Analytic closure of sets of real hyperbolic polynomials with separated roots

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Abstract We characterize closures (in the topology of local uniform convergence on $\mathbb{C}$) of certain classes of hyperbolic polynomials (real polynomials with no complex roots) in terms of restricted versions of the classical Laguerre–Pólya class. Our results are motivated by recent considerations of Fisk and others of the interlacing-preserving linear operators on hyperbolic polynomials. We discuss closures of classes of hyperbolic polynomials with mesh (minimal root separation) bounded from below and of hyperbolic polynomials with logarithmic mesh (minimal ratio of successive roots) bounded from below. We also provide some examples and applications.

Keywords Linear operators on hyperbolic polynomials · Zero separation · Laguerre–Pólya class · Approximation of entire functions · Growth estimation

Mathematics Subject Classification 30C15 · 30D15 · 47B38

1 Introduction

Denote by $\mathcal{HP} \subset \mathbb{R}[x]$ the set of all *hyperbolic* (having all its zeros on the real line) polynomials. As usual we will treat the zero polynomial as hyperbolic. A linear operator $T: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ is called a *hyperbolicity preserver* if it preserves $\mathcal{HP}$.

We say that polynomials P and Q of the same degree with roots correspondingly $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ *strictly interlace* if one of the chains of inequalities $x_1 > y_1 > x_2 > y_2 > \dots > x_n > y_n$ or $y_1 > x_1 > y_2 > x_2 > \dots > y_n > x_n$

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holds. Following Fisk [6], we write $P \ll Q$ if P and Q interlace and P has the largest root. There exist other types of interlacing, including not strict interlacing and cases when polynomials' degrees are different. For more details, see [6]. We say that a linear operator T *preserves interlacing* if whenever P and Q interlace, so do TP and TQ . It is important to note that if T maps $\mathcal{HP}$ to itself, then T preserves interlacing [6, Theorem 1.43].

Let A be an affine transformation, i.e. $A(x) = qx + b$, $q \neq 0$. Then it acts on polynomials as $AP(x) = P(qx + b)$. With every affine transformation we associate a subset in $\mathcal{HP}$, denoted by $\mathcal{HP}^A = \{P \in \mathcal{HP} : P \ll AP\}$.

The following property of $\mathcal{HP}^A$ due to Fisk [6] stimulated our further explorations of certain classes $\mathcal{HP}^A$ and their closures, which we present in this paper.

Theorem A ([6, Lemma 8.25]) *If T preserves interlacing, A is an affine transformation and $TA = AT$, then $T : \mathcal{HP}^A \rightarrow \mathcal{HP}^A$.*

This theorem has turned out to be an important result for problems concerning root separation, as can be seen from the recent results in [3–5, 7]. We explain some of its consequences below and this will help to explain our motivation.

Definition 1.1 Given a polynomial $P \in \mathcal{HP}$, $\deg P \geq 2$, denote by $\text{mesh}(P)$ the minimal distance between its roots

$$\text{mesh}(P) = \min_{j \neq k} |x_j - x_k|, \quad P = C(x - x_1)(x - x_2) \cdots (x - x_n).$$

Note that if P has a double real root, then $\text{mesh}(P) = 0$. For $\deg P \leq 1$ we put $\text{mesh}(P) = +\infty$.

Shift operator $(S_b P)(x) = P(x + b)$, $b > 0$. The space $\mathcal{HP}^{S_b}$ can be easily described. Indeed, polynomials $P(x)$ and $P(x + b)$ interlace if and only if the minimal distance between two roots of the polynomial P is greater than b , i.e. $P \in \mathcal{HP}^{S_b}$ if and only if $\text{mesh}(P) > b$.

If we denote by $\mathcal{HP}^{>b}$ the set of polynomials with mesh bounded from below by b , then Theorem A for $A = S_b$ can be formulated in the following way.

Theorem A1 *If a hyperbolicity preserver T commutes with the shift operator S_b , $S_b P(x) = P(x + b)$, $b > 0$, then it preserves $\mathcal{HP}^{>b}$.*

Hence

Theorem A1' *If a hyperbolicity preserver T commutes with all shift operators, then it does not decrease mesh of polynomials in $\mathcal{HP}$.*

As a consequence of Theorem A we get straightforwardly the following earlier known facts.

Example 1.2 Differentiation operator does not decrease mesh. Indeed, differentiation commutes with all shift operators.

Example 1.3 An operator of the form $T = I + \lambda D$ does not decrease the mesh. This is a classic theorem, due to Riesz, but it was recorded by Stoyanoff [16]: *For any hyperbolic polynomial P and any real λ , $\text{mesh}(P + \lambda P') \geq \text{mesh}(P)$.* The natural extension of this result can be obtained using the Hermite–Poulain theorem [13, p. 4], which states that a finite order linear differential operator $T = a_0 + a_1 d/dx + \cdots + a_k d^k/dx^k$ with constant coefficients is a hyperbolicity preserver if and only if its symbol polynomial $Q(t) = a_0 + a_1 t + \cdots + a_k t^k$ is hyperbolic.

Definition 1.4 For a polynomial P of degree k , $k \geq 2$, with all roots positive $0 < x_1 \leq x_2 \leq \cdots \leq x_k$ define its logarithmic mesh as

$$\text{lmesh}(P) = \min_{j=1, \dots, k-1} \frac{x_{j+1}}{x_j}.$$

For $\deg P \leq 1$ we put $\text{lmesh}(P) = +\infty$.

Obviously, for any polynomial with all roots real and positive one has $\text{lmesh}(P) \geq 1$ and $\text{lmesh}(P) = 1$ if and only if P has a multiple (real) root.

Dilatation operator $(T_q P)(x) = P(qx)$, $q > 1$. The space $\mathcal{HP}^{T_q}$ can be easily described as well. First of all, $P \in \mathcal{HP}^{T_q}$ necessarily implies that all roots of P are positive. We will denote by $\mathcal{HP}_+$ the set of polynomials with all roots positive. Secondly, the polynomials $P(x)$ and $P(qx)$ interlace if and only if the minimal ratio of two roots of polynomial P is greater than q , i.e. $P \in \mathcal{HP}^{T_q}$ if and only if $\text{lmesh}(P) > q$.

Similarly to mesh, we will denote by $\mathcal{HP}_+^{>q}$ the set of all polynomials with roots positive and with logarithmic mesh bounded from below by q . For $A = T_q$, Theorem A states

Theorem A2 *If a hyperbolicity preserver T commutes with the dilatation operator T_q , then it preserves $\mathcal{HP}_+^{>q}$.*

Hence

Theorem A2' *If a hyperbolicity preserver commutes with all dilatation operators, then it does not decrease logarithmic mesh of polynomials in $\mathcal{HP}_+$.*

Example 1.5 Consider the Schur–Szegő composition of polynomials $P(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ and $Q(x) = \sum_{k=0}^n \binom{n}{k} b_k x^k$ which is defined as

$$(P * Q)(x) = \sum_{k=0}^n \binom{n}{k} a_k b_k x^k.$$

For more information about the Schur–Szegő composition, see [12, 17]. Its well-known property is the following: if P and Q are hyperbolic and one of them has all roots of the same sign, then $P * Q$ also has all its roots real.

Katkova, Shapiro, and Vishnyakova established in [11] the following result: *Given two polynomials P and Q of degree k with all roots of the same sign, then $\text{lmesh}(P * Q) \geq \max\{\text{lmesh}(P), \text{lmesh}(Q)\}$.*

Theorem A again gives a straightforward proof of this fact. Let Q be fixed and have all roots negative. Then we get a linear operator, preserving hyperbolicity and, therefore, preserving interlacing, and mapping $\mathcal{HP}_+$ to $\mathcal{HP}_+$. Such an operator commutes with dilatation operators (see definition of the Schur–Szegő composition). Then, by Theorem A2', it does not decrease logarithmic mesh.

Example 1.6 Differentiation does not decrease logarithmic mesh. We know that differentiation preserves interlacing, however, it does not commute with dilatation operators. Instead the next equality holds: $DT_q P = qT_q DP$. But it turns out that such weaker commutation property is sufficient for Theorem A to remain true.

Remark 1.7 The result of Theorem A remains true if we substitute commutation by the condition that there exists $\lambda \neq 0$ such that $TA = \lambda AT$. This follows from the fact that the polynomials P and λP have the same roots, and it is only the roots of polynomials which are essential for the proof of Theorem A.

Theorems A1 and A2 can be extended in a straightforward way to analytic closures of $\mathcal{HP}^{>b}$ and $\mathcal{HP}_+^{>q}$. Before we state the corresponding results, we need to recall some facts about limits of polynomials in the topology of convergence on compact subsets of $\mathbb{C}$.

Definition 1.8 A real entire function f is said to be in the *Laguerre–Pólya class*, denoted $f \in \mathcal{L-P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k},$$

where $c, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. The product in the right-hand side can be finite or empty (in which case it is set equal to 1).

This class is important in the theory of entire functions due to the following property due to Laguerre and Pólya, see, e.g. [9, pp. 42–46]:

- Let $(P_n)_{n=1}^{\infty} \subset \mathcal{HP}$ be a sequence of polynomials, and let $(P_n)_{n=1}^{\infty}$ converge uniformly in the disk $|z| \leq A$, for some $A > 0$. Then the sequence converges uniformly on compact subsets in $\mathbb{C}$ to an entire function $f \in \mathcal{L-P}$.
- Conversely, for any $f \in \mathcal{L-P}$ there exists a sequence $(P_n)_{n=1}^{\infty} \subset \mathcal{HP}$ which converges to f uniformly on compact sets of $\mathbb{C}$.

For various properties and characterizations of the Laguerre–Pólya class, see Pólya and Schur [1, p. 100], [15] or [13, Kapitel II].

Definition 1.9 A real entire function f is said to be of *type I in the Laguerre–Pólya class*, written $f \in \mathcal{L-PI}$, if one of functions $f(z)$, $-f(z)$, $f(-z)$ or $-f(-z)$ can be expressed in the form

$$cx^n e^{\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right),$$

where $c, \beta, x_k \in [0; +\infty)$, $x_k \neq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k < \infty$.

As shown in [1, p. 100] and [15], real entire functions from $\mathcal{L}\text{-}\mathcal{P}\mathcal{I}$, and only them, are the uniform limits on compact subsets of $\mathbb{C}$ of polynomials with all zeros real and of the same sign.

Now we can formulate analogs of Theorems A1 and A2 for entire functions.

Theorem A1'' *If a linear operator T , preserving $\mathcal{L}\text{-}\mathcal{P}$, maps $\mathcal{HP}^{>b}$ to $\mathcal{HP}$ and commutes with the shift operator S_b , then it preserves $\overline{\mathcal{HP}^{>b}}$.*

Theorem A2'' *If a linear operator T , preserving $\mathcal{L}\text{-}\mathcal{P}$, maps $\mathcal{HP}_+^{>q}$ to $\mathcal{HP}$ and commutes with the dilatation operator T_q , then it preserves $\overline{\mathcal{HP}_+^{>q}}$.*

The aim of this paper is to give complete descriptions of the classes $\overline{\mathcal{HP}^{>b}}$ and $\overline{\mathcal{HP}_+^{>q}}$, Theorems 1.10 and 1.11.

Theorem 1.10 *A function f belongs to $\overline{\mathcal{HP}_+^{>q}}$ if and only if f is of the form*

$$f(x) = cx^r \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$, $r \in \mathbb{N} \cup \{0\}$, $x_k > 0$, $\sum_{k=1}^{\infty} 1/x_k < \infty$ and $\inf_k x_{k+1}/x_k \geq q$.

The characterization of $\overline{\mathcal{HP}^{>b}}$ was suggested by Fisk [6]: as long as a function f in $\mathcal{L}\text{-}\mathcal{P}$ has no factor $e^{-\alpha x^2}$ and minimal distance between roots is at least b , f belongs to $\overline{\mathcal{HP}^{>b}}$. However, the proof of this fact in [6] is not complete. In [6] it is shown that for the function $e^{-\alpha x^2}$ the differentiation operator decreases mesh, and hence, according to Theorem A1, $e^{-\alpha x^2}$ cannot be in $\overline{\mathcal{HP}^{>b}}$. Next it is shown that functions without the factor $e^{-\alpha x^2}$ do belong to $\overline{\mathcal{HP}^{>b}}$. But these two facts taken together, strictly speaking, still do not show that only functions without the factor $e^{-\alpha x^2}$ are in $\overline{\mathcal{HP}^{>b}}$. In this paper we prove directly that indeed no other functions belong to $\overline{\mathcal{HP}^{>b}}$.

Theorem 1.11 *A function f belongs to $\overline{\mathcal{HP}^{>b}}$ if and only if f is of the form*

$$f(x) = cx^{\theta} e^{\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-x/x_k},$$

where $c, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, θ is zero or 1, $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$, and the distance between two subsequent roots is greater than or equal to b .

The paper is organized as follows. In Sects. 2 and 3 we prove Theorems 1.10 and 1.11. In Sect. 4, we provide examples of linear operators commuting with a shift operator or a dilatation operator and investigate the action of such operators on the Laguerre–Pólya class. While operators commuting with all shifts $P(x) \mapsto P(x+b)$, $b > 0$, do not decrease mesh of polynomials, the obvious extension of this result to the Laguerre–Pólya class fails. We also give similar examples for Imesh .

2 Limits of sequences of polynomials with logarithmic mesh bounded from below

Assume that $(P_n(z))_{n=1}^{\infty}$ is a sequence of hyperbolic polynomials of the form

$$P_n(z) = \prod_{j=1}^{m_n} \left(1 - \frac{z}{x_j(n)}\right),$$

where $x_j(n) > 0$, and for each n we have $x_1(n) < x_2(n) < x_3(n) < \dots$. We also suppose that $\text{lmesh}(P_n) > q$. This sequence converges uniformly on compacts to an entire function $f(z)$ (denote its roots by $x_1 \leq x_2 \leq x_3 \leq \dots$). Our aim is to understand what kind of function f is.

First, let us locate the roots of f (the case where f has no roots will be included below). If f has a root of degree r at the origin, then we will modify the sequence of polynomials $P_n(z) = \prod_{j=r+1}^{m_n} (1 - z/x_j(n))$ so that it converges to a function without a root at the origin.

It can be easily seen that, for each j , the sequence $x_j(n)$ converges to x_j . We will show that this is true for the smallest root. Indeed, if not, then there is a subsequence $x_1(n_k)$ of $x_1(n)$ which converges to some $a \neq x_1$. Taking the corresponding subsequence of polynomials P_{n_k} we see that a is the minimal root of the limit of P_{n_k} , therefore, P_{n_k} cannot converge to f . It is impossible as P_{n_k} is a subsequence of a converging sequence. Thus, $x_1(n)$ converges to x_1 and similarly $x_j(n)$ converges to x_j .

This argument implies that $\text{lmesh}(f) \geq q$, hence $\prod_{j=1}^{\infty} (1 - z/x_j)$ converges. Thus we conclude that $f(z) = g(z)z^{\theta} \prod_{j=1}^{\infty} (1 - z/x_j)$, where $g(z)$ does not have any roots (therefore, $g(z) \equiv 1$ or $g(z) = e^{\alpha x}$). We will show that $g(z) \equiv 1$.

Consider the sequence of polynomials $P_{nk}(z) = \prod_{j=k+1}^{\infty} (1 - z/x_j(n))$. When k is fixed, the sequence converges to $f_k(z) = g(z) \prod_{j=k+1}^{\infty} (1 - z/x_j)$. This fact will be shown with the help of the next lemma.

Lemma 2.1 *The sequence of polynomials $P_{n1}(z) = \prod_{j=2}^{\infty} (1 - z/x_j(n))$ converges to the function $f_1(z)$ uniformly in $A = \{|z| \leq x_1/2\}$.*

Proof We have

$$\begin{aligned} \left| P_{n1}(z) - \frac{f(z)}{1 - z/x_1} \right| &\leq \left| P_{n1}(z) - \frac{P_n(z)}{1 - z/x_1} \right| + \left| \frac{P_n(z)}{1 - z/x_1} - \frac{f(z)}{1 - z/x_1} \right| \\ &\leq |P_n(z)| \left| \frac{1}{1 - z/x_1(n)} - \frac{1}{1 - z/x_1} \right| + \left| \frac{1}{1 - z/x_1} \right| |P_n(z) - f(z)| \\ &< M\varepsilon_1 + C\varepsilon_2 \rightarrow 0. \end{aligned}$$

The last inequality follows from the following simple observations:

- convergence of the polynomials implies that they are uniformly bounded in A , $|P_n(z)| < M$ for all n ,

•

$$\left| \frac{1}{1 - z/x_1(n)} - \frac{1}{1 - z/x_1} \right| \rightarrow 0$$

because $x_1(n) \rightarrow x_1$,

- $|1/(1 - z/x_1)|$ is bounded by a constant C on A ,
- P_n converge uniformly on compacts to f , so $|P_n(z) - f(z)| \rightarrow 0$ in A . $\square$

Lemma 2.1 implies, that for each $k \in \mathbb{N}$ the sequence of polynomials $P_{nk}(z) = \prod_{j=k+1}^{\infty} (1 - z/x_j(n))$ converges to the function $f_k(z) = f(z)/\prod_{j=1}^k (1 - z/x_j)$ uniformly in $A = \{|z| \leq x_1/2\}$.

So, $P_{nk}(z) = \prod_{j=k+1}^{m_n} (1 - z/x_j(n))$ converge to $f_k(z) = g(z) \prod_{j=k+1}^{\infty} (1 - z/x_j)$. Take $g(z) = e^{\alpha z}$, and let us present P_{nk} and f_k as power series,

$$P_{nk}(z) = 1 - \sum_{j=k+1}^{m_n} \frac{1}{x_j(n)} z + \sum_{i,j=k+1}^{m_n} \frac{1}{x_j(n)x_i(n)} z^2 + \dots,$$

$$f_k(z) = e^{\alpha z} \prod_{j=k+1}^{\infty} \left(1 - \frac{z}{x_j}\right) = (1 + \alpha z + \dots) \left(1 - \sum_{j=k+1}^{\infty} \frac{1}{x_j} z + \dots\right).$$

Convergence of functions implies that the corresponding coefficients converge. Therefore, when k is fixed, $\sum_{j=k+1}^{m_n} 1/x_j(n) \rightarrow \sum_{j=k+1}^{\infty} 1/x_j - \alpha$ as n tends to infinity. However, by the choice of k each sum can be made very small

$$\sum_{j=k+1}^{m_n} \frac{1}{x_j(n)} \leq \sum_{j=k+1}^{m_n} \frac{1}{x_1(n)q^{j-1}} = \frac{1}{x_1(n)q^k} \sum_{j=0}^{m_n} \frac{1}{q^j}.$$

As $x_1(n)$ converges to x_1 , for sufficiently large n , we have $x_1(n) \geq x_1/2$, and finally choosing k big enough, we get

$$\sum_{j=k+1}^{m_n} \frac{1}{x_j(n)} \leq \frac{2}{x_1 q^k} \frac{q}{q-1} < \varepsilon,$$

similarly

$$\sum_{j=k+1}^{\infty} \frac{1}{x_j} \leq \frac{1}{x_1 q^k} \frac{q}{q-1} < \varepsilon.$$

Hence, in $\sum_{j=k+1}^{\infty} 1/x_j(n) \rightarrow \sum_{j=k+1}^{\infty} 1/x_j - \alpha$ the left-hand side is close to zero and the right-hand side is close to α , thus $\alpha = 0$. So $g(z) \equiv 1$ and $f(z) = \prod_{j=1}^{\infty} (1 - z/x_j)$. This proves the first implication in Theorem 1.10.

Let us show that the converse is also true. That is, if f is of the form $f(x) = cx^r \prod_{k=1}^{\infty} (1 - x/x_k)$, then it can be approximated by polynomials whose logarithmic

mesh is bounded from below by q . If f has no root at the origin, then approximation can be achieved just by partial products $P_n = c \prod_{k=1}^n (1 - x/x_k)$. As for x^r , it can be approximated by $Q_n(x) = (x - 1/n)(x - q/n)(x - q^2/n) \cdots (x - q^k/n)$. The proof of Theorem 1.10 is complete.

Here is an important corollary of Theorem 1.10.

Corollary 2.2 *The function $e^{\alpha x}$, $\alpha \neq 0$, cannot be the limit of a sequence of polynomials with logarithmic mesh bounded from below by $q > 1$.*

3 Limits of sequences of polynomials with mesh bounded from below

Here we are going to modify the proof of Theorem 1.10 so that to establish Theorem 1.11. We start with the sequence of polynomials

$$P_n(z) = \prod_{j=-r_n}^{m_n} \left(1 - \frac{z}{x_j(n)}\right),$$

where $x_j(n) \in \mathbb{R}$, $0 < x_0(n) < x_1(n) < x_2(n) < x_3(n) < \cdots$ and $\cdots < x_{-3}(n) < x_{-2}(n) < x_{-1}(n) < 0$, and the distance between any two roots is greater than b .

The sequence $\{P_n(z)\}$ converges to an entire function $f(z)$ from the Laguerre–Pólya class

$$f(z) = e^{-\alpha z^2 + \beta z} x^\theta \prod_{j=-\infty}^{\infty} \left(1 - \frac{z}{x_j}\right) e^{z/x_j}.$$

If f has a root at the origin, we can shift the sequence of polynomials so that f has no root at the origin. Similarly, we have $x_j(n) \rightarrow x_j$. Again we will consider polynomials P_{nk} which have no roots in disks around zero,

$$P_{nk}(z) = \prod_{j=-k}^{-r_n} \left(1 - \frac{z}{x_j(n)}\right) \prod_{j=k}^{m_n} \left(1 - \frac{z}{x_j(n)}\right).$$

They converge to functions

$$f_k(z) = e^{-\alpha z^2 + \beta z} \prod_{j=-k}^{-\infty} \left(1 - \frac{z}{x_j}\right) e^{z/x_j} \prod_{j=k}^{\infty} \left(1 - \frac{z}{x_j}\right) e^{z/x_j}.$$

The sequence of functions $f_k(z)$ converges to $e^{-\alpha z^2 + \beta z}$, so using the standard diagonal process, we can find a subsequence $P_{n_l k_l}$ that converges to $e^{-\alpha z^2 + \beta z}$.

We will show that $\alpha = 0$. Present $P_{n_l k_l}$ and f_k as power series,

$$P_{n_l k_l}(z) = 1 - \left(\sum_{j=k_l}^{m_{n_l}} \frac{1}{x_j(n_l)} + \sum_{j=-k_l}^{-r_{n_l}} \frac{1}{x_j(n_l)} \right) z + \sum_{\substack{i, j=-r_{n_l} \\ |i| \geq k_l, |j| \geq k_l}}^{m_{n_l}} \frac{1}{x_j(n_l)x_i(n_l)} z^2 - \dots,$$

and

$$\begin{aligned} e^{-\alpha z^2 + \beta z} &= 1 + (-\alpha z^2 + \beta z) + \frac{(-\alpha z^2 + \beta z)^2}{2} + \dots \\ &= 1 + \beta z + \left(\frac{\beta^2}{2} - \alpha \right) z^2 + \dots \end{aligned}$$

Now we consider the sum of the inverse squares of roots

$$\sum_{j=k_l}^{m_{n_l}} \frac{1}{x_j(n_l)^2} + \sum_{j=-k_l}^{-r_{n_l}} \frac{1}{x_j(n_l)^2}.$$

It can be expressed using the coefficients of the series for $P_{n_l k_l}$ as $a_1^2 - 2a_2$, where a_1 is the coefficient of z and a_2 is the coefficient of z^2 . As $P_{n_l k_l}$ converges to $e^{-\alpha z^2 + \beta z}$, we have

$$\sum_{j=k_l}^{m_{n_l}} \frac{1}{x_j(n_l)^2} + \sum_{j=-k_l}^{-r_{n_l}} \frac{1}{x_j(n_l)^2} \rightarrow \beta^2 - 2 \left(\frac{\beta^2}{2} - \alpha \right) = 2\alpha.$$

Now we will estimate the sums in the left-hand side. Consider the first of them, the second one is estimated analogously,

$$\sum_{j=k_l}^{m_{n_l}} \frac{1}{x_j(n_l)^2} \leq \sum_{j=k_l}^{\infty} \frac{1}{(x_1(n_l) + b(j-1))^2} \leq \sum_{j=k_l}^{\infty} \frac{1}{(x_1/2 + b(j-1))^2},$$

the last inequality holds as $x_1(n)$ converges to x_1 , therefore, for sufficiently large n , $x_1(n)$ is bounded from below by $x_1/2$. The right-hand side is the tail of a converging series, therefore, it can be made arbitrarily small. So, $\alpha = 0$ and $f(z) = e^{\beta z} \prod_{j=1}^{\infty} (1 - z/x_j) e^{z/x_j}$.

Now let us prove the converse. That is, if f is of the form $f(z) = e^{\beta z} \prod_{j=1}^{\infty} (1 - z/x_j) e^{z/x_j}$, then it can be approximated by polynomials with mesh bounded from below by b . This becomes obvious if we recall that $e^{\alpha z}$ is in $\overline{\mathcal{HCP}^{>b}}$, see [6, Lemma 8.47]. The proof of Theorem 1.11 is complete.

4 Examples of linear operators commuting with shift or dilatation operators

In this section we give various examples of polynomials, entire functions and linear operators, illustrating how mesh and logarithmic mesh change after transformations. These examples may help to understand how Theorem A can be applied.

First we will show how certain linear operators increase mesh or logarithmic mesh of entire functions using Theorem A and its extension to entire functions.

Example 4.1 Consider the function $f(x) = \sin x - 1/2$, $\text{mesh}(f) = 2\pi/3$, and the differential operator D . Then $(Df)(x) = \cos x$ and $\text{mesh}(Df) = \pi$. We see that $\text{mesh}(Df) > \text{mesh}(f)$. Note that the distance between the roots of f is $2\pi/3$ or $4\pi/3$, but the maximal distance between two roots of Df is π . This example shows that not all pairwise distances between neighboring roots increase, but only the minimal ones do.

Example 4.2 Consider the function $f(x) = \sin x + a$, where $|a| \leq 1$, $\text{mesh}(f) = 2 \arccos |a|$, and the operator $T = I + bD$, $b \in \mathbb{R}$. Then

$$(Tf)(x) = \sin x + a + b \cos x = \sqrt{1+b^2} \left(\sin(x+\phi) + \frac{a}{\sqrt{1+b^2}} \right)$$

and $\text{mesh}(Tf) = 2 \arccos(|a|/\sqrt{1+b^2})$. We see that $\text{mesh}(Tf) > \text{mesh}(f)$.

Example 4.3 The function

$$g_a(x) = \prod_{j=1}^{\infty} \left(1 + \frac{x}{a^j} \right) = \sum_{k=0}^{\infty} \frac{x^k}{(a^k - 1)(a^{k-1} - 1) \cdots (a - 1)}, \quad a > 1,$$

is called the partial theta-function. One can easily see that $\text{lmesh}(g_a) = a$.

Consider also the function

$$h_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{k! (a^k + 1)(a^{k-1} + 1) \cdots (a + 1)}.$$

For large a it has all roots real due to the following criterion due to Hutchinson [10, p. 327] who in 1926 extended the work of Pétrovitch [14] and Hardy [8] or [2, pp. 95–100]: Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k . Then $a_n^2 / (a_{n-1} a_{n+1}) \geq 4$ holds for all $n \geq 1$ if and only if the following two properties hold:

- zeros of f are all real, simple and negative,
- zeros of any polynomial $\sum_{k=m}^n a_k z^k$, formed by the two-sided truncations of f , are all real and nonpositive.

Consider $a_n^2/(a_{n-1}a_{n+1})$ for the above function $h_a(x)$, it is equal to

$$\begin{aligned} \frac{(n-1)!(a^{n-1}+1)\cdots(a+1)(n+1)!(a^{n+1}+1)\cdots(a+1)}{(n!(a^n+1)(a^{n-1}+1)\cdots(a+1))^2} \\ = \frac{(n+1)(a^{n+1}+1)}{n(a^n+1)} \geq 4. \end{aligned}$$

The last inequality holds for all n when $a \geq 4$, so, for these a the function $h_a(x)$ has all roots real, simple and negative. So, the sequence

$$\left\{ \frac{1}{(a^k+1)(a^{k-1}+1)\cdots(a+1)} \right\}_{k=0}^{\infty}$$

is a multiplier sequence for $a \geq 4$, which means that the corresponding operator T which multiplies coefficients of function by these numbers preserves the $\mathcal{L}\text{-}\mathcal{P}$ class and commutes with dilatation operators. Let us apply this operator to $g_a(x)$. We get

$$\sum_{k=0}^{\infty} \frac{x^k}{(a^{2k}-1)(a^{2k-2}-1)\cdots(a^2-1)} = \prod_{j=1}^{\infty} \left(1 + \frac{x}{a^{2j}}\right) = g_{a^2},$$

and $\text{lmesh}(Tg_a) = \text{lmesh}(g_{a^2}) = a^2$.

The next example shows how mesh of Chebyshev polynomials increases after differentiation.

Example 4.4 Let $T_n(x)$ be Chebyshev polynomials of the first kind. The roots of $T_n(x)$ are $x_k = \cos \pi(2k-1)/2n$, $k = \overline{1, n}$. We can see that the minimal distance is attained on the first two roots: $\text{mesh}(T_n(x)) = \cos \pi/2n - \cos 3\pi/2n$. Now let us differentiate $T_n(x)$. We get $T'_n(x) = nU_{n-1}(x)$, where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind, and its roots are $x_k = \cos \pi k/n$, $k = \overline{1, n-1}$. Similarly, $\text{mesh}(U_{n-1}(x)) = \cos \pi/n - \cos 2\pi/n$. We have

$$\frac{\text{mesh}(T'_n(x))}{\text{mesh}(T_n(x))} = \frac{\cos \pi/n - \cos 2\pi/n}{\cos \pi/2n - \cos 3\pi/2n} = \frac{\sin \pi/n \sin 3\pi/n}{\sin \pi/n \sin 2\pi/n} \rightarrow \frac{3}{2}, \quad n \rightarrow \infty.$$

Now we show that if $f \in \mathcal{L}\text{-}\mathcal{P}$ does not belong to $\overline{\mathcal{H}\mathcal{P}^{>b}}$, then a linear operator that commutes with the shift operators may decrease its mesh.

Example 4.5 Take the function $f(x) = (x^2 - 1)e^{-x^2} \in \mathcal{L}\text{-}\mathcal{P}$, $\text{mesh}(f) = 2$, and $D = d/dx$ (obviously D commutes with shift operators). We have $(Df)(x) = 2x(2 - x^2)e^{-x^2}$, hence $\text{mesh}(Df) = \sqrt{2} < 2 < \text{mesh}(f)$.

Example 4.6 Consider the function $f(x) = (x^2 - x)e^{-(n^3+2n^2)x^2/(2(n+1))} \in \mathcal{L}\text{-}\mathcal{P}$, $\text{mesh}(f) = 1$, and $D = d/dx$. Then

$$Df(x) = \left((2x-1) + (x^2-x) \left(-\frac{n^3+2n^2}{2(n+1)} 2x \right) \right) e^{-(n^3+2n^2)x^2/(2(n+1))}.$$

One can check that $Df(1/n) = 0$, $Df(0) = -1 < 0$, $Df(-1/n) = 2/(n+1) > 0$, which means that there is a root of Df on the interval $(-1/n; 0)$. Therefore, Df has two roots on the interval $[-1/n; 1/n]$, implying $\text{mesh}(Df) < 2/n$. This example shows that for all $\varepsilon > 0$ there exists $f_\varepsilon \in \mathcal{L}\text{-}\mathcal{P}$ with $\text{mesh}(f_\varepsilon) = 1$ and $\text{mesh}(Df_\varepsilon) < \varepsilon$, hence it is impossible to estimate how mesh decreases.

Example 4.7 Consider the function $f(x) = (x-1)(x-2)e^{-(n^4-3n^2)x/(n^2-1)} \in \mathcal{L}\text{-}\mathcal{PI}$, $\text{lmesh}(f) = 2$, and the operator $A = I + xD/(n^2-2)$, which commutes with dilatation operators. Then

$$Af(x) = \left((x-1)(x-2) + \frac{x}{n^2-1} \left(2x-3 + (x-1)(x-2) \left(-\frac{n^4-3n^2}{n^2-1} \right) \right) \right) \cdot e^{-(n^4-3n^2)x/(n^2-1)}.$$

One can check that $Af(1+1/n) = 0$, and $Af(1-1/n) = 0$, which means that $\text{lmesh}(Af) < (1+1/n)/(1-1/n) = (n+1)/(n-1)$. As $\lim_{n \rightarrow \infty} (n+1)/(n-1) = 1$, this example shows that for all $\varepsilon > 0$ there is $f_\varepsilon \in \mathcal{L}\text{-}\mathcal{P}$ with $\text{lmesh}(f_\varepsilon) = 2$ and $A_\varepsilon: \mathcal{L}\text{-}\mathcal{PI} \rightarrow \mathcal{L}\text{-}\mathcal{PI}$ such that $\text{lmesh}(A_\varepsilon f_\varepsilon) < 1 + \varepsilon$, hence it is impossible to estimate how lmesh decreases.

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