $C(X, \mathbb{R})$.

If $<$ is a linear order of a set X , then the sets $(a, b) = \{x \in X : a < x < b\}$ as well as $(a, \rightarrow) = \{x \in X : x > a\}$ and $(\leftarrow, b) = \{x \in X : x < b\}$ are called *intervals*. It is said that X is a *linearly ordered space* or *LOTS* if $\tau(X)$ is generated by the intervals of a linear order on X . We will also use the sets $(\leftarrow, a] = \{x \in X : x \leq a\}$ and $[a, \rightarrow) = \{x \in X : x \geq a\}$. Recall that X is a *GO space* or *generalized ordered space* if X embeds in a linearly ordered space.

The rest of our notation is standard and follows the book [3].

3 Reflections and order structure

We will show that *i-weight* reflects in small continuous images for all Tychonoff spaces while several properties that are not reflectable even in compact spaces, do reflect in GO spaces.

It was proved in [11] that compactness, normality and Lindelöf property are not reflectable in continuous images of weight ω_1 . It turns out that a similar result holds for linearly ordered spaces.

Example 3.1 Let X be the ordinal ω_2 with its order topology. Then any continuous image of X of weight $\leq \omega_1$ is compact. Therefore, compactness, paracompactness, Lindelöf property and realcompactness are not reflected in continuous images of weight $\leq \omega_1$ for linearly ordered spaces.

Proof Suppose that $f: X \rightarrow Y$ is a continuous onto map and $w(Y) \leq \omega_1$. Take a dense set $A \subset Y$ with $|A| \leq \omega_1$ and choose a set $B \subset X$ such that $|B| \leq \omega_1$ and $f(B) = A$. The set $K = \overline{B}$ is easily seen to be compact so $Y = f(K)$ is compact as well. $\square$

It was established in [11] that separability is not reflectable in continuous images of weight $\leq \omega_1$ even in compact spaces. The following theorem shows that the situation is different in GO spaces.

Theorem 3.2 *If X is a GO space such that every continuous image of X of weight κ^+ has density $\leq \kappa$, then $d(X) \leq \kappa$, i.e., in GO spaces, density is reflectable in small continuous images.*

Proof Take a linearly ordered compactification K of the space X . If L is a continuous image of K of weight $\leq \kappa^+$ then the image of X is a dense subspace of L so L has a dense subspace of density $\leq \kappa$ which shows that $d(L) \leq \kappa$. The Souslin number reflects in continuous images of weight κ^+ by [11, Theorem 2.6] so $c(K) \leq \kappa$ and hence $\chi(K) \leq \kappa$.

Therefore $t(K) \leq \chi(K) \leq \kappa$; since tightness in compact spaces is invariant under continuous images, we have $\pi\chi(L) \leq t(L) \leq t(K) \leq \kappa$ whenever L is a continuous image of the space K with $w(L) \leq \kappa^+$. This, together with $d(L) \leq \kappa$ implies that $\pi w(L) \leq \kappa$. Since π -weight is reflectable in continuous images of weight $\leq \kappa^+$ for all spaces (see [9, Theorem 3.5]), we conclude that $\pi w(K) \leq \kappa$. As a consequence, $d(X) \leq \pi w(X) = \pi w(K) \leq \kappa$. $\square$

If $hl(X) \leq \kappa$, then $hl(Y) \leq \kappa$ and hence $\psi(Y) \leq \kappa$ for every continuous image Y of the space X . We will show that the converse is true for GO spaces in a stronger form, i.e., if all continuous images of weight $\leq \kappa^+$ of a GO space X have pseudocharacter at most κ , then $hl(X) \leq \kappa$. It is not clear whether this can be proved for all Tychonoff spaces.

Proposition 3.3 *Assume that κ is an infinite cardinal and all continuous images of weight $\leq \kappa^+$ of a space X have pseudocharacter at most κ . Then*

- (a) *every closed discrete C -embedded subset of X has cardinality $\leq \kappa$;*
- (b) *every closed subset $F \subset X$ is a G_κ -set.*

Proof (a) Suppose D is a closed discrete C -embedded subset of X such that $|D| = \kappa^+$. Take a set $K \subset \mathbb{R}^{\kappa^+}$ homeomorphic to the one-point compactification of a discrete

space of cardinality κ^+ and let $p_\alpha: \mathbb{R}^{\kappa^+} \rightarrow \mathbb{R}$ be the projection onto the α -th factor for every $\alpha < \kappa^+$.

Take a surjective map $f: D \rightarrow K$ which is automatically continuous. Since D is C -embedded in X , for every $\alpha < \kappa^+$ we can find a continuous function $g_\alpha: X \rightarrow \mathbb{R}$ such that $g_\alpha|_D = p_\alpha \circ f$. The diagonal product $g = \Delta\{g_\alpha: \alpha < \kappa^+\}$ maps X continuously onto a set $Y \subset \mathbb{R}^{\kappa^+}$ so $w(Y) \leq \kappa^+$. It follows from our choice of the functions g_α that $g|_D = f$ so $Y = g(X) \supset g(D) = f(D) = K$. Then $\psi(K) \leq \psi(Y) \leq \kappa$ while $\psi(K) = \kappa^+$; this contradiction proves (a).

(b) Let $\Phi = \{f \in C(X) : f(F) \subset \{0\}\}$ and consider the map $\varphi = \Delta\Phi$. It is straightforward that there is a point $y \in Y = \varphi(X)$ such that $\varphi^{-1}(y) = F$. If Z is a continuous image of Y with $w(Z) \leq \kappa^+$, then Z is also a continuous image of X so $\psi(Z) \leq \kappa$. It was proved in [11] that pseudocharacter is reflectable in small continuous images, so $\psi(Y) \leq \kappa$. Now, it follows from $\psi(y, Y) \leq \kappa$ and $F = \varphi^{-1}(y)$ that $\psi(F, X) \leq \kappa$, i.e., F is a G_κ -subset of X . $\square$

Corollary 3.4 *Assume that κ is an infinite cardinal and all continuous images of weight $\leq \kappa^+$ of a normal space X have pseudocharacter at most κ . Then $s(X) \leq \kappa$.*

Proof It follows from Proposition 3.3(a) and normality of X that $\text{ext}(X) \leq \kappa$. A standard argument can be used to show that $\text{ext}(X) \leq \kappa$ together with the fact that every closed subset of X is a G_κ -set (see Proposition 3.3(b)) imply that $s(X) \leq \kappa$. $\square$

Corollary 3.5 *Given an infinite cardinal κ , all continuous images of weight $\leq \kappa^+$ of a GO space X have pseudocharacter at most κ if and only if $\text{hl}(X) \leq \kappa$.*

Proof If X is any space with $\text{hl}(X) \leq \kappa$ then $\psi(Y) \leq \text{hl}(Y) \leq \kappa$ for any continuous image Y of the space X ; this proves necessity.

Now, if all continuous images of weight $\leq \kappa^+$ of a GO space X have pseudocharacter at most κ , then $s(X) \leq \kappa$ by Corollary 3.4 because GO spaces are normal. Since spread coincides with hereditary Lindelöf number in any GO space (see [5, Proposition 2.10(b)]), we have $\text{hl}(X) = s(X) \leq \kappa$. $\square$

Theorem 3.6 *Suppose that κ is an infinite cardinal and all continuous images of weight $\leq \kappa^+$ of a space X have i -weight at most κ .*

- (a) *If $\text{fl}(X) \leq \kappa$, then $\text{iw}(X) \leq \kappa$.*
- (b) *If there exists a closed discrete C -embedded set $A \subset X$ such that $|A| = \kappa$, then $\text{iw}(X) \leq \kappa$.*

Proof We will prove statements (a) and (b) simultaneously indicating the differences in the respective reasonings. For the sake of brevity and only in this proof, we will say that a space Y is *small* if $w(Y) \leq \kappa^+$. It suffices to derive a contradiction from the assumption that either the hypothesis of (a) or the hypothesis of (b) holds and $\text{iw}(X) > \kappa$.

If we prove (b), fix a closed discrete C -embedded faithfully enumerated set $A = \{a_\alpha : \alpha < \kappa\} \subset X$. Observe, for both (a) and (b), that $\psi(Y) \leq \text{iw}(Y) \leq \kappa$ for any small continuous image Y of the space X . Since pseudocharacter reflects in small

continuous images by [11, Corollary 2.15], we have $\psi(X) \leq \kappa$. Recall that a set $E \subset C(X)$ separates a set $A \subset X$ from $X \setminus A$ if for any points $x \in A$ and $y \in X \setminus A$, there exists $f \in E$ such that $f(x) \neq f(y)$. If E separates $\{x\}$ from $X \setminus \{x\}$ for any $x \in X$, we say that E separates the points of X . We will make use of the following statements whose proofs are an easy exercise:

- ($\star$) a space Z has i-weight at most κ if and only if there exists a set $E \subset C(Z)$ of cardinality at most κ which separates the points of Z ;
- ($\star\star$) if $\psi(Z) \leq \kappa$ and $|A| \leq \kappa$ for some set $A \subset Z$, then there exists a set $E \subset C(X)$ of cardinality at most κ , that separates A from $Z \setminus A$.

For the case (b) only, choose a faithfully indexed dense set $B = \{b_\alpha : \alpha < \kappa\}$ in the space $\mathbb{R}^{\kappa^+}$. We are going to construct by transfinite induction a set $\{f_\alpha : \alpha < \kappa^+\} \subset C(X)$. It is an easy exercise that $A \cup K$ is C -embedded in X for any finite $K \subset X$. Since $|X| \geq \text{iw}(X) > \kappa$, we can find distinct points $x_0, y_0 \in X \setminus A$ and a function $f_0 \in C(X)$ such that $f_0(x_0) \neq f_0(y_0)$ and $f_0(a_\alpha) = b_\alpha(0)$ for all $\alpha < \kappa$. Assume that $\alpha < \kappa^+$ and we have constructed a set $\{f_\beta : \beta < \alpha\} \subset C(X)$ and a set $\{x_\beta, y_\beta : \beta < \alpha\} \subset X \setminus A$ such that

- (1) $f_\beta(a_\gamma) = b_\gamma(\beta)$ for all $\beta < \alpha$ and $\gamma < \kappa$;
- (2) for any $\beta < \alpha$ we have the equality $f_\gamma(x_\beta) = f_\gamma(y_\beta)$ for all $\gamma < \beta$ while $f_\beta(x_\beta) \neq f_\beta(y_\beta)$.

If we are proving (a), then we need only to ensure that (2) holds. Since i-weight of X is greater than κ , no subset of $C(X)$ of cardinality $\leq \kappa$ can separate the points of X by ($\star$). If the family $Q = \{f_\beta : \beta < \alpha\}$ separates the points of $X \setminus A$, then we can use ($\star\star$) to find a set $E \subset C(X)$ of cardinality $\leq \kappa$ such that the set $Q \cup E$ separates the points of X which is a contradiction. Therefore there exist distinct points $x_\alpha, y_\alpha \in X \setminus A$ such that $f_\beta(x_\alpha) = f_\beta(y_\alpha)$ for all $\beta < \alpha$.

The set $A \cup \{x_\alpha, y_\alpha\}$ being C -embedded in the space X , we can find a function $f_\alpha \in C(X)$ such that $f_\alpha(x_\alpha) \neq f_\alpha(y_\alpha)$ and (only for the case (b)) $f_\alpha(x_\gamma) = b_\gamma(\alpha)$ for all $\gamma < \kappa$. It is clear that conditions (1) and (2) are now satisfied for all $\beta \leq \alpha$ so our inductive procedure can be continued to construct sets $\{f_\beta : \beta < \kappa^+\} \subset C(X)$ and $\{x_\beta, y_\beta : \beta < \kappa^+\} \subset X \setminus A$ for which properties (1) and (2) hold for all $\alpha < \kappa^+$.

If $f = \Delta\{f_\alpha : \alpha < \kappa^+\}$ then the space $Y = f(X) \subset \mathbb{R}^{\kappa^+}$ is a small continuous image of X . If we are proving (b), then property (1) implies that $f(A) = B$ so $B \subset Y$ and hence Y is dense in the space $\mathbb{R}^{\kappa^+}$. Our next step is to observe that any continuous map from Y to a space of weight $\leq \kappa$ factorizes through a face of cardinality $\leq \kappa$. In the case (b) this follows from density of Y in $\mathbb{R}^{\kappa^+}$. If we are proving (a), then Y is not necessarily dense in $\mathbb{R}^{\kappa^+}$ but the above-mentioned factorization property follows from $\text{l}(Y) \leq \kappa$.

We have the inequality $\text{iw}(Y) \leq \kappa$ so there exists a condensation $\varphi : Y \rightarrow M$ of Y onto a space M with $\text{w}(M) \leq \kappa$. Apply [8, Corollary 1] for the case (a) and [2, Theorem 1] for the case (b) to find an ordinal $\alpha < \kappa^+$ and a continuous map $h : p_\alpha(Y) \rightarrow M$ such that $h \circ p_\alpha = \varphi$. Here $p_\alpha : \mathbb{R}^{\kappa^+} \rightarrow \mathbb{R}^\alpha$ is the projection. It is worth mentioning that [8, Corollary 1] is formulated for $\kappa = \omega$ but Tkachenko's proof works for any infinite cardinal κ . However, a complete proof for the general case can be found in [10, Problem 298].

Since φ is a condensation, the map p_α has to be injective on Y . To see that this is false, note that $t = f(x_\alpha)$ and $z = f(y_\alpha)$ are distinct points of Y because $t(\alpha) = f_\alpha(x_\alpha) \neq f_\alpha(y_\alpha) = z(\alpha)$ by property (2). However, $t(\beta) = f_\beta(x_\alpha) = f_\beta(y_\alpha) = z(\beta)$ for every $\beta < \alpha$ which shows that $p_\alpha(t) = p_\alpha(z)$, i.e., the map p_α is not injective on Y . This contradiction shows that $i\text{-weight}$ of X does not exceed κ . $\square$

Corollary 3.7 *If X is a Lindelöf space and every continuous image of X of weight $\leq \kappa^+$ has $i\text{-weight} \leq \kappa$ then $i\text{w}(X) \leq \kappa$.*

Corollary 3.8 *If X is a space and every continuous image of X of weight $\leq \omega_1$ has countable $i\text{-weight}$, then $i\text{w}(X) \leq \omega$.*

Proof Assume first that X is pseudocompact. Then every continuous image of X of weight $\leq \omega_1$ is compact and metrizable being a pseudocompact space of countable $i\text{-weight}$ (see [10, Problem 140]). Therefore every small continuous image of X has countable weight, so $w(X) \leq \omega$ by [7, Theorem 2] and hence $i\text{w}(X) \leq w(X) \leq \omega$.

Now, if X is not pseudocompact, then there exists an infinite discrete family $\mathcal{U}$ of non-empty open subsets of X ; pick a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $A = \{x_U : U \in \mathcal{U}\}$ is easily seen to be closed discrete and C -embedded in X so we can apply Theorem 3.6 for $\kappa = \omega$ to conclude that $i\text{w}(X) \leq \omega$. $\square$

Corollary 3.9 *Suppose that X is a GO space such that every continuous image of X of weight $\leq \kappa^+$ has diagonal number at most κ . Then $hl(X) \leq \kappa$ and $i\text{w}(X) \leq \kappa$ which, of course, implies that the diagonal number of X does not exceed κ .*

Proof Suppose that Y is a continuous image of the space X of weight $\leq \kappa^+$. Since pseudocharacter does not exceed the diagonal number, we have the inequality $\psi(Y) \leq \kappa$; therefore we can apply Corollary 3.5 to see that $hl(X) \leq \kappa$. Therefore $l(Y) \leq \kappa$ and the diagonal number of the space Y does not exceed κ whenever Y is a continuous image of X of weight $\leq \kappa^+$; this implies $i\text{w}(Y) \leq \kappa$ (see [1, Theorem 2.1.8]). Applying Theorem 3.6(a), we convince ourselves that $i\text{w}(X) \leq \kappa$ and hence the diagonal number of X is not greater than κ . $\square$

Corollary 3.10 *Suppose that X is a linearly ordered space such that every continuous image of X of weight $\leq \omega_1$ has diagonal G_δ . Then X is second countable and hence has a G_δ -diagonal.*

Proof Apply Corollary 3.9 to see that $hl(X) \leq \omega$ and the diagonal of X is a G_δ -set. Every linearly ordered space with a G_δ -diagonal is metrizable (see [6, Theorem 5.1]) which, together with the Lindelöf property of X , implies that $w(X) \leq \omega$. $\square$

It was proved in [9] that it is independent of ZFC whether countable network weight is reflectable in continuous images of weight at most ω_1 . However, network weight behaves better in GO spaces.

Theorem 3.11 *Given a GO space X , suppose that $nw(Y) \leq \kappa$ for any continuous image Y of the space X such that $w(Y) \leq \kappa^+$. Then $w(X) \leq \kappa$.*

Proof Let $<$ be a linear order on X which witnesses that X is a GO space, i.e., X has a base whose elements are order convex. Recall that a set $C \subset X$ is convex if $x, y \in C$ implies $z \in C$ whenever $x < z < y$. Say that a point $x \in X$ is *left splitting* if the set $(\leftarrow, x]$ is open in X (this is a short way of saying that x is either a left point of a jump, or a right pseudogap). Analogously, $x \in X$ is *right splitting* if $[x, \rightarrow)$ is open in X . It is well known and easy to prove that

- (3) for any GO space Z , if $d(Z) \leq \kappa$ and the set $S_Z = \{z \in Z : z \text{ is either left splitting or right splitting}\}$ has cardinality at most κ , then $w(Z) \leq \kappa$.

Apply Theorem 3.2 to see that $d(X) \leq \kappa$; the above property (3) shows that it is sufficient to prove that $|S_X| \leq \kappa$. To obtain a contradiction, assume that $|S_X| > \kappa$. Without loss of generality, we can consider that there exists a faithfully enumerated set $\{z_\alpha : \alpha < \kappa^+\}$ such that every z_α is a left splitting point and hence the characteristic function f_α of the set $(\leftarrow, z_\alpha]$ is continuous for each $\alpha < \kappa^+$.

If $f = \Delta\{f_\alpha : \alpha < \kappa^+\}$ and $Y = f(X)$, then $Y \subset \{0, 1\}^{\kappa^+}$ and therefore $w(Y) \leq \kappa^+$. By our assumption about continuous images of X , we have $nw(Y) \leq \kappa$; let $\{F_\alpha : \alpha < \kappa\}$ be a network of Y and consider the projection $p_\alpha : Y \rightarrow \{0, 1\}$ of Y onto the α th factor for each $\alpha < \kappa^+$.

The set $V_\alpha = p_\alpha^{-1}(1)$ is open in Y and $z_\alpha \in f^{-1}(V_\alpha) = (\leftarrow, z_\alpha]$; choose $\mu(\alpha) < \kappa$ such that $f(z_\alpha) \in F_{\mu(\alpha)} \subset V_\alpha$ for every $\alpha < \kappa^+$. There are distinct ordinals $\alpha, \beta < \kappa^+$ such that $\mu(\alpha) = \mu(\beta) = \mu$; observe that there is no loss of generality to consider that $z_\alpha < z_\beta$. Then $f^{-1}(F_\mu) \subset f^{-1}(V_\alpha) = (\leftarrow, z_\alpha]$. However, $f(z_\beta) \in F_{\mu(\beta)} = F_{\mu(\alpha)} = F_\mu$ so $z_\beta \in f^{-1}(F_\mu) \setminus (\leftarrow, z_\alpha]$ which is a contradiction. $\square$

Corollary 3.12 *Suppose that X is a GO space such that $nw(Y) \leq \omega$ whenever Y is a continuous image of X of weight $\leq \omega_1$. Then X is second countable.*

It is still an open problem whether countable tightness reflects in continuous images of weight $\leq \omega_1$ in all Tychonoff spaces (see [11, Problem 3.1]). However, there is a positive answer for GO spaces.

Theorem 3.13 *Given an infinite cardinal κ , suppose that X is a GO space such that every continuous image of X of weight $\leq \kappa^+$ has tightness $\leq \kappa$. Then $\chi(X) \leq \kappa$ and hence $t(X) \leq \kappa$.*

Proof We will split the proof in several steps. Say that *projective tightness* of a space Z is at most κ if for any continuous image Y of the space Z we have $t(Y) \leq \kappa$ whenever $w(Y) \leq \kappa^+$. Our purpose is to show that if a GO space X has projective tightness $\leq \kappa$ then $t(X) \leq \kappa$ and hence we have the inequality $\chi(X) \leq \kappa$ (see [1, Theorem 1.3.1]).

Observe first that

- (4) if a space X has projective tightness $\leq \kappa$, then any continuous image of X has projective tightness $\leq \kappa$.

The following step outlines a class of spaces of projective tightness $\leq \kappa$ in which we must have tightness $\leq \kappa$. Recall that, for any space X , if $A \subset X$, then $[A]_\kappa = \bigcup \{\overline{B} : B \subset A, |B| \leq \kappa\}$.

- (5) If X has projective tightness $\leq \kappa$ and we have a set $A \subset X$ with $|A| \leq \kappa^+$, then $\overline{A} = [A]_\kappa$.

To prove statement (5) assume that $x \in \overline{A} \setminus [A]_\kappa$ and choose an enumeration $\{a_\alpha : \alpha < \kappa^+\}$ of the set A . For every $\alpha < \kappa^+$, we can find a function $f_\alpha \in C(X)$ such that $f_\alpha(\{a_\beta : \beta < \alpha\}) = \{0\}$ and $f_\alpha(x) = 1$. If $f = \Delta\{f_\alpha : \alpha < \kappa^+\}$ then the space $Y = f(X) \subset \mathbb{R}^{\kappa^+}$ has weight at most κ^+ so $t(Y) \leq \kappa$.

It follows from $x \in \overline{A}$ that $y = f(x) \in \overline{B}$ where $B = f(A)$ so there exists $\alpha < \kappa^+$ such that $y \in \overline{\{f(a_\beta) : \beta < \alpha\}}$. However, $f(a_\beta)(\alpha) = f_\alpha(a_\beta) = 0$ for every $\beta < \alpha$ by the choice of f_α so it follows from $y(\alpha) = f_\alpha(x) = 1$ that $y \notin \overline{\{f(a_\beta) : \beta < \alpha\}}$; this contradiction shows that (5) is proved.

Now take a GO space X with projective tightness $\leq \kappa$. We can consider that X is a dense subspace of a linearly ordered compact space K ; let $<$ be an order that generates the topology of K . Assume that there exists a point $x \in X$ such that $t(x, X) > \kappa$. There is no loss of generality to consider that $x \in \overline{L} \setminus [L]_\kappa$ where $L = \{y \in X : y < x\}$.

Using the fact that $x \notin \overline{A}$ for any $A \subset L$ with $|A| \leq \kappa$ it is easy to construct a set $Z = \{z_\alpha : \alpha < \kappa^+\} \subset L$ such that $\alpha < \beta$ implies $z_\alpha < z_\beta$. Let z be the least upper bound of the set Z in K . If $z \in X$ then it is immediate that $z \in \overline{Z} \setminus [Z]_\kappa$ which contradicts property (5).

This shows that $z \in K \setminus X$ and hence both sets $U = \{y \in X : y < z\}$ and $V = \{y \in X : y > z\}$ are clopen in X and $X = U \cup V$. Let $f(y) = y$ for every $y \in U$ and $f(y) = z$ for each $y \in V$. The map $f : X \rightarrow Y = U \cup \{z\}$ is continuous and onto because $x \in V$ which shows that V is non-empty. Therefore Y has projective tightness $\leq \kappa$ being a continuous image of the space X . However, for the set $Z \subset Y$ we have $z \in \overline{Z} \setminus [Z]_\kappa$; this contradiction with (5) proves that $t(X) \leq \kappa$ and hence $\chi(X) \leq \kappa$ by [1, Theorem 1.3.1]. $\square$

Corollary 3.14 *If X is a GO space and every continuous image of X of weight $\leq \omega_1$ has countable tightness, then $\chi(X) \leq \omega$ and hence $t(X) \leq \omega$.*

Corollary 3.15 *If X is a GO space and every continuous image of X of weight $\leq \omega_1$ is sequential, then $\chi(X) \leq \omega$ and hence X is sequential.*

Corollary 3.16 *If X is a GO space and every continuous image of X of weight $\leq \omega_1$ is Fréchet–Urysohn, then $\chi(X) \leq \omega$ and hence X is Fréchet–Urysohn.*

4 Open problems

A considerable amount of information on reflexivity of topological properties in small continuous images has been obtained in recent years. However, we are still far away from having a complete picture of the topic as can be seen from the following list of open questions.

Question 4.1 Suppose that X is a space such that $\text{iw}(Y) \leq \omega_1$ whenever Y is a continuous image of X of weight $\leq \omega_2$. Is it true that $\text{iw}(X) \leq \omega_1$?

Question 4.2 Suppose that X is a space such that $\psi(Y) \leq \omega$ whenever Y is a continuous image of X of weight $\leq \omega_1$. Is it true that X is Lindelöf?

Question 4.3 Suppose that X is a space such that $\psi(Y) \leq \omega$ whenever Y is a continuous image of X of weight $\leq \omega_1$. Is it true that $\text{ext}(X) \leq \omega$?

Question 4.4 Suppose that X is a pseudocompact space such that $\psi(Y) \leq \omega$ whenever Y is a continuous image of X of weight $\leq \omega_1$. Must X be compact?

Question 4.5 Suppose that X is a GO space such that every continuous image of X of weight $\leq \omega_1$ has a weaker compact topology. Must X have a weaker compact topology?

Question 4.6 Suppose that X is a GO space such that every continuous image of X of weight $\leq \omega_1$ is σ -pseudocompact. Must X be σ -pseudocompact?

Question 4.7 Suppose that X is a GO space such that every continuous image of X of weight $\leq \omega_1$ is ω -monolithic ($\equiv$ the closure of every countable subset of X has countable network). Is it true in ZFC that X is ω -monolithic?

Question 4.8 Suppose that X is a monotonically normal space such that every continuous image of X of weight $\leq \omega_1$ has countable network. Is it true in ZFC that X has a countable network? The definition of a monotonically normal space can be found in [4, Definition 5.15].

Question 4.9 Suppose that X is a monotonically normal space such that every continuous image of X of weight $\leq \omega_1$ has countable tightness. Must X have countable tightness?

Question 4.10 Suppose that X is a monotonically normal space such that every continuous image of X of weight $\leq \omega_1$ is Fréchet–Urysohn. Must the space X be Fréchet–Urysohn?

Question 4.11 Suppose that X is a countably compact space such that every continuous image of X of weight $\leq \omega_1$ has countable tightness. Must X have countable tightness?

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