Infrnilmanifolds which admit complex contact structures

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Abstract We study the complex nilpotent Lie group $\mathcal{L}$ which admits a complex contact structure and whose real subgroup admits a (real) contact structure. The group $\mathcal{L}$ is called the Iwasawa complex nilpotent Lie group. The homogeneous model of complex contact similarity geometry is introduced as $(\mathcal{L}, \text{Sim}(\mathcal{L}))$ where $\text{Sim}(\mathcal{L})$ is a transitive group of complex contact similarity transformations of $\mathcal{L}$. A smooth manifold locally modelled on this geometry is said to be a complex contact similarity manifold. We show that a compact complex contact similarity manifold can be determined under the conditions of developing pair.

Keywords Complex contact structure · Infrnil-manifold · Iwasawa complex Lie group · Complex Boothby–Wang fibration · Complex affine group · Holonomy group · Developing map · Similarity geometry

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1 Introduction

Let $(\text{PSp}(n+1, 1), \mathbb{H}_{\mathbb{H}}^{n+1})$ be the *homogeneous model* of quaternionic hyperbolic geometry, where $\mathbb{H}$ is the quaternion number field. The hyperbolic action of $\text{PSp}(n+1, 1)$ extends to the quaternionic projective action on the boundary sphere $\partial\mathbb{H}_{\mathbb{H}}^{n+1} = S^{4n+3}$ for which $\partial\mathbb{H}_{\mathbb{H}}^{n+1} = \text{PSp}(n+1, 1)/\text{PSp}(n+1, 1)_{\infty}$. Here $\text{PSp}(n+1, 1)_{\infty}$ is the stabilizer of the point at infinity $\infty \in \partial\mathbb{H}_{\mathbb{H}}^{n+1}$. The quaternionic

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CR -structure identifies the complement $S^{4n+3} - \{\infty\}$ with the quaternionic Heisenberg nilpotent Lie group $\mathcal{M}$. The group $\mathcal{M}$ has the three dimensional center $\mathbb{R}^3 = \text{Im } \mathbb{H}$ which gives a central group extension $1 \rightarrow \mathbb{R}^3 \rightarrow \mathcal{M} \rightarrow \mathbb{H}^n \rightarrow 1$, where $\mathbb{H}^n$ is the quaternionic vector space. Taking a 1-dimensional central group $\mathbb{R} (= \mathbb{R}i)$ from $\mathbb{R}^3$, we obtain a quotient Lie group

$$\mathcal{L} = \mathcal{M}/\mathbb{R}.$$

We call $\mathcal{L}$ the *Iwasawa group*. It is shown that $\mathcal{L}$ is a complex nilpotent Lie group (see Sect. 2). Recall that a complex contact structure on a complex $(2n+1)$ -dimensional complex manifold M is a collection of local forms $\{U_\alpha, \omega_\alpha\}_{\alpha \in \Lambda}$ satisfying that $\bigcup_{\alpha \in \Lambda} U_\alpha = M$, each ω_α is a holomorphic 1-form defined on U_α such that $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ on U_α and if $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a nonzero holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $f_{\alpha\beta} \cdot \omega_\alpha = \omega_\beta$.

We prove that $\mathcal{L}$ admits an invariant *complex contact structure* canonically by constructing a global holomorphic 1-form η endowed with the group of complex contact transformations $A(\mathcal{L})$ in Sects. 3 and 4 (see [10, Section 4]). In this paper we take a specific subgroup $\text{Sim}(\mathcal{L})$ called a *complex contact similarity group* from $A(\mathcal{L})$. In general, the pair $(\text{Sim}(\mathcal{L}), \mathcal{L})$ is said to be the homogeneous model of *complex contact similarity geometry* (cf. Sect. 6). We study *compact* complex contact manifolds modelled on $(\text{Sim}(\mathcal{L}), \mathcal{L})$. As usual, given a uniformization with respect to $(\text{Sim}(\mathcal{L}), \mathcal{L})$ on a manifold M , there exist a developing immersion $\text{dev}: \tilde{M} \rightarrow \mathcal{L}$ and a holonomy homomorphism $\rho: \pi_1(M) \rightarrow \text{Sim}(\mathcal{L})$ respectively. Here $\pi_1(M)$ is the fundamental group of M . The image $\rho(\pi_1(M))$ is said to be the holonomy group for M . We prove the following uniformization.

Theorem A *Let M be a $(4n+2)$ -dimensional compact complex contact similarity manifold. Suppose that either of the following holds.*

- (a) *The holonomy group is virtually nilpotent.*
- (b) *The holonomy group is discrete.*

Then M is biholomorphic to either a complex contact Iwasawa infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact infra-Hopf manifold $\mathcal{L} - \{0\}/\Gamma$. Here Γ is the holonomy group isomorphic to either a discrete cocompact virtually nilpotent subgroup in $\mathcal{L} \rtimes (\text{Sp}(n) \cdot S^1)$ or the product $\mathbb{Z}_p \times \mathbb{Z}^+$ in $\text{Sp}(n) \cdot S^1 \times \mathbb{R}^+$ respectively.

Theorem A may be true without condition (a) or (b). The supporting examples were obtained by Fried (Euclidean similarity manifold) [5] and Miner (Heisenberg similarity manifold) [17]. However, there is an embedding of $\text{Sim}(\mathcal{L})$ into the complex affine group $\text{Aff}(\mathbb{C}^{2n+1})$ (cf. (5)). By Proposition 5.3, M is also a complex affinely flat manifold. Thus, the determination of M is also related to the Auslander–Milnor conjecture (cf. [6]).

We discuss the deformation of the complex contact Iwasawa infranilmanifold $\mathcal{L}/\Gamma$ as a holomorphic Seifert fibering (cf. [15]) in Sect. 6.1. As in Theorem A, the developing pair (ρ, dev) is a geometric invariant for complex contact similarity manifolds. In Sect. 7, we study a compactification which shows that $\mathcal{L} \cup \{\infty\}$ is homeomorphic to the $(4n+2)$ -sphere. By using developing maps we characterize M (see Theorem 7.2).

Theorem B *If the developing map of a compact $(4n+2)$ -dimensional complex contact similarity manifold M is not surjective onto $\mathcal{L}$, then dev is a biholomorphism of $\tilde{M}$ onto $\mathcal{L} - \{0\}$.*

2 Iwasawa Lie group

Let J_n be the complex structure $\overbrace{J \oplus \cdots \oplus J}^n$ on $\mathbb{C}^{2n}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The quaternionic vector space $\mathbb{H}^n$ is identified with $\mathbb{C}^{2n}$ by the isomorphism

$$(z_1 + z_2 j, \dots, z_{2n-1} + z_{2n} j) \mapsto (\bar{z}_1, z_2, \dots, \bar{z}_{2n-1}, z_{2n}).$$

Our group $\mathcal{L}$ is defined to be the product $\mathbb{C}^{2n+1} = \mathbb{C} \times \mathbb{H}^n$ with the group law

$$\begin{aligned} (x, z) \cdot (y, w) &= \left(x + y + \sum_{i=1}^n z_{2i-1} w_{2i} - z_{2i} w_{2i-1}, z + w \right) \\ &= (x + y + {}^t z J_n w, z + w), \quad n \geq 1, \end{aligned}$$

where $x, y \in \mathbb{C}$, $z_k \in \mathbb{C}$, $w_k \in \mathbb{C}$, $k = 1, \dots, 2n$, ${}^t z = (z_1, \dots, z_{2n}) = (\bar{z}_1 + z_2 j, \dots, \bar{z}_{2n-1} + z_{2n} j) \in \mathbb{H}^n$ and ${}^t w = (w_1, \dots, w_{2n}) = (\bar{w}_1 + w_2 j, \dots, \bar{w}_{2n-1} + w_{2n} j) \in \mathbb{H}^n$.

It is easy to see that $[(x, z), (y, w)] = (2 \sum_{i=1}^n z_{2i-1} w_{2i} - z_{2i} w_{2i-1}, 0)$ so $[\mathcal{L}, \mathcal{L}] = (\mathbb{C}, (0, \dots, 0)) = \mathbb{C}$ which is the center of $\mathcal{L} = \mathcal{L}_n$. It has a central group extension $1 \rightarrow \mathbb{C} \rightarrow \mathcal{L}_n \rightarrow \mathbb{C}^{2n} \rightarrow 1$.

Moreover, let $L_h: \mathcal{L} \rightarrow \mathcal{L}$ be a left translation for $h = (x, z) \in \mathcal{L}$ defined by $L_h(y, w) = h \cdot (y, w) = (x, z) \cdot (y, w)$. Consider the differentiable map $L_{h*}: T_{(y,w)}\mathcal{L} \rightarrow T_{h \cdot (y,w)}\mathcal{L}$. For $(X, Y) \in T_{(y,w)}\mathcal{L} = T_y\mathbb{C} \times T_w\mathbb{C}^{2n}$, it follows

$$L_{h*}(X, Y) = \lim_{t \rightarrow 0} \frac{1}{t} (L_h(y + tX, w + tY) - L_h(y, w)) = (X + {}^t z J_n Y, Y).$$

Let J_0 be the standard complex structure on $\mathcal{L} = \mathbb{C}^{2n+1}$. As $J_0(X, Y) = (iX, iY)$, the above equation implies $L_{h*} \circ J_0 = J_0 \circ L_{h*}$ obviously, i.e. every left translation is a holomorphic transformation on $\mathcal{L}$. Similar to this argument, it is easily checked that the correspondences $((x, z), (y, w)) \mapsto (x, z) \cdot (y, w) = (x + y + {}^t z J_n w, z + w)$ and $(x, z) \mapsto (x, z)^{-1} = (-x, -z)$ are also holomorphic. Thus $\mathcal{L}$ is a complex Lie group.

Definition 2.1 A complex $(2n+1)$ -dimensional complex nilpotent Lie group $\mathcal{L}_n$ is said to be the *Iwasawa Lie group*.

Remark 2.2 It is easy to check that $\mathcal{L}_1$ is isomorphic to the Iwasawa group consisting of 3×3 -upper triangular unipotent complex matrices by the correspondence

$$(x, (z_1, z_2)) = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad z_3 = \frac{z_1 z_2 + x}{2}.$$

3 Existence of complex contact structure

3.1 Holomorphic 1-form

Let $(z_0, z_1, \dots, z_{2n})$ be the standard coordinates of $\mathcal{L}$. We define a complex 1-form η as follows:

$$\eta = dz_0 - \sum_{i=1}^n z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1} = dz_0 - {}^t z \mathbf{J}_n dz,$$

where ${}^t z = (z_1, \dots, z_{2n})$, ${}^t dz = (dz_1, \dots, dz_{2n})$. Since $\eta \wedge (d\eta)^n$ is a non-vanishing form $n(-1)^n(-2)^n dz_0 \wedge \dots \wedge dz_{2n}$ on $\mathcal{L}$, η is a complex contact structure (globally defined) on $\mathcal{L}$ (see [12] for the definition).

3.2 Real points of $\mathcal{L}$

The group $\mathcal{L}$ contains the *real point set* $\mathcal{N} = \mathbb{R} \times \mathbb{C}^n$ which has an induced group structure

$$(a, x) \cdot (b, y) = \left(a + b + \sum_{i=1}^n x_{2i-1} y_{2i} - x_{2i} y_{2i-1}, x + y \right),$$

where ${}^t x = (x_1 + x_2 j, \dots, x_{2n-1} + x_{2n} j) \in \mathbb{C}^n$, $x_k \in \mathbb{R}$, and ${}^t y = (y_1 + y_2 j, \dots, y_{2n-1} + y_{2n} j) \in \mathbb{C}^n$, $y_k \in \mathbb{R}$ (here we use j as a complex structure instead of i). Obviously, $\mathcal{N}$ is also a 2-step nilpotent Lie subgroup of $\mathcal{L}$ such that $T\mathcal{N} \oplus i T\mathcal{N} = T\mathcal{L}$. By the definition, $\mathcal{N}$ is the *Heisenberg nilpotent* Lie group. The complex contact structure η restricts it to a (real) 1-form

$$\omega = dx_0 - \sum_{i=1}^n x_{2i-1} \cdot dx_{2i} - x_{2i} \cdot dx_{2i-1}$$

which satisfies $\omega \wedge (d\omega)^n \equiv dx_0 \wedge dx_1 \wedge \dots \wedge dx_{2n} \neq 0$ (up to constant). Moreover, the pair $(\ker \eta, J)$ is a CR -structure on $\mathcal{N}$.

Remark 3.1 Let $1 \rightarrow \mathbb{R} \rightarrow \mathcal{M} \xrightarrow{\nu} \mathcal{L} \rightarrow 1$ be the central group extension as before. The canonical quaternionic CR -structure on $\mathcal{M}$ defines a Carnot–Carathéodory structure $(B, \{I, J, K\})$. Then one of the almost complex structures I is integrable on $D = T\mathbb{C} \oplus B$ (cf. [1]). In [10] we have shown that the holomorphic $2n$ -dimensional bundle $D^{1,0}$ maps isomorphically onto a complex contact subbundle $\nu_*(D)^{1,0}$ on $\mathcal{L}$, i.e. $[\nu_*(D)^{1,0}, \nu_*(D)^{1,0}] \equiv \mathbb{C} \bmod \nu_*(D)^{1,0}$. Note that the above extension is a principal bundle but not a 3-Sasakian space. So this complex contact structure on $\mathcal{L}$ does not come from any Sasakian 3-structure or any twistor space (cf. [7, 18]). As a consequence, η turns out to be a representative (global) holomorphic 1-form on $\mathcal{L}$.

4 The group of complex contact transformations

Let $\text{hol}(\mathcal{L})$ be the group of biholomorphic transformations of $(\mathcal{L}, J_0)$. The group of complex contact transformations on $(\mathcal{L}, \eta)$ is denoted by

$$\text{hol}(\mathcal{L}, \eta) = \{f \in \text{hol}(\mathcal{L}) : f^* \eta = \tau \cdot \eta\} \quad (1)$$

for some holomorphic function τ on $\mathcal{L}$. Let $\text{Sp}(n; \mathbb{C}) = \{A \in M(2n, \mathbb{C}) : {}^t A J_n A = J_n\}$ be the complex symplectic group. Put

$$A(\mathcal{L}) = \mathcal{L} \rtimes (\text{Sp}(n; \mathbb{C}) \cdot \mathbb{C}^*)$$

which is the semidirect product; let $\lambda \cdot A, \mu \cdot B \in \text{Sp}(n; \mathbb{C}) \cdot \mathbb{C}^*$ for $A, B \in \text{Sp}(n; \mathbb{C})$, $\lambda, \mu \in \mathbb{C}^*$ and $(a, w), (b, z) \in \mathcal{L}$. This forms a group by defining

$$\begin{aligned} ((a, w), \lambda \cdot A) \cdot ((b, z), \mu \cdot B) \\ &= ((a, w), I) \cdot ((\lambda^2 b, \lambda A z), \lambda \cdot A \cdot \mu \cdot B) \\ &= ((a + \lambda^2 b + {}^t w J_n (\lambda A z), w + \lambda A z), \lambda \mu \cdot AB). \end{aligned}$$

The group $A(\mathcal{L})$ acts on $\mathcal{L}$ as follows. For $((a, w), \lambda \cdot A) \in A(\mathcal{L})$, $(z_0, z) \in \mathcal{L}$, define

$$\begin{aligned} ((a, w), \lambda \cdot A) \cdot (z_0, z) &= (a, w) \cdot (\lambda^2 z_0, \lambda A z) \\ &= (a + \lambda^2 z_0 + {}^t w J_n (\lambda A z), w + \lambda A z). \end{aligned} \quad (2)$$

If $h = ((b, w), \mu \cdot B) \in A(\mathcal{L})$ with $\mu \in \mathbb{C}^*$, then it is easy to see that

$$h^* \eta = \mu^2 \cdot \eta. \quad (3)$$

We have to check that each element of $A(\mathcal{L})$ is *holomorphic*. Consider the complex affine group $\text{Aff}(\mathbb{C}^{2n+1}) = \mathbb{C}^{2n+1} \rtimes \text{GL}(2n+1, \mathbb{C})$. We assign to each $((a, w), \lambda \cdot A) \in A(\mathcal{L})$ an element

$$\left(\begin{bmatrix} a \\ w \end{bmatrix}, \begin{pmatrix} \lambda^2 & \lambda^t w J_n A \\ 0 & \lambda A \end{pmatrix} \right) \in \text{Aff}(\mathbb{C}^{2n+1}). \quad (4)$$

It is easy to see that this correspondence is an injective homomorphism. Since $\mathcal{L} = \mathbb{C}^{2n+1}$, by this assignment the action (2) of $((a, w), \lambda \cdot A)$ on $\mathcal{L}$ coincides with the complex affine transformation (4) of $\mathbb{C}^{2n+1}$. In particular, we obtain

$$A(\mathcal{L}) \leq \text{Aff}(\mathbb{C}^{2n+1}) (\leq \text{hol}(\mathcal{L})). \quad (5)$$

Combined with (3), $A(\mathcal{L})$ preserves the complex contact structure on $(\mathcal{L}, \eta)$ (see (1)).

Remark 4.1 Since $\eta|_{\mathcal{N}} = \omega$, $A(\mathcal{L})$ restricted to $\mathcal{N}$ is the group of CR -transformations $\text{Aut}_{CR}(\mathcal{N}) = \{f \in \text{Diff}(\mathcal{N}) : f^* \omega = \mu \cdot \omega\}$ for some smooth function μ on $\mathcal{N}$.

5 Uniformization over $(A(\mathcal{L}), \mathcal{L})$

5.1 Complex contact structure and transformations

Recall from [12] that a complex contact structure on a complex $(2n+1)$ -dimensional complex manifold M is a collection of local forms $\{U_\alpha, \omega_\alpha\}_{\alpha \in \Lambda}$ satisfying that (a) $\bigcup_{\alpha \in \Lambda} U_\alpha = M$, (b) each ω_α is a holomorphic 1-form defined on U_α such that $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ on U_α and (c) if $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a nonzero holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $f_{\alpha\beta} \cdot \omega_\alpha = \omega_\beta$.

Let $h: M \rightarrow M$ be a biholomorphism. Suppose that $h(U_\alpha) \cap U_\beta \neq \emptyset$ for some $\alpha, \beta \in \Lambda$. If there exists a holomorphic function $f_{\alpha\beta}$ defined on a neighborhood of U_α such that $h^* \omega_\beta = f_{\alpha\beta} \cdot \omega_\alpha$, then we call h a *complex contact transformation* of M . This is consistent with the definition of (1).

Denote by $\text{Aut}_{\text{cc}}(M)$ the group of complex contact transformations (when M is compact, note that $\text{Aut}_{\text{cc}}(M)$ is a finite dimensional Lie group). Let $\tilde{M}$ be the universal covering manifold of M and π its fundamental group $\pi_1(M)$. Naturally, by taking a lift of complex contact structure of M , $\tilde{M}$ is a complex contact manifold for which $\pi \leq \text{Aut}_{\text{cc}}(\tilde{M})$. If $N_{\text{Aut}_{\text{cc}}(\tilde{M})}(\pi)$ is the normalizer of π , then there is an exact sequence

$$1 \rightarrow \pi \rightarrow N_{\text{Aut}_{\text{cc}}(\tilde{M})}(\pi) \rightarrow \text{Aut}_{\text{cc}}(M) \rightarrow 1.$$

5.2 Uniformization and developing map

Let M be a smooth (not necessarily complex) manifold. Suppose that there exists a maximal collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ whose coordinate changes belong to $A(\mathcal{L})$. More precisely, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, $\varphi_\alpha: U_\alpha \rightarrow \mathcal{L}$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $g_{\alpha\beta} \in A(\mathcal{L})$ such that $g_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_\alpha \cap U_\beta)$. We say that M is locally modelled on $(A(\mathcal{L}), \mathcal{L})$ (compare [14]). The equivalence class of such a maximal collection of charts is said to be a *uniformization* with respect to $(A(\mathcal{L}), \mathcal{L})$.

Given a uniformization on a manifold M , the monodromy argument shows that there exist a developing immersion $\text{dev}: \tilde{M} \rightarrow \mathcal{L}$ and a holonomy homomorphism $\rho: \text{Aut}_{\text{cc}}(\tilde{M}) \rightarrow A(\mathcal{L})$ respectively. By the construction (cf. Proposition 5.1), there exists a complex structure J on M such that $\text{dev}_* \circ J = J_0 \circ \text{dev}_*$ (we use the same notation J on $\tilde{M}$). So dev is a *holomorphic immersion*. Here is a sufficient condition for the existence on complex contact structure.

Proposition 5.1 *If a $(4n+2)$ -dimensional smooth manifold M is locally modelled on $(A(\mathcal{L}), \mathcal{L})$, then M admits a complex contact structure.*

Proof First of all we define a complex structure on M . Let J_0 be the standard complex structure on $\mathcal{L} = \mathbb{C}^{2n+1}$. Define a complex structure J_α on U_α by setting $\varphi_{\alpha*} J_\alpha = J_0 \varphi_{\alpha*}$ on U_α for each $\alpha \in \Lambda$. When $h \in A(\mathcal{L})$, note that $h_* J_0 = J_0 h_*$ from (5). If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $g_{\alpha\beta} \in A(\mathcal{L})$ such that $g_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$. A calculation shows that $\varphi_{\beta*} J_\alpha = g_{\alpha\beta*} \varphi_{\alpha*} J_\alpha = J_0 \varphi_{\beta*}$. Since $\varphi_{\beta*} J_\beta = J_0 \varphi_{\beta*}$ by the

definition, it follows $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$. This defines a complex structure J on M . In particular, each $\varphi_\alpha: (U_\alpha, J) \rightarrow (\mathcal{L}, J_0)$ is a holomorphic embedding.

Let η be the holomorphic 1-form on $\mathcal{L}$ as before. Define a family of local holomorphic 1-forms $\{\omega_\alpha, U_\alpha\}_{\alpha \in \Lambda}$ by

$$\omega_\alpha = \varphi_\alpha^* \eta \quad \text{on } U_\alpha.$$

From (3), $g_{\alpha\beta}^* \eta = \mu_{\alpha\beta}^2 \cdot \eta$ for some $\mu_{\alpha\beta} \in \mathbb{C}^*$. It follows $\omega_\beta = \mu_{\alpha\beta}^2 \cdot \omega_\alpha$. Thus the family $\{\omega_\alpha, U_\alpha\}_{\alpha \in \Lambda}$ gives rise to a complex contact structure on (M, J) . $\square$

Remark 5.2 When a subgroup $\Gamma \leq A(\mathcal{L})$ acts properly discontinuously and freely on a domain Ω of $\mathcal{L}$ with compact quotient, we obtain a compact complex contact manifold Ω/Γ by this proposition. In this case, when we put $M = \Omega/\Gamma$, the developing map is the covering map $\tilde{M} = \tilde{\Omega} \rightarrow \Omega \subset \mathcal{L}$ and the holonomy homomorphism is a projection $p: \tilde{\Gamma} \rightarrow \Gamma$, where $1 \rightarrow \pi_1(\Omega) \rightarrow \tilde{\Gamma} \xrightarrow{p} \Gamma \rightarrow 1$ is an exact sequence. In general it is too difficult to find such a properly discontinuous cocompact subgroup Γ .

On the other hand, we have the embedding (5) (cf. (4)), this induces another uniformization simultaneously.

Proposition 5.3 *A $(4n+2)$ -dimensional smooth manifold M locally modelled on $(A(\mathcal{L}), \mathcal{L})$ supports a complex affine flat structure.*

When the developing map $\text{dev}: \tilde{M} \rightarrow \mathcal{L} = \mathbb{C}^{2n+1}$ is a diffeomorphism, $\mathbb{C}^{2n+1}/\Gamma$ is said to be a compact *complete complex affinely flat* manifold. Concerning the Auslander–Milnor conjecture, we do not know whether the fundamental group $\Gamma \leq A(\mathcal{L})$ is virtually *polycyclic* (compare [16]).

6 Complex contact similarity geometry

In view of Remark 5.2 and the Auslander–Milnor conjecture below Proposition 5.3, we may find a smaller subgroup of $A(\mathcal{L})$ in order to get compact complex contact manifolds with nontrivial fundamental groups. First of all, $\text{Sp}(n, \mathbb{C})$ contains a maximal compact symplectic subgroup $\text{Sp}(n) = \{A \in \text{U}(2n) : {}^t A J_n A = J_n\}$ where $\text{Sp}(n, \mathbb{C}) \cong \text{Sp}(n) \times \mathbb{R}^{n(2n+1)}$.

Definition 6.1 Put $\text{Sim}(\mathcal{L}) = \mathcal{L} \rtimes (\text{Sp}(n) \cdot \mathbb{C}^*) \leq A(\mathcal{L})$. The pair $(\text{Sim}(\mathcal{L}), \mathcal{L})$ is called the homogeneous model of *complex contact similarity geometry*. A manifold locally modelled on this geometry is said to be a complex contact similarity manifold (cf. Proposition 5.1). The Euclidean subgroup $E(\mathcal{L})$ of $\text{Sim}(\mathcal{L})$ is defined to be $\mathcal{L} \rtimes (\text{Sp}(n) \cdot S^1)$.

We give two typical examples of compact complex contact similarity manifolds.

(I) Infra-Hopf manifold. Choose $\lambda \in \mathbb{C}^*$ with $|\lambda| \neq 1$ and put $r = ((0, 0), \lambda) \in \text{Sim}(\mathcal{L})$. Denote by $\mathbb{Z}^+$ an infinite cyclic group generated by r . Let $\mathcal{L} - \{0\}$ be the

complement with the origin $0 = (0, 0)$ removed. Then it is easy to see that $\mathbb{Z}^+$ acts freely and properly discontinuously on $\mathcal{L} - \{0\}$ by (2):

$$r(z_0, z) = (\lambda^2 z_0, \lambda z).$$

The quotient $\mathcal{L} - \{0\}/\mathbb{Z}^+$ is diffeomorphic to $S^1 \times S^{4n+1}$. By Proposition 5.1 (cf. Remark 5.2), $S^1 \times S^{4n+1}$ is a complex contact similarity manifold on which $\mathrm{Sp}(n) \cdot S^1 \times S^1 = \mathrm{Sp}(n) \cdot \mathbb{C}^*/\langle r \rangle$ acts properly as complex contact transformations. If $\mathbb{Z}_p \leq \mathrm{Sp}(n) \cdot S^1 \times S^1$ is a cyclic subgroup of order $p > 1$ acting freely on $S^1 \times S^{4n+1}$, then the quotient by the action $\mathbb{Z}_p$ is an infra-Hopf manifold.

(II) Infranilmanifold. Let Γ be a torsionfree discrete cocompact subgroup of $E(\mathcal{L})$. By the Auslander–Bieberbach theorem, $\Gamma \cap \mathcal{L}$ is discrete uniform in $\mathcal{L}$. Since $\mathbb{C}$ is the central subgroup of $\mathcal{L}$, $\Gamma \cap \mathbb{C}$ is a discrete uniform subgroup $\mathbb{Z}^2$ in $\mathbb{C}$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{C} & \longrightarrow & E(\mathcal{L}) & \xrightarrow{p} & \mathbb{H}^n \rtimes (\mathrm{Sp}(n) \cdot S^1) \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{p} & Q \longrightarrow 1. \end{array}$$

As a consequence, the quotient group $Q = p(\Gamma)$ is a discrete uniform subgroup in $\mathbb{H}^n \rtimes (\mathrm{Sp}(n) \cdot S^1) \leq E(\mathbb{H}^n)$. In particular, the quotient $\mathbb{H}^n/Q$ is a compact *quaternionic Euclidean orbifold*. Letting $T_{\mathbb{C}}^1 = \mathbb{C}/\mathbb{Z}^2$, we obtain a compact $(4n+2)$ -dimensional complex contact Iwasawa infranilmanifold $\mathcal{L}/\Gamma$ which supports a holomorphic Seifert fibering $T_{\mathbb{C}}^1 \rightarrow \mathcal{L}/\Gamma \rightarrow \mathbb{H}^n/Q$ (compare [15]).

Remark 6.2 Since there exists a finite index normal free abelian subgroup $\mathbb{Z}^{4n} \leq \mathbb{H}^n$ of Q by the Bieberbach theorem, this induces a central group extension $1 \rightarrow \mathbb{Z}^2 \rightarrow \Delta \rightarrow \mathbb{Z}^{4n} \rightarrow 1$ such that $\Delta \leq \mathcal{L}$. A complex contact nilmanifold $\mathcal{L}/\Delta$ admits a holomorphic principal $T_{\mathbb{C}}^1$ -bundle over a complex torus $T_{\mathbb{C}}^{2n} = \mathbb{H}^n/\mathbb{Z}^{4n}$. This example is a special case of Foreman’s T^2 -connection bundle over $T_{\mathbb{C}}^{2n}$ (cf. [3]). As a consequence, a compact complex contact Iwasawa infranilmanifold $\mathcal{L}/\Gamma$ is finitely covered by $\mathcal{L}/\Delta$.

6.1 Deformation of Iwasawa infranilmanifold $\mathcal{L}/\Gamma$

Let $1 \rightarrow \mathbb{Z}^2 \rightarrow \Gamma \rightarrow Q \rightarrow 1$ be a group extension which is associated with a holomorphic Seifert fibering $T_{\mathbb{C}}^1 \rightarrow \mathcal{L}/\Gamma \rightarrow \mathbb{H}^n/Q$. As $\mathcal{L}/\mathbb{Z}^2$ is biholomorphic to the product $T_{\mathbb{C}}^1 \times \mathbb{H}^n$, the holomorphic action of Q can be described as

$$\alpha(t, w) = (\phi(\alpha)(t) \cdot \chi(\alpha)(\alpha w), \alpha w),$$

where $\chi(\alpha): \mathbb{H}^n = \mathbb{C}^{2n} \rightarrow T_{\mathbb{C}}^1$, $\alpha \in Q$, is a holomorphic map. When $\mathcal{L}/\Gamma$ is described as $T_{\mathbb{C}}^1 \times_Q \mathbb{H}^n$, we introduce the following equivalence.

Definition 6.3 Two such manifolds $\mathcal{L}/\Gamma_1, \mathcal{L}/\Gamma_2$ are *equivalent* if there is an equivariant fiber preserving holomorphic map $H(t, w) = (t \cdot h(w), w)$ for a holomorphic map $h: \mathbb{H}^n \rightarrow T_{\mathbb{C}}^1$ with the following commutative diagram:

$$\begin{array}{ccc} T_{\mathbb{C}}^1 \times \mathbb{H}^n & \xrightarrow{H} & T_{\mathbb{C}}^1 \times \mathbb{H}^n \\ \text{pro} \searrow & & \swarrow \text{pro} \\ & \mathbb{H}^n & \end{array}$$

Let $\text{hol}(\mathbb{H}^n, \mathbb{C})$ be the set of all holomorphic maps from $\mathbb{H}^n = \mathbb{C}^{2n}$ to $\mathbb{C}$. It is a Q -module naturally. Similarly for $\text{hol}(\mathbb{H}^n, T_{\mathbb{C}}^1)$ and $\mathbb{Z}^2$. Then a short exact sequence $1 \rightarrow \mathbb{Z}^2 \xrightarrow{i} \text{hol}(\mathbb{H}^n, \mathbb{C}) \xrightarrow{j} \text{hol}(\mathbb{H}^n, T_{\mathbb{C}}^1) \rightarrow 1$ induces a long cohomology exact sequence [15]

$$\begin{aligned} \longrightarrow H^1(Q; \mathbb{Z}^2) &\xrightarrow{i} H^1(Q; \text{hol}(\mathbb{H}^n, \mathbb{C})) \xrightarrow{j} H^1(Q; \text{hol}(\mathbb{H}^n, T_{\mathbb{C}}^1)) \\ &\xrightarrow{\delta} H^2(Q; \mathbb{Z}^2) \longrightarrow \cdots \end{aligned}$$

The above group extension determines a 2-cocycle $[f] \in H^2(Q; \mathbb{Z}^2)$. Consider the pre-image $\delta^{-1}([f]) \leq H^1(Q; \text{hol}(\mathbb{H}^n, T_{\mathbb{C}}^1))$ which is thought of as a Jacobian variety. Then we have shown

Theorem 6.4 ([9]) *The Q -module $\delta^{-1}([f])$ is the space of all equivalence classes (in the sense of Definition 6.3) representing $\mathcal{L}/\Gamma$.*

- Each element of $\delta^{-1}([f])$ gives a complex contact Iwasawa infranilmanifold $(\mathcal{L}/\Gamma, J')$ which supports a holomorphic Seifert fibering $T_{\mathbb{C}}^1 \rightarrow \mathcal{L}/\Gamma \rightarrow \mathbb{H}^n/Q$.
- $(\mathcal{L}/\Gamma, J')$ is a deformation of $(\mathcal{L}/\Gamma, J)$ as a holomorphic $T_{\mathbb{C}}^1$ -bundle.

We end this section with the proof of Theorem A.

6.2 Proof of Theorem A

As M is a $(4n+2)$ -dimensional compact complex contact similarity manifold, by Definition 6.1 and Proposition 5.1, there is a developing pair

$$(\rho, \text{dev}): (\text{Aut}_{\text{cc}}(\tilde{M}), \tilde{M}, J) \rightarrow (\text{Sim}(\mathcal{L}), \mathcal{L}, J_0).$$

Note that $\pi = \pi_1(M) \leq \text{Aut}_{\text{cc}}(\tilde{M})$ and dev is a holomorphic immersion, i.e. $\text{dev}_* J = J_0 \text{dev}_*$. Put $\Gamma = \rho(\pi)$.

Assume (a). Taking a finite index subgroup we may assume Γ is nilpotent. We prove this case by the following steps.

Lemma 6.5 Suppose that some $\gamma = ((a, w), \lambda \cdot A) \in \Gamma$ has a nontrivial summand $\lambda, \neq 1$, in $\mathbb{C}^* \leq \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*) = \mathrm{Sim}(\mathcal{L})$. Then $\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^*$.

Proof As $\mathbb{C}^* = S^1 \times \mathbb{R}^+$ acts as multiplication on $\mathcal{L}$, it is noted that $[\mathcal{L}, \mathbb{R}^+] = \mathcal{L}$ (and also $[\mathcal{L}, S^1] = \mathcal{L}$). So if Γ has a nontrivial summand x in $\mathcal{L}$, then $(1 - \lambda \cdot A)x \in \Gamma_1 = [\Gamma, \Gamma]$ and so $(1 - \lambda \cdot A)^i x \in \Gamma_i = [\Gamma_{i-1}, \Gamma]$ (possibly $A = I$). So $\Gamma_i = [\Gamma_{i-1}, \Gamma]$ cannot be trivial for any $i > 0$. This is impossible because Γ is nilpotent. Thus Γ has no summand in $\mathcal{L}$ and hence

$$\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^* \leq \mathrm{Sim}(\mathcal{L}). \quad \square$$

In particular, Γ has the fixed point 0 at the origin $(0, 0) \in \mathcal{L}$ (in this case M is called *radiant* (compare [6])). We can assume that there exists an element $\gamma = (k, t) \in \Gamma$ such that

$$k \in \mathrm{Sp}(n) \cdot S^1, \quad t \in \mathbb{R}^+, \quad t < 1. \quad (6)$$

Lemma 6.6 The immersion dev misses $\{0\}$, i.e. $\{0\} \notin \mathrm{dev}(\tilde{M})$.

Proof Assume $\{0\} \in \mathrm{dev}(\tilde{M})$.

Case 1. If the complement $\mathcal{L} - \mathrm{dev}(\tilde{M})$ is not empty, then it is a Γ -invariant closed subset which satisfies that $(\mathcal{L} - \mathrm{dev}(\tilde{M})) \cap \{0\} = \emptyset$. Since both $\{0\}$ and $\mathcal{L} - \mathrm{dev}(\tilde{M})$ are Γ -invariant, it follows

$$\gamma^i(\mathcal{L} - \mathrm{dev}(\tilde{M})) \cap \{0\} = \emptyset.$$

Choose a point $p = (z_0, z) \in \mathcal{L} - \mathrm{dev}(\tilde{M})$. Then $\gamma^i p = (t^{4i} z_0, t^{2i} k^{2i} z)$ by definition. It follows that $\lim_{i \rightarrow \infty} \gamma^i p = (0, 0) = 0 \in \mathcal{L}$ by (6). Since $\gamma^i p \in \mathcal{L} - \mathrm{dev}(\tilde{M})$ which is a closed subset, it follows $0 \in \mathcal{L} - \mathrm{dev}(\tilde{M})$. This yields a contradiction. Hence Case 1 does not occur.

Case 2. Suppose $\mathcal{L} - \mathrm{dev}(\tilde{M}) = \emptyset$, i.e. dev is surjective. Recall that M is viewed as an affinely flat manifold; $(\rho, \mathrm{dev}): (\pi, \tilde{M}) \rightarrow (\Gamma, \mathbb{C}^{2n+1})$ where

$$\Gamma = \left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda A \end{pmatrix} \right) \right\} \leq \mathrm{Aff}(\mathbb{C}^{2n+1}). \quad (7)$$

If we note that Γ is nilpotent, then it follows from [6, Theorem A] that M is complete, i.e. the developing map $\mathrm{dev}: \tilde{M} \rightarrow \mathbb{C}^{2n+1}$ is a diffeomorphism. Then $\Gamma \leq \mathrm{Sp}(n) \cdot \mathbb{C}^* \leq \mathrm{Sim}(\mathcal{L})$ is discrete and so a finite index subgroup of Γ is an infinite cyclic subgroup. Then $\mathbb{C}^{2n+1}/\Gamma$ cannot be compact and so Case 2 does not occur. As a consequence, combining Cases 1 and 2 shows that dev misses $\{0\}$. $\square$

Lemma 6.7 The manifold M is biholomorphic to $S^{4n+1} \times \mathbb{R}^+ / \Gamma$ which is diffeomorphic to an infra-Hopf manifold $S^{4n+1} / \mathbb{Z}_p \times S^1$.

Proof First note that the complement $\mathcal{L} - \{0\}$ admits an $\mathrm{Sp}(n) \cdot \mathbb{C}^*$ -invariant Riemannian metric. In fact, $\mathbb{R}^+ \leq \mathbb{C}^*$ acts on $\mathcal{L} - \{0\}$ as

$$\lambda(z_0, z) = (\lambda^2 z_0, \lambda z)$$

so $\mathbb{R}^+$ acts properly. Since $\mathrm{Sp}(n) \cdot S^1$ is compact, $\mathrm{Sp}(n) \cdot \mathbb{C}^* = \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$ acts properly on $\mathcal{L} - \{0\}$. Then there exists an $\mathrm{Sp}(n) \cdot \mathbb{C}^*$ -invariant Riemannian metric on $\mathcal{L} - \{0\}$ (see [13] for example). By Lemma 6.6, the developing image $\mathrm{dev}(\tilde{M})$ misses 0, i.e. $\mathrm{dev}(\tilde{M}) \subset \mathcal{L} - \{0\}$. Taking a pullback metric by the developing immersion, $\mathrm{dev}: \tilde{M} \rightarrow \mathcal{L} - \{0\}$ is a local isometry. Since this metric is Γ -invariant and dev is ρ -equivariant, the pullback metric on $\tilde{M}$ is π -invariant which induces a Riemannian metric on the quotient M . As M is compact, M is complete with respect to this metric so is $\tilde{M}$. Then $(\rho, \mathrm{dev}): (\pi, \tilde{M}) \rightarrow (\Gamma, \mathcal{L} - \{0\})$ becomes an equivariant covering map. Since $\mathcal{L} - \{0\} \cong S^{4n+1} \times \mathbb{R}^+$ is simply connected, $\mathrm{dev}: \tilde{M} \rightarrow \mathcal{L} - \{0\}$ is a diffeomorphism. Taking the quotient, M is holomorphically diffeomorphic to $\mathcal{L} - \{0\}/\Gamma$.

On the other hand, Γ becomes discrete in $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^+ \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & F & \longrightarrow & \Gamma & \longrightarrow & \mathbb{Z}^+ \longrightarrow 1. \end{array}$$

The image of Γ to $\mathbb{R}^+$ is nontrivial (and so an infinite cyclic subgroup $\mathbb{Z}^+$) because $\mathcal{L} - \{0\}/\Gamma$ is compact. Choosing an infinite cyclic subgroup $\mathbb{Z}^+$ from Γ which maps onto $\mathbb{Z}^+ \leq \mathbb{R}^+$, it follows that $\Gamma = F \rtimes \mathbb{Z}^+$. Consider the exact sequences further,

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Sp}(n) & \longrightarrow & \mathrm{Sp}(n) \cdot S^1 & \xrightarrow{p} & S^1 \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & F \cap \mathrm{Sp}(n) & \longrightarrow & F & \longrightarrow & p(F) \longrightarrow 1. \end{array}$$

As $F \leq \Gamma$ acts freely on $\mathcal{L} - \{0\}$ in which $A \in \mathrm{Sp}(n)$ acts on $\mathcal{L} - \{0\}$ by $A(z_0, z) = (z_0, Az)$ (cf. (7) of Case 2), it follows that $F \cap \mathrm{Sp}(n) = \{1\}$. In particular, $F \cong p(F)$ is a finite cyclic subgroup of S^1 . Put $F = \mathbb{Z}_p$, for some $p \in \mathbb{Z}$. As a consequence,

$$\Gamma = \mathbb{Z}_p \times \mathbb{Z}^+ \leq T^{2n+1} \times \mathbb{R}^+$$

which acts on $\mathcal{L} - \{0\}$ by

$$\lambda(z_0, z_1, \dots, z_{2n}) = (\lambda^2 z_0, \lambda z_1, \dots, \lambda z_{2n}).$$

As the $\mathbb{Z}_p$ -action preserves the standard sphere $S^{4n+1} \subset \mathcal{L} - \{0\} = S^{4n+1} \times \mathbb{R}^+$, the Γ -action on $\mathcal{L} - \{0\}$ induces a properly discontinuous action of $\mathbb{Z}^+$ on $S^{4n+1}/\mathbb{Z}_p \times \mathbb{R}^+$ (here $S^{4n+1}/\mathbb{Z}_p$ is the lens space $L(p, 2, 1, \dots, 1)$). Hence $\mathcal{L} - \{0\}/\Gamma$ is biholomorphic to an infra-Hopf complex contact manifold $S^{4n+1}/\mathbb{Z}_p \times S^1$. $\square$

There is the remaining case to Lemma 6.5.

Lemma 6.8 Suppose $\Gamma \leq \mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1 \leq \mathrm{Sim}(\mathcal{L})$, i.e. every γ has no summand in $\mathbb{R}^+$. Then M is biholomorphic to an infranilmanifold $\mathcal{L}/\Gamma$.

Proof Since $\mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1$ acts properly on $\mathcal{L}$, there is an $\mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1$ -invariant Riemannian metric on $\mathcal{L}$. Taking the pullback metric, it follows similarly as above that $(\rho, \mathrm{dev}): (\pi, \tilde{M}) \rightarrow (\mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1, \mathcal{L})$ becomes an equivariant isometry. Hence M is holomorphically diffeomorphic to $\mathcal{L}/\Gamma$. Auslander–Bieberbach’s theorem implies that $\mathcal{L}/\Gamma$ is finitely covered by a holomorphic nilmanifold $\mathcal{L}/\Delta$, where $\Delta = \Gamma \cap \mathcal{L}$ is a finite index normal subgroup of Γ . Hence $\mathcal{L}/\Gamma$ is a holomorphic infranilmanifold. $\square$

All together with Lemmas 6.5, 6.6, 6.7 and 6.8, this finishes the proof of Theorem A under the condition (a).

Assume (b). Since $\mathrm{Sim}(\mathcal{L}) = \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$ is an amenable Lie group, any discrete subgroup is virtually polycyclic (see [16]). We may assume that Γ is polycyclic. Consider the exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*) & \xrightarrow{L} & \mathrm{Sp}(n) \cdot \mathbb{C}^* \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \xrightarrow{L} & L(\Gamma) \longrightarrow 1 \end{array}$$

where we put $\Delta = \mathcal{L} \cap \Gamma$. As $L(\Gamma)$ is solvable, it follows that $L(\Gamma) \leq T^n \times S^1 \times \mathbb{R}^+$. In particular, $[\Gamma, \Gamma] \leq \Delta$. We can assume that Δ is nontrivial. Otherwise, $[\Gamma, \Gamma] = \{1\}$, Γ is abelian. So the condition Theorem A(a) is satisfied. Then some finite cover of M is either a complex contact nilmanifold $\mathcal{L}/\Gamma$ or a complex contact Hopf manifold $S^{4n+1} \times S^1$. As Γ is abelian, some finite cover of M must be a complex contact Hopf manifold $S^{4n+1} \times S^1$.

Suppose that there is an element $\gamma \in \Gamma$ such that $L(\gamma)$ has a nontrivial summand λ in $\mathbb{R}^+$. We may assume that $\lambda < 1$. Put $\gamma = ((a, w), \lambda A)$ and $n = (b, z) \in \Delta$. We can check that

$$\begin{aligned} \gamma^i n \gamma^{-i} &= \left(\lambda^{2i} b + 2 \sum_{k=1}^i \lambda^{2i-k} ({}^t w J_n A^k z), \lambda^i A^i z \right) \\ &= \left(\lambda^i \left(\lambda^i b + 2 \sum_{k=1}^i \lambda^{i-k} ({}^t w J_n A^k z) \right), \lambda^i A^i z \right). \end{aligned}$$

As $\gamma^i n \gamma^{-i} \in \Delta$, it follows

$$\lim_{i \rightarrow \infty} \gamma^i n \gamma^{-i} = (0, 0) = 1 \in \Delta,$$

which contradicts that Δ is discrete. Hence $L(\Gamma) \leq T^n \times S^1$ and so $\Gamma \leq \mathcal{L} \rtimes T^n \times S^1 \leq \mathcal{L} \rtimes \mathrm{Sp}(n) \cdot S^1$. As in the proof of Lemma 6.8, M is biholomorphic to an infranilmanifold $\mathcal{L}/\Gamma$.

This proves Theorem A with conditions (a) or (b) respectively. We proved this theorem in [10] when $S^1 \leq \text{Aut}_{\text{cc}}(M)$ acts on M without fixed points.

7 Compactification of $\mathcal{L}$ and developing maps

7.1 $(4n+2)$ -sphere $\mathcal{L} \cup \{\infty\}$

Let $(\text{PSp}(n+1, 1), \mathbb{H}_{\mathbb{H}}^{n+1})$ be the homogeneous model of the quaternionic hyperbolic geometry. There is a compactification $\overline{\mathbb{H}}_{\mathbb{H}}^{n+1}$ of $\mathbb{H}_{\mathbb{H}}^{n+1}$ whose boundary $\partial\mathbb{H}_{\mathbb{H}}^{n+1}$ is diffeomorphic to the $(4n+3)$ -sphere S^{4n+3} . The isometric group $\text{PSp}(n+1, 1)$ extends to a projective action on S^{4n+3} whose stabilizer at the point at infinity $\infty \in S^{4n+3}$ is isomorphic to the maximal amenable Lie group $\mathcal{M} \rtimes \text{Sp}(n, 1) \cdot \text{Sp}(1) \times \mathbb{R}^+$, where $\mathcal{M}$ is the quaternionic Heisenberg Lie group whose center is $\mathbb{R}^3$. There is the parabolic decomposition $S^{4n+3} = \mathcal{M} \cup \{\infty\}$. As $\mathbb{R} \leq \mathcal{M}$ acts as isometries of $\mathbb{H}_{\mathbb{H}}^{n+1}$, there is the principal bundle $\mathbb{R} \rightarrow \mathbb{H}_{\mathbb{H}}^{n+1} \xrightarrow{p} B$, where $B = \mathbb{H}_{\mathbb{H}}^{n+1}/\mathbb{R}$. Then the projection p naturally extends to a map $\overline{p}$ of $\overline{\mathbb{H}}_{\mathbb{H}}^{n+1}$ in such a way that

$$\overline{p}: \mathbb{H}_{\mathbb{H}}^{n+1} \cup (\mathcal{M} \cup \{\infty\}) \rightarrow B \cup (\mathcal{L} \cup \{\infty\}),$$

where $\mathbb{R} \rightarrow \mathcal{M} \xrightarrow{p} \mathcal{L}$ is the exact sequence as before. Then note that $\overline{B} = B \cup (\mathcal{L} \cup \{\infty\})$ is homeomorphic to a $(4n+3)$ -ball such that the boundary $\partial B = \mathcal{L} \cup \{\infty\}$ is a sphere S^{4n+2} . Thus we have an equivariant projection

$$\mathbb{R} \rightarrow (\mathcal{M} \rtimes \text{Sp}(n) \cdot \mathbb{C}^*, \overline{\mathbb{H}}_{\mathbb{H}}^{n+1}) \xrightarrow{\overline{p}} (\mathcal{L} \rtimes \text{Sp}(n) \cdot \mathbb{C}^*, \overline{B}).$$

7.2 A proper action on a domain of $\overline{B}$

Suppose that a group $\Gamma \leq \mathcal{L} \rtimes \text{Sp}(n) \cdot \mathbb{C}^*$ acts on a domain Ω of $\mathcal{L}$. We note that the action of Γ naturally extends to an action on $\partial B = \mathcal{L} \cup \{\infty\}$ such that Γ leaves ∞ fixed. Define

$$\partial\Omega = \partial B - \Omega (= (\mathcal{L} - \Omega) \cup \{\infty\}).$$

Then $\partial\Omega$ is a closed subset in ∂B . Let $\overline{\Gamma} \leq \mathcal{L} \rtimes \text{Sp}(n, 1) \cdot \mathbb{C}^*$ be the closure of Γ . As $\partial\Omega$ is closed and Ω is Γ -invariant, note that $\overline{\Gamma}$ leaves $\partial\Omega$ invariant.

Proposition 7.1 *Suppose that $\mathcal{L} - \Omega \neq \emptyset$. Then either one of the following holds: (i) $\overline{\Gamma} \leq \mathcal{L} \rtimes \text{Sp}(n) \cdot S^1$ or (ii) there exists a $\overline{\Gamma}$ -invariant simply connected domain $Y \subset \mathcal{L}$ containing Ω .*

Proof Let $1 \rightarrow \mathbb{R} \rightarrow \mathcal{M} \rtimes \text{Sp}(n) \cdot \mathbb{C}^* \xrightarrow{\nu} \mathcal{L} \rtimes \text{Sp}(n) \cdot \mathbb{C}^* \rightarrow 1$ be the exact sequence. Put $G = \nu^{-1}(\overline{\Gamma})$ and $\Lambda = \overline{p}^{-1}(\partial\Omega)$. Then Λ is a G -invariant closed subset in S^{4n+3} .

Since $\mathcal{L} - \Omega \neq \emptyset$, note that $\Lambda \subsetneq S^{4n+3}$. Let $L(G)$ be the limit set for G (compare [2]). As Λ contains more than one point, it follows $L(G) \subset \Lambda$. Consider the exact sequence

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{M} \rtimes \mathrm{Sp}(n) \cdot S^1 & \longrightarrow & \mathcal{M} \rtimes \mathrm{Sp}(n) \cdot \mathbb{C}^* & \xrightarrow{q} & \mathbb{R}^+ \longrightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & \mathfrak{q}(G) \longrightarrow 1 \end{array}$$

where $H = G \cap (\mathcal{M} \rtimes \mathrm{Sp}(n) \cdot S^1)$.

Suppose that $\mathfrak{q}(G) \neq \{1\}$ otherwise (i) holds. Let G^0 be the identity component of G .

Case 1. Suppose that $\mathfrak{q}(G^0) = \mathbb{R}^+$. Then $G = H \rtimes \mathbb{R}^+$. As $G^0 = H^0 \rtimes \mathbb{R}^+$, it follows (cf. [11])

$$L(G) = L(H^0 \rtimes \mathbb{R}^+).$$

Case 2. Suppose that $G^0 \leq \mathcal{M} \rtimes \mathrm{Sp}(n, 1) \cdot S^1$. The above exact sequence induces a group extension

$$1 \rightarrow H/G^0 \rightarrow G/G^0 \xrightarrow{q} \mathfrak{q}(G) \rightarrow 1.$$

As $\mathfrak{q}(G) \leq \mathbb{R}^+$ and H/G^0 is discrete, noting that $\mathbb{R}^+$ acts on $\mathcal{M}$ as multiplication: $\lambda(a, z) = (\lambda^2 a, \lambda z)$, $\lambda \in \mathbb{R}^+$, $(a, z) \in \mathcal{M}$, it follows $H/G^0 = \{1\}$, or $H = G^0$. Then $\mathfrak{q}(G)$ is discrete in $\mathbb{R}^+$ and so $\mathfrak{q}(G) \cong \mathbb{Z}^+$. It follows $G = H^0 \rtimes \mathbb{Z}^+$ and so

$$L(G) = L(H^0 \rtimes \mathbb{Z}^+).$$

Let $(G, \overline{\mathbb{H}}_{\mathbb{H}}^{n+1})$ be the action as above, where $G \leq \mathcal{M} \rtimes \mathrm{Sp}(n, 1) \cdot \mathbb{C}^*$. The totally geodesic subspace of $\overline{\mathbb{H}}_{\mathbb{H}}^{n+1}$ is $\mathbb{H}_{\mathbb{K}}^m$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $1 \leq m \leq n+1$. If we note that $\mathbb{R} \leq H^0 \leq \mathcal{M} \rtimes \mathrm{Sp}(n, 1) \cdot S^1$, then $H^0 \rtimes \mathbb{R}^+$ is viewed as a warped product which acts transitively in $\mathbb{H}_{\mathbb{K}}^m$. Similarly, $H^0 \rtimes \mathbb{Z}^+$ acts in $\mathbb{H}_{\mathbb{K}}^m$ for which the orbit (at a point in $\mathbb{H}_{\mathbb{K}}^m$) approaches to the boundary $\partial \mathbb{H}_{\mathbb{K}}^m$. By the definition of the limit set (cf. [2]), both Cases 1 and 2 imply that

$$L(G) = \partial \mathbb{H}_{\mathbb{K}}^m$$

for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $1 \leq m \leq n+1$, $1 \leq m \leq n+1$, $1 \leq m \leq n$ respectively. For the last case, $m = n+1$ is excluded because $\Lambda \subsetneq S^{4n+3}$ (as $\mathbb{R} \leq H^0$, note that $\partial(\mathbb{R} \rtimes \mathbb{R}^+ \cdot p) = \partial(\mathbb{R} \rtimes \mathbb{Z}^+ \cdot p) = \partial \mathbb{H}_{\mathbb{R}}^2 = S^1$ for $p \in \overline{\mathbb{H}}_{\mathbb{H}}^{n+1}$. In particular, $\{\infty\} \in S^1 \subset L(G)$). Then the complement $S^{4n+3} - L(G)$ is homeomorphic to $\mathbb{H}_{\mathbb{K}}^m \times S^\ell$, where $\ell = 4n+3 - m|\mathbb{K}|$ for $|\mathbb{K}| = 1, 2$ or 4 . Since $\ell \geq 3n+2, 2n+1, 3$ respectively for $|\mathbb{K}| = 1, 2$ or 4 , $S^{4n+3} - L(G)$ is a simply connected G -invariant submanifold.

Moreover, as $S^{4n+3} - L(G) = S^{4n+3} - S^{m|\mathbb{K}|-1}$, if $\mathrm{Aut}(S^{4n+3} - L(G))$ is a subgroup of $\mathrm{PSp}(n+1, 1)$ leaving $S^{4n+3} - L(G)$ invariant, then it acts properly on $S^{4n+3} - L(G)$ (compare [8, 14]). This follows also from the result of Frances [4] which says that if

a domain D of S^{4n+3} is not $S^{4n+3} \setminus \mathcal{M}$, then $\text{Aut}(D)$ acts properly on D . Since G is a closed subgroup of $\text{Aut}(S^{4n+3} \setminus L(G))$, G acts properly on $S^{4n+3} \setminus L(G)$.

As $\infty \in L(G)$, note that $S^{4n+3} \setminus L(G) = (\mathcal{M} \cup \infty) \setminus L(G) \subset \mathcal{M}$. Since $\mathbb{R} (\leq G)$ acts properly and freely on $\mathcal{M}$, we have a simply connected manifold $Y = S^{4n+3} \setminus L(G)/\mathbb{R}$ which is a submanifold of $\mathcal{L}$. There induces an equivariant principal bundle:

$$\mathbb{R} \rightarrow (G, S^{4n+3} \setminus L(G)) \xrightarrow{(v, \bar{p})} (\bar{\Gamma}, Y).$$

Then it is easy to see that $\bar{\Gamma}$ acts properly on Y . There is a $\bar{\Gamma}$ -invariant Riemannian metric on Y (compare [13]). Recall that $L(G) \subset \Lambda$, then

$$\Omega \subset S^{4n+2} \setminus \partial\Omega = \bar{p}(S^{4n+3} \setminus \Lambda) \subset \bar{p}(S^{4n+3} \setminus L(G)) = Y.$$

We obtain $(\Gamma, \Omega) \subset (\bar{\Gamma}, Y)$. □

Theorem 7.2 *If the developing map of a compact $(4n+2)$ -dimensional complex contact similarity manifold M is not surjective onto $\mathcal{L}$, then dev is a biholomorphism of $\tilde{M}$ onto $\mathcal{L} \setminus \{0\}$. In particular, M is a complex contact infra-Hopf manifold.*

Proof We have a developing pair $\text{dev}: (\pi, \tilde{M}) \rightarrow (\Gamma, \mathcal{L})$. Suppose that dev is not surjective onto $\mathcal{L}$. Let $\Omega = \text{dev}(\tilde{M})$. Then $\mathcal{L} \setminus \Omega = \partial\Omega \neq \emptyset$. By Proposition 7.1, there is a $\bar{\Gamma}$ -invariant Riemannian metric on Y such that the pull back metric by dev induces a local isometry $\text{dev}: (\pi, \tilde{M}) \rightarrow (\bar{\Gamma}, Y)$. As M is compact and Y is simply connected, $\text{dev}: \tilde{M} \rightarrow Y$ is an isometry. Hence $(\Gamma, \Omega) = (\bar{\Gamma}, Y)$. In particular, Γ is discrete. Then the result follows from Theorem A. □

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