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RESEARCH ARTICLE

Around the Mukai conjecture for Fano manifolds

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Abstract As a generalization of the Mukai conjecture, we conjecture that the Fano manifolds X which satisfy the property $\rho_X(r_X - 1) \geq \dim X - 1$ have very special structure, where ρ_X is the Picard number of X and r_X is the index of X. In this paper, we classify those X with $\rho_X \leq 3$ or $\dim X \leq 5$.

Keywords Fano manifold · Mukai conjecture · Extremal ray

Mathematics Subject Classification 14J45 · 14E30

1 Introduction

Let X be a Fano manifold, that is, a smooth projective variety such that the anticanonical divisor is ample. In this paper, we study the relationship between the Picard number ρ_X, the *index*

$$r_X = \max\left\{r \in \mathbb{Z}_{>0} : -K_X \sim rL \text{ for some Cartier divisor } L\right\}$$

and the *pseudoindex*

$$\iota_X = \min\left\{(-K_X \cdot C) : C \text{ is a rational curve on } X\right\}.$$

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Clearly, ι_X is divisible by r_X. In particular, we have $\iota_X \geq r_X$.

The following conjecture due to Mukai [22] is one of the most famous conjectures related to the relationship between the Picard number and the index of a Fano manifold.

Conjecture 1.1 (Mukai conjecture) *Let X be a Fano manifold then $\rho_X(r_X - 1) \leq \dim X$ and equality holds if and only if $X \simeq (\mathbb{P}^{r_X-1})^{\rho_X}$.*

Based on an earlier work due to Wiśniewski [30], Bonavero et al. [4] generalized Conjecture 1.1 by replacing r_X with ι_X.

Conjecture 1.2 (generalized Mukai conjecture) *Let X be a Fano manifold then $\rho_X(\iota_X - 1) \leq \dim X$ and equality holds if and only if $X \simeq (\mathbb{P}^{\iota_X-1})^{\rho_X}$.*

As in [9], we will consider split versions of the Mukai and generalized Mukai conjectures, see Sect. 5.

Conjecture 1.3 *Let n and ρ be positive integers.*

- (Conjecture M^n_ρ) *Let X be an n-dimensional Fano manifold. If $\rho_X \geq \rho$ and $r_X \geq (n+\rho)/\rho$, then X is isomorphic to $(\mathbb{P}^{r_X-1})^{\rho}$.*
- (Conjecture GM^n_ρ) *Let X be an n-dimensional Fano manifold. If $\rho_X \geq \rho$ and $\iota_X \geq (n+\rho)/\rho$, then X is isomorphic to $(\mathbb{P}^{\iota_X-1})^{\rho}$.*

It is obvious that the Mukai conjecture (resp. generalized Mukai conjecture) is true if and only if Conjecture M^n_ρ (resp. GM^n_ρ) is true for all positive integers n, ρ.

We conjecture that n-dimensional Fano manifolds X with $\rho_X(r_X - 1) \geq n-1$ (resp. $\rho_X(\iota_X - 1) \geq n-1$) have a very special structure.

Conjecture 1.4 *Let n and ρ be positive integers.*

- (Conjecture AM^n_ρ) *Let X be an n-dimensional Fano manifold. If $\rho_X \geq \rho$ and $r_X \geq (n+\rho-1)/\rho$, then X is isomorphic to one of the following:*
  - (i) $(\mathbb{P}^{r_X-1})^{\rho}$,
  - (ii) $\mathbb{Q}^{r_X} \times (\mathbb{P}^{r_X-1})^{\rho-1}$,
  - (iii) $\mathbb{P}_{\mathbb{P}^{r_X}}\left(\mathcal{O}^{\oplus r_X-1} \oplus \mathcal{O}(1)\right) \times (\mathbb{P}^{r_X-1})^{\rho-2}$,
  - (iv) $\mathbb{P}_{\mathbb{P}^{r_X}}(T_{\mathbb{P}^{r_X}}) \times (\mathbb{P}^{r_X-1})^{\rho-2}$.
- (Conjecture AGM^n_ρ) *Let X be an n-dimensional Fano manifold. If $\rho_X \geq \rho$ and $\iota_X \geq (n+\rho-1)/\rho$, then X is isomorphic to one of the following:*
  - (i) $(\mathbb{P}^{\iota_X-1})^{\rho}$,
  - (ii) $\mathbb{Q}^{\iota_X} \times (\mathbb{P}^{\iota_X-1})^{\rho-1}$,
  - (iii) $\mathbb{P}_{\mathbb{P}^{\iota_X}}\left(\mathcal{O}^{\oplus \iota_X-1} \oplus \mathcal{O}(1)\right) \times (\mathbb{P}^{\iota_X-1})^{\rho-2}$,
  - (iv) $\mathbb{P}_{\mathbb{P}^{\iota_X}}(T_{\mathbb{P}^{\iota_X}}) \times (\mathbb{P}^{\iota_X-1})^{\rho-2}$,
  - (v) $\mathbb{P}^{\iota_X} \times (\mathbb{P}^{\iota_X-1})^{\rho-1}$.

In particular, Conjecture AM^n_1 (resp. Conjecture AGM^n_1) asserts that an n-dimensional Fano manifold X with $r_X \geq n$ (resp. $\iota_X \geq n$) is isomorphic to either $\mathbb{P}^n$ or $\mathbb{Q}^n$. The "A" in AM^n_ρ and AGM^n_ρ stands for "advanced". We note that Conjecture 1.4 asserts in particular that the variety $\mathbb{P}^{\iota_X} \times (\mathbb{P}^{\iota_X-1})^{\rho-1}$ is characterized by the Fano manifold such that the gap between index and pseudoindex is the "largest".

Remark 1.5 Clearly, Conjecture GM^n_ρ (resp. Conjecture AGM^n_ρ) implies Conjecture M^n_ρ (resp. Conjectures AM^n_ρ and GM^n_ρ). We also note that Conjecture GM^n_ρ is true if $n \le 5$ [1] or $\rho \le 3$ [6,13,24], Conjecture AGM^n_ρ is true if $n \le 3$ [11,12,20,21,28], Conjecture AM^n_ρ is true if $n \le 4$ [29] or $\rho \le 2$ [14,32], and Conjecture AGM^n_1 is proved in [18].

In this paper, we prove Conjecture AM^n_ρ provided $\rho \le 3$ or $n \le 5$.

Theorem 1.6 (main theorem) *Conjecture* AM^n_ρ *is true if* $\rho \le 3$ *or* $n \le 5$.

In other words, we classify Fano manifolds X which satisfy the property $\rho_X(r_X - 1) \ge \dim X - 1$ under the condition $\rho_X \le 3$ or $\dim X \le 5$. We note that, as a corollary of [23, Theorem 5.1], any n-dimensional Fano manifold X with $\rho_X \ge 3$ and $r_X \ge (n+2)/3$ satisfies either $\rho_X = 3$ or $X \simeq (\mathbb{P}^1)^4$. Let us rephrase Theorem 1.6 for reader's convenience.

Theorem 1.7 *Let X be an n-dimensional Fano manifold. Suppose that $\rho_X(r_X - 1) \ge n - 1$. Suppose also that either $\rho_X \le 3$ or $n \le 5$. Then X is isomorphic to one of* $(\mathbb{P}^{r_X-1})^{\rho_X}$, $\mathbb{Q}^{r_X} \times (\mathbb{P}^{r_X-1})^{\rho_X-1}$ *with* $r_X \ge 3$, $\mathbb{P}_{\mathbb{P}^{r_X}}\left(\mathcal{O}^{\oplus r_X-1} \oplus \mathcal{O}(1)\right) \times (\mathbb{P}^{r_X-1})^{\rho_X-2}$ *or* $\mathbb{P}_{\mathbb{P}^{r_X}}(T_{\mathbb{P}^{r_X}}) \times (\mathbb{P}^{r_X-1})^{\rho_X-2}$.

In order to prove Theorem 1.6, we consider a certain inductive process. We will prove the following proposition.

Proposition 1.8 (a) *Let $n \ge 2$ and $\rho \in \{2, 3\}$. Then Conjectures* $\mathrm{AGM}^{n'}_{\rho-1}$ *for all* $n' \le n - (n-1)/\rho$ *imply Conjecture* AGM^n_ρ.
(b) *Conjecture* AGM^n_ρ *is true if* $n \le 5$ *and* $\rho \ge 2$.

Remark 1.9 We do not use the deep result [18] in order to prove Theorem 1.6 and Proposition 1.8. Obviously, if we combine Proposition 1.8 and [18, Theorem 0.1], then we can show that Conjecture AGM^n_ρ is true for $\rho \le 3$ or $n \le 5$.

The paper is organized as follows. In Sects. 2 and 3, we recall definitions and some properties of families of rational curves and chains of rational 1-cycles on Fano manifolds, see also [24, Sections 2–3]. In Sect. 4, we study some vector bundles on special Fano manifolds. This step is crucial in the inductive approach for proving Conjecture GM^n_ρ or AGM^n_ρ. In Sect. 5, we consider a certain inductive step on ρ to prove Conjecture GM^n_ρ or AGM^n_ρ under an additional assumption that there exists a special extremal ray. This assumption is strong, and it might be one of reasons why we cannot prove neither Conjecture GM^n_ρ nor AGM^n_ρ for the general case. We show in Sect. 6 that such an extremal ray does exist under the assumption that there exist many numerically independent dominating and unsplit families of rational curves. The argument is a standard technique; see e.g., [32, Lemma 4] and [25, Theorem 1.1]. We show in Sect. 7 that, if $\rho \le 3$ or $n \le 5$, then there exist many numerically independent dominating and unsplit families of rational curves. In Sect. 8, we prove Theorem 1.6 making use of techniques developed in preceding sections.

Notation and terminology

We always work in the category of algebraic varieties (integral, separated and of finite type scheme) over the complex number field $\mathbb{C}$. For a normal projective variety X, we denote the normalization of the space of irreducible and reduced rational curves on X by $\mathrm{RatCurves}^n(X)$, see [15, Definition II.2.11]. For the theory of extremal contraction, we refer the reader to [17]. For a smooth projective variety X and a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$,

$$l(R) = \min\left\{(-K_X \cdot C) : C \text{ is a rational curve with } [C] \in R\right\}$$

is called the *length* of R. The contraction morphism of R is denoted by $\phi_R \colon X \to X_R$.

For a morphism of varieties $f \colon X \to Y$, we define the *exceptional locus* $\mathrm{Exc}(f)$ *of* f by

$$\mathrm{Exc}(f) = \left\{x \in X : f \text{ is not an isomorphism around } x\right\}.$$

For a complete variety X, an invertible sheaf $\mathcal{L}$ on X and for a nonnegative integer i, we denote the dimension of the $\mathbb{C}$-vector space $H^i(X, \mathcal{L})$ by $h^i(X, \mathcal{L})$. We also define $h^i(X, L)$ as $h^i(X, \mathcal{O}_X(L))$ for a Cartier divisor L on X.

For a complete variety X, the Picard number of X is denoted by ρ_X. For a complete variety X and a closed subvariety $Y \subset X$, we denote the image of the homomorphism $\mathrm{N}_1(Y) \to \mathrm{N}_1(X)$ by $\mathrm{N}_1(Y, X)$.

For an algebraic scheme X and a locally free sheaf of finite rank $\mathcal{E}$ on X, let $\mathbb{P}_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck and $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological invertible sheaf. We usually denote the projection by $p \colon \mathbb{P}_X(\mathcal{E}) \to X$. We use the terms "vector bundle" and "locally free sheaf of finite rank" interchangeably. For a smooth projective variety X, let T_X be the tangent bundle of X.

The symbol $\mathbb{Q}^n$ means a smooth hyperquadric in $\mathbb{P}^{n+1}$ for $n \geq 2$. We write $\mathcal{O}_{\mathbb{Q}^n}(1)$ as the invertible sheaf which is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ under the natural embedding. We sometimes write $\mathcal{O}(m)$ instead of $\mathcal{O}_{\mathbb{Q}^n}(m)$ on $\mathbb{Q}^n$ (or $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathbb{P}^n$) for simplicity.

2 Families of rational curves

Recall the definition and properties of a family of rational curves for a fixed smooth projective variety, for details see [15].

Definition 2.1 Let X be a smooth projective variety. We define a *family of rational curves* on X to be an irreducible component $V \subset \mathrm{RatCurves}^n(X)$ with the induced universal family. For any $x \in X$, let V_x be the subvariety of V parameterizing rational curves passing through x. We define $\mathrm{Locus}(V)$ (resp. $\mathrm{Locus}(V_x)$) to be the union of rational curves parametrized by V (resp. V_x). For a Cartier divisor L on X, the intersection number $(L \cdot C)$ for any rational curve C whose class belongs to V is denoted by $(L \cdot V)$. We also denote by $[V] \in \mathrm{N}_1(X)$ the numerical class of any rational curve among those parametrized by V.

For a family V of rational curves on X, the family V is said to be *dominating* if $\overline{\mathrm{Locus}(V)} = X$, *unsplit* if V is projective, and *locally unsplit* if V_x is projective for a general $x \in \mathrm{Locus}(V)$. If V is a locally unsplit family, then $(-K_X \cdot V) \leq \dim X + 1$ holds by [19, Theorem 4].

If X is a Fano manifold, then by [19, Theorem 6], X admits a dominating family of rational curves. If for a dominating family V of rational curves on X the intersection number $(-K_X \cdot V)$ attains its minimum on the set of dominating families of rational curves on X, then the family V is called by a *minimal dominating family* of X. We note that a minimal dominating family is locally unsplit.

Definition 2.2 Let X be a Fano manifold, $U \subset X$ be an open subvariety and $\pi : U \to Z$ be a proper surjective morphism to a quasiprojective variety Z of positive dimension. A family V of rational curves on X is a *horizontal dominating family with respect to* π if $\mathrm{Locus}(V)$ dominates Z and curves parametrized by V are not contracted by π. We know that such a family always exists by [16, Theorem 2.1]. A horizontal dominating family V of rational curves on X with respect to π is called a *minimal horizontal dominating family* with respect to π if the intersection number $(-K_X \cdot V)$ attains its minimum on the set of horizontal dominating families of rational curves on X with respect to π. We note that a minimal horizontal dominating family is locally unsplit.

Definition 2.3 Let X be a smooth projective variety. We define a *Chow family* $\mathcal{W}$ *of rational 1-cycles* on X to an irreducible component of the Chow variety $\mathrm{Chow}(X)$ of X parameterizing rational and connected 1-cycles. We define $\mathrm{Locus}(\mathcal{W})$ to be the union of the supports of 1-cycles parametrized by $\mathcal{W}$. We say that $\mathcal{W}$ is a *covering family* if $\mathrm{Locus}(\mathcal{W}) = X$.

For a family V of rational curves on X, the closure of the image of V in $\mathrm{Chow}(X)$ is denoted by $\mathcal{V}$ and called the *Chow family associated to* V. If V is unsplit, then V is the normalization of $\mathcal{V}$ by [15, II.2.11].

For a family V of rational curves on X, we say that V (and also $\mathcal{V}$) is *quasi-unsplit* if any component of any reducible cycle parametrized by $\mathcal{V}$ is numerically proportional to the class of curves parametrized by V.

If families $V^1, \ldots, V^k$ of rational curves on X are such that the dimension of the vector space $\sum_{i=1}^{k} \mathbb{R}[V^i]$ in $\mathrm{N}_1(X)$ is equal to k, then we say that $V^1, \ldots, V^k$ are *numerically independent*.

Definition 2.4 Let X be a smooth projective variety, $V^1, \ldots, V^k$ be families of rational curves on X and $Y \subset X$ be a closed subvariety. We define

$$\mathrm{Locus}(V^1)_Y = \bigcup_{\substack{[C] \in V^1 \\ Y \cap C \neq \emptyset}} C,$$

and we inductively define $\mathrm{Locus}(V^1, \ldots, V^k)_Y = \mathrm{Locus}(V^k)_{\mathrm{Locus}(V^1, \ldots, V^{k-1})_Y}$. Analogously, we define $\mathrm{Locus}(\mathcal{W}^1, \ldots, \mathcal{W}^k)_Y$ for Chow families $\mathcal{W}^1, \ldots, \mathcal{W}^k$ of rational 1-cycles. For any point $x \in X$, we define $\mathrm{Locus}(V^1, \ldots, V^k)_x = \mathrm{Locus}(V^1, \ldots, V^k)_{\{x\}}$.

The following assertions are well known. We omit the proof.

Proposition 2.5 [15, Corollary IV.2.6] *Let X be a smooth projective variety, V be a family of rational curves on X and $x \in \mathrm{Locus}(V)$ be a (closed) point such that V_x is projective. Then the dimension of any irreducible component of $\mathrm{Locus}(V_x)$ is greater than or equal to*

$$\dim X - \dim \mathrm{Locus}(V) + (-K_X \cdot V) - 1.$$

Proposition 2.6 [24, Proposition 2] *Let V be a dominating and locally unsplit family of rational curves on a smooth projective variety X and $\mathcal{V}$ be the associated Chow family. Assume that $\dim \mathrm{Locus}(V_x) \geq s$ for a general $x \in X$ and some integer s, then for any $x \in X$ every irreducible component of $\mathrm{Locus}(\mathcal{V})_x$ has dimension greater than or equal to s.*

Lemma 2.7 [1, Lemma 5.4] *Let X be a smooth projective variety, $Y \subset X$ be a closed subvariety and $V^1, \ldots, V^k$ be numerically independent unsplit families of rational curves on X. Assume that $\left(\sum_{i=1}^{k} \mathbb{R}[V^k]\right) \cap \mathrm{N}_1(Y, X) = 0$ and $\mathrm{Locus}(V^1, \ldots, V^k)_Y \neq \emptyset$. Then we have*

$$\dim \mathrm{Locus}(V^1, \ldots, V^k)_Y \geq \dim Y + \sum_{i=1}^{k} \big((-K_X \cdot V^i) - 1\big).$$

Lemma 2.8 [1, Lemma 4.1] *Let X be a smooth projective variety, $Y \subset X$ be a closed subvariety and $\mathcal{W}$ be a Chow family of rational 1-cycles on X. Then any curve in $\mathrm{Locus}(\mathcal{W})_Y$ is numerically equivalent to a linear combination of rational coefficients of curves in Y and of irreducible components of cycles parametrized by $\mathcal{W}$ which meet Y.*

Lemma 2.9 [24, Corollary 1] *Let X be a smooth projective variety, V^1 be a locally unsplit family of rational curves on X and $V^2, \ldots, V^k$ be unsplit families of rational curves on X. Then for a general $x \in \mathrm{Locus}(V^1)$, we have the following results.*

(a) $\mathrm{N}_1(\mathrm{Locus}(V^1)_x, X) = \mathbb{R}[V^1]$.
(b) *If* $\mathrm{Locus}(V^1, \ldots, V^k)_x \neq \emptyset$, *then* $\mathrm{N}_1\big(\mathrm{Locus}(V^1, \ldots, V^k)_x, X\big) = \sum_{i=1}^{k} \mathbb{R}[V^i]$.

3 Rationally connected fibrations

In this section, we recall the theory of rationally connected fibrations, for details see [15] and [24, Section 3].

Definition 3.1 [15, IV.4], [1, Section 3] Let X be a smooth projective variety, $Y \subset X$ be a closed subvariety, m be a positive integer and $\mathcal{W}^1, \ldots, \mathcal{W}^k$ be Chow families of rational 1-cycles on X. We define $\mathrm{ChLocus}(\mathcal{W}^1, \ldots, \mathcal{W}^k)_Y$ to be the set of points $y \in X$ such that there exist cycles $\Gamma_1, \ldots, \Gamma_m$ with the following properties:

- the cycle Γ_i belongs to one of the families $\mathcal{W}^1, \ldots, \mathcal{W}^k$ for any $1 \leq i \leq m$,
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ for any $1 \leq i \leq m-1$,
- $\Gamma_1 \cap Y \neq \emptyset$ and $y \in \Gamma_m$.

For a point $x \in X$, we define $\mathrm{ChLocus}_m(\mathcal{W}^1, \ldots, \mathcal{W}^k)_x = \mathrm{ChLocus}_m(\mathcal{W}^1, \ldots, \mathcal{W}^k)_{\{x\}}$.

We say that two points $x, y \in X$ are $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$-*equivalent* if there exists $m \in \mathbb{Z}_{>0}$ such that $y \in \mathrm{ChLocus}_m(\mathcal{W}^1, \ldots, \mathcal{W}^k)_x$.

We say that X is $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$-*connected* if $X = \mathrm{ChLocus}_m(\mathcal{W}^1, \ldots, \mathcal{W}^k)_x$ holds for some m and for some (hence any) $x \in X$.

Theorem 3.2 [15, Theorem IV.4.16] *Let X be a smooth projective variety and $\mathcal{W}^1, \ldots, \mathcal{W}^k$ be Chow families of rational 1-cycles on X. Then there exists an open subvariety $X^0 \subset X$ and a proper surjective morphism with connected fibers $\pi : X^0 \to Z^0$ to a quasiprojective variety Z^0 such that the following holds.*

- *The equivalence relation obtained by the* $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$*-equivalence restricts to an equivalence relation on X^0.*
- *$\pi^{-1}(z)$ coincides with an* $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$*-equivalence class for any $z \in Z^0$.*
- *For any $z \in Z^0$ and $x, y \in \pi^{-1}(z)$, we have $y \in \mathrm{ChLocus}_m(\mathcal{W}^1, \ldots, \mathcal{W}^k)_x$ for some $m \leq 2^{\dim X - \dim Z^0} - 1$.*

We call this morphism the $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$-fibration *and often write $\pi : X \dashrightarrow Z$ for simplicity* (*where Z is a projective variety*).

Proposition 3.3 [1, Corollary 4.4] *Let X be a smooth projective variety and $\mathcal{W}^1, \ldots, \mathcal{W}^k$ be Chow families of rational 1-cycles on X. If X is* $\mathrm{rc}(\mathcal{W}^1, \ldots, \mathcal{W}^k)$*-connected, then $\mathrm{N}_1(X)$ is spanned by the classes of irreducible components of cycles in $\mathcal{W}^1, \ldots, \mathcal{W}^k$. In particular, if $\mathcal{W}^i$ is the Chow family associated to some quasi-unsplit family W^i of rational curves on X for any $1 \leq i \leq k$, then $\rho_X \leq k$ and equality holds if and only if $W^1, \ldots, W^k$ are numerically independent.*

Theorem 3.4 [24, Theorem 2] *Let X be a Fano manifold and V be a dominating and locally unsplit family of rational curves on X. Assume that X is* $\mathrm{rc}(\mathcal{V})$*-connected and $(-K_X \cdot V) < 3\iota_X$.*

(a) *If V is a minimal dominating family and $(-K_X \cdot V) > \dim X + 1 - \iota_X$, then $\rho_X = 1$.*
(b) *If $(-K_X \cdot V) > \dim X + 1 - \iota_X$, then $\rho_X \leq 2$.*
(c) *If $(-K_X \cdot V) \geq \dim X + 1 - \iota_X$ and $\iota_X \geq 2$, then $\rho_X \leq 3$.*

Proof The proof is almost the same as that of [24, Theorem 2]. Fix a general point $x \in X$. There exists $m \in \mathbb{Z}_{>0}$ such that $X = \mathrm{ChLocus}_m(\mathcal{V})_x$ since X is $\mathrm{rc}(\mathcal{V})$-connected. Since $(-K_X \cdot V) < 3\iota_X$, any reducible cycle Γ of $\mathcal{V}$ has only two irreducible components. Hence either both of them are numerically proportional to $[V] \in \mathrm{N}_1(X)$ or neither of them is numerically proportional to $[V] \in \mathrm{N}_1(X)$.

If any irreducible component of an m-chain $\Gamma_1 \cup \cdots \cup \Gamma_m$ which satisfies

- $x \in \Gamma_1$ and
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ for any $1 \leq i \leq m-1$

is numerically proportional to $[V] \in N_1(X)$, then $\rho_X = 1$ by Proposition 3.3.

We can assume that there exists an m-chain $\Gamma_1 \cup \cdots \cup \Gamma_m$ which satisfies the above properties and there exists an integer $1 \leq j \leq m$ such that the irreducible components Γ_j^1 and Γ_j^2 of Γ_j are not numerically proportional to $[V] \in N_1(X)$. Let $1 \leq j_0 \leq m$ be the minimum integer for which such a chain exists. We have $j_0 \geq 2$ since $x \in X$ is general. If $j_0 = 2$ then set $x_1 = x$, otherwise let $x_1 \in X$ be a point in $\Gamma_{j_0-2} \cap \Gamma_{j_0-1}$. Take an irreducible component Y of $\mathrm{Locus}(V_{x_1})$ which meets Γ_{j_0}. We can assume that $\Gamma_{j_0}^1 \cap Y \neq \emptyset$. We know that $N_1(Y, X) = \mathbb{R}[V]$ by Lemma 2.8 and the minimality of j_0. Take a family W of rational curves on X such that the class of $\Gamma_{j_0}^1$ is in W. Then W is unsplit by the property $(-K_X \cdot V) < 3\iota_X$. By Lemma 2.7, Propositions 2.5 and 2.6, we have $\dim \mathrm{Locus}(W)_Y \geq \dim Y + (-K_X \cdot W) - 1 \geq (-K_X \cdot V) + \iota_X - 2$.

(a) We have $\mathrm{Locus}(W)_Y = X$ since $(-K_X \cdot V) > \dim X + 1 - \iota_X$. In particular, W is a dominating family. However, this leads to a contradiction since V is a minimal dominating family and $(-K_X \cdot V) > (-K_X \cdot W)$. Thus $\rho_X = 1$.

(b) We have $\mathrm{Locus}(W)_Y = X$ by the same reason. We know that $N_1(\mathrm{Locus}(W)_Y, X) = \mathbb{R}[V] + \mathbb{R}[W]$ by Lemma 2.8. Thus $\rho_X \leq 2$.

(c) We have $\mathrm{Locus}(W)_Y$ is a divisor or equal to X and $N_1(\mathrm{Locus}(W)_Y, X) = \mathbb{R}[V] + \mathbb{R}[W]$ by the same reason. If $\mathrm{Locus}(W)_Y$ is equal to X, then $\rho_X \leq 2$. If $\mathrm{Locus}(W)_Y$ is a divisor, then $\rho_X \leq 3$ by [5, Theorem 1.2]. □

We recall the following argument due to Novelli and Occhetta.

Construction 3.5 [24, Construction 1] *Let X be a Fano manifold. Take a minimal dominating family V^1 of rational curves on X. If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, take a minimal horizontal dominating family V^2 of rational curves on X with respect to the $\mathrm{rc}(\mathcal{V}^1)$-fibration $\pi^1 : X \dashrightarrow Z^1$. If X is not $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected, take a minimal horizontal dominating family V^3 of rational curves on X with respect to the $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-fibration $\pi^2 : X \dashrightarrow Z^2$, and so on. Since $\dim Z^{i+1} < \dim Z^i$, for some integer k we have that X is $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connected. We note that the families $V^1, \ldots, V^k$ are numerically independent by construction.*

Lemma 3.6 [24, Lemma 4] *Let X be a Fano manifold with $\iota_X \geq 2$ and $V^1, \ldots, V^k$ be families of rational curves as in Construction 3.5. Then we have*

$$\begin{aligned}
\dim X &\geq \sum_{i=1}^{k} \dim\bigl((\pi^i)^{-1}(\pi^i(x_i))\bigr) \geq \sum_{i=1}^{k} \dim \mathrm{Locus}(V^i)_{x_i} \\
&\geq \sum_{i=1}^{k} \bigl(\dim X - \dim \mathrm{Locus}(V^i) + (-K_X \cdot V^i) - 1\bigr) \geq \sum_{i=1}^{k} \bigl((-K_X \cdot V^i) - 1\bigr)
\end{aligned}$$

for any general $x_i \in \mathrm{Locus}(V^i)$.

Lemma 3.7 [23, Lemma 4.5] *Let X be a Fano manifold with $\iota_X \geq 2$ and $V^1, \ldots, V^k$ be families of rational curves as in Construction 3.5. Assume that at least one of these families, say V^j, is non-unsplit. Then $k(\iota_X - 1) \leq \dim X - \iota_X$. Moreover,*

- *if* $j = (\dim X - \iota_X)/(\iota_X - 1)$*, then* $j = k$ *and* $\rho_X(\iota_X - 1) = \dim X - \iota_X$*;*
- *if* $j = (\dim X - \iota_X - 1)/(\iota_X - 1)$*, then* $j = k$ *and either* $\rho_X(\iota_X - 1) = \dim X - \iota_X - 1$*, or* $\iota_X = 2$ *and* $\rho_X = \dim X - 2$*.*

4 Special vector bundles

In this section, we consider vector bundles on some special Fano manifolds whose projectivizations are also Fano manifolds with large pseudoindices.

Definition 4.1 A morphism $f: X \to Y$ is called a $\mathbb{P}^m$*-fibration* if f is a proper and smooth morphism such that the scheme theoretic fiber of f is isomorphic to $\mathbb{P}^m$ for any (closed) point in Y.

The following lemma from [4] is fundamental.

Lemma 4.2 [4, Lemme 2.5 (a)] *Let* $f: X \to Y$ *be a* $\mathbb{P}^m$*-fibration between smooth projective varieties. If* X *is a Fano manifold, then* Y *is also a Fano manifold and* $\iota_Y \geq \iota_X$*.*

We give a sufficient condition that a given $\mathbb{P}^m$-fibration is isomorphic to a projective space bundle.

Proposition 4.3 *Let* $f: X \to Y$ *be a* $\mathbb{P}^m$*-fibration between smooth projective varieties. If* Y *is a rational variety, i.e., birational to a projective space, then* f *is a projective space bundle. More precisely, there exists a locally free sheaf* $\mathcal{E}$ *of rank* $m + 1$ *on* Y *such that* X *is isomorphic to* $\mathbb{P}_Y(\mathcal{E})$ *over* Y*.*

Proof Since Y is a smooth projective rational variety, the cohomological Brauer group $H^2_{\text{ét}}(Y, \mathbb{G}_{\text{m}})$ of Y is equal to zero, see for example [8, Section 5]. Thus the homomorphism $H^1_{\text{ét}}(Y, \text{GL}_{m+1}) \to H^1_{\text{ét}}(Y, \text{PGL}_{m+1})$ is surjective. □

We introduce the notion of minimal horizontal curves of projective space bundles over rational curves. The idea focusing on these curves has been developed in [4, Section 2].

Definition 4.4 Let Y be a smooth projective variety, let $\mathcal{E}$ be a locally free sheaf on Y of rank $m + 1$ and let $X = \mathbb{P}_Y(\mathcal{E})$ with the projection $p: X \to Y$. Let $C \subset Y$ be a rational curve with the normalization morphism $\nu: \mathbb{P}^1 \to C \hookrightarrow Y$. Consider the fiber product

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{E}) & \xrightarrow{\nu'} & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{\nu} & Y. \end{array}$$

There exists an isomorphism

$$\nu^*\mathcal{E} \simeq \bigoplus_{0 \leq i \leq m} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

with $a_0 \leq \cdots \leq a_m$. Let $C' \subset \mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{E})$ be the section of p' corresponding to the canonical projection

$$\nu^*\mathcal{E} \simeq \bigoplus_{0 \leq i \leq m} \mathcal{O}_{\mathbb{P}^1}(a_i) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(a_0)$$

and let $C^{p,0} \subset X$ be the image of C' in X. We call this $C^{p,0}$ a *minimal horizontal curve of p over C*. The choice of $C^{p,0}$ is not unique in general. However, we have

$$\left(-K_X \cdot C^{p,0}\right) = (-K_Y \cdot C) - \sum_{i=1}^{m}(a_i - a_0) \qquad (1)$$

since $\left(-K_X \cdot C^{p,0}\right) = \left(p^*\left(\mathcal{O}_Y(-K_Y) \otimes (\det \mathcal{E})^{\vee}\right) \otimes \mathcal{O}_{\mathbb{P}}(m+1) \cdot C^{p,0}\right)$, $\deg(\det(\nu^*\mathcal{E})) = \sum_{i=0}^{m} a_i$ and $(\mathcal{O}_{\mathbb{P}}(1) \cdot C')_{\mathbb{P}_{\mathbb{P}^1}(\nu^*\mathcal{E})} = a_0$. This value does not depend on the choice of $C^{p,0}$.

Lemma 4.5 *Let Z be a smooth projective variety and $Y = \mathbb{P}^m \times Z$ (we allow the case Z is a point). We write the projections $p_1 \colon Y \to \mathbb{P}^m$ and $p_2 \colon Y \to Z$. Let $\mathcal{E}$ be a locally free sheaf on Y of rank $m+1$ and $X = \mathbb{P}_Y(\mathcal{E})$ with the projection $p \colon X \to Y$. Assume that X is a Fano manifold with $\iota_X \geq m+1$. Then there exist an integer a and a locally free sheaf $\mathcal{E}_Z$ on Z of rank $m+1$ such that $\mathcal{E} \simeq p_1^*\mathcal{O}_{\mathbb{P}^m}(a) \otimes p_2^*\mathcal{E}_Z$. Moreover, $X_Z = \mathbb{P}_Z(\mathcal{E}_Z)$ satisfies $X \simeq X_Z \times \mathbb{P}^m$. In particular, X_Z is also a Fano manifold with $\iota_{X_Z} \geq m+1$.*

Proof Pick any (closed) point $z \in Z$ and any line $l \subset p_2^{-1}(z)(= \mathbb{P}^m) \subset Y$. Then we have $\mathcal{E}|_l \simeq \bigoplus_{0 \leq i \leq m} \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l) = m+1$ and $(-K_X \cdot l^{p,0}) \geq m+1$. The integer a does not depend on the choices of z and l since the value $(\det \mathcal{E} \cdot l) = (m+1)a$ is independent of the choices of z and l. Thus $\mathcal{E}' = \mathcal{E} \otimes p_1^*\mathcal{O}_{\mathbb{P}^m}(-a)$ satisfies $\mathcal{E}'|_l \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus m+1}$ for any (closed) point $z \in Z$ and any line $l \subset p_2^{-1}(z) \subset Y$. Thus $\mathcal{E}'|_{p_2^{-1}(z)} \simeq \mathcal{O}_{\mathbb{P}^m}^{\oplus m+1}$ by [3, Proposition (1.2)]. We have $h^0\left(p_2^{-1}(z), \mathcal{E}'|_{p_2^{-1}(z)}\right) = m+1$ and $h^1\left(p_2^{-1}(z), \mathcal{E}'|_{p_2^{-1}(z)}\right) = 0$. Hence $\mathcal{E}_Z = (p_2)_*\mathcal{E}'$ is a locally free sheaf on Z of rank $m+1$ and $p_2^*\mathcal{E}_Z \simeq \mathcal{E}'$ holds by the cohomology and base change theorem. Therefore we have $\mathcal{E} \simeq p_1^*\mathcal{O}_{\mathbb{P}^m}(a) \otimes p_2^*\mathcal{E}_Z$. The remaining assertions are trivial. □

Applying Lemma 4.5 and induction on k, one concludes

Corollary 4.6 *Let $Y = (\mathbb{P}^m)^k$ for some $m, k \geq 1$, let $\mathcal{E}$ be a locally free sheaf on Y of rank $m+1$ and let $X = \mathbb{P}_Y(\mathcal{E})$ with the projection $p \colon X \to Y$. If X is a Fano manifold with $\iota_X \geq m+1$, then X is isomorphic to $(\mathbb{P}^m)^{k+1}$.*

Corollary 4.7 *Fix $m, k \geq 1$. Let Y be a smooth projective variety, let $\mathcal{E}$ be a locally free sheaf on Y of rank $m+1$ and let $X = \mathbb{P}_Y(\mathcal{E})$ with the projection $p \colon X \to Y$. Assume that X is a Fano manifold with $\iota_X \geq m+1$.*

(a) *If $Y = \mathbb{Q}^{m+1} \times (\mathbb{P}^m)^{k-1}$, then $X \simeq Y \times \mathbb{P}^m$.*
(b) *If $Y = \mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}^{\oplus m} \oplus \mathcal{O}(1)\right) \times (\mathbb{P}^m)^{k-1}$, then $X \simeq Y \times \mathbb{P}^m$.*

(c) *If* $Y = \mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}) \times (\mathbb{P}^m)^{k-1}$*, then* $X \simeq Y \times \mathbb{P}^m$*.*
(d) *If* $Y = \mathbb{P}^{m+1} \times (\mathbb{P}^m)^{k-1}$*, then* X *is isomorphic to one of the following:*
- $Y \times \mathbb{P}^m$,
- $\mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}^{\oplus m} \oplus \mathcal{O}(1)\right) \times (\mathbb{P}^m)^{k-1}$,
- $\mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}) \times (\mathbb{P}^m)^{k-1}$.

Proof We can assume $k = 1$ by Lemma 4.5.

(d) Take any line $l \subset Y = \mathbb{P}^{m+1}$. Then the locally free sheaf $\mathcal{E}|_l$ is either isomorphic to

(d_1) $\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m+1}$ or
(d_2) $\mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^1}(a+1)$

for some $a \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l) = m+2$ and $(-K_X \cdot l^{p,0}) \geq m+1$. Moreover, the possibility (d_1) or (d_2) and the integer a do not depend on the choice of l. If the case (d_1) occurs, then $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^{m+1}}(-a) \simeq \mathcal{O}_{\mathbb{P}^{m+1}}^{\oplus m+1}$ by [3, Proposition (1.2)]. Thus X is isomorphic to $\mathbb{P}^{m+1} \times \mathbb{P}^m$. If the case ($d_2$) occurs, then $\mathcal{E}$ is isomorphic to either $\mathcal{O}_{\mathbb{P}^{m+1}}(a)^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^{m+1}}(a+1)$ or $T_{\mathbb{P}^{m+1}} \otimes \mathcal{O}_{\mathbb{P}^{m+1}}(a-1)$ by [27, Main Theorem 2 (ii)]. Thus X is isomorphic to either $\mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}^{\oplus m} \oplus \mathcal{O}(1)\right)$ or $\mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}})$.

(a) If $m = 1$, then the assertion is true by Corollary 4.6. We can assume that $m \geq 2$. Take any line $l \subset Y = \mathbb{Q}^{m+1}$. Then we have $\mathcal{E}|_l \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m+1}$ for some $a \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l) = m+1$ and $(-K_X \cdot l^{p,0}) \geq m+1$. Moreover, the integer a does not depend on the choice of l. Then $\mathcal{E} \otimes \mathcal{O}_{\mathbb{Q}^{m+1}}(-a) \simeq \mathcal{O}_{\mathbb{Q}^{m+1}}^{\oplus m+1}$ by [3, Proposition (1.2)]. Thus X is isomorphic to $\mathbb{Q}^{m+1} \times \mathbb{P}^m$.

(b) Let $p' \colon Y = \mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}^{\oplus m} \oplus \mathcal{O}(1)\right) \to \mathbb{P}^{m+1}$ be the projection and $q \colon Y \to \mathbb{P}^{2m+1}$ be the blowing up along an $(m-1)$-dimensional linear subspace. Take any (closed) point $z \in \mathbb{P}^{m+1}$ and any line $l \subset (p')^{-1}(z)(\simeq \mathbb{P}^m) \subset Y$. Then we have $\mathcal{E}|_l \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m+1}$ for some $a \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l) = m+1$ and $(-K_X \cdot l^{p,0}) \geq m+1$. Moreover, the integer a does not depend on the choices of z and l. Then $\mathcal{E}' = \mathcal{E} \otimes q^*\mathcal{O}_{\mathbb{P}^{2m+1}}(-a)$ satisfies $\mathcal{E}'|_{(p')^{-1}(z)} \simeq \mathcal{O}_{\mathbb{P}^m}^{\oplus m+1}$ for any (closed) point $z \in \mathbb{P}^{m+1}$. Thus $\mathcal{E}_1 = p'_*\mathcal{E}'$ is a locally free sheaf on $\mathbb{P}^{m+1}$ of rank $m+1$ and $(p')^*\mathcal{E}_1 \simeq \mathcal{E}'$ holds by the cohomology and base change theorem. Hence $\mathcal{E}_1$ is isomorphic to one of the following:

(b_1) $\mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m+1}$,
(b_2) $\mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^{m+1}}(b+1)$ or
(b_3) $T_{\mathbb{P}^{m+1}} \otimes \mathcal{O}_{\mathbb{P}^{m+1}}(b-1)$

for some $b \in \mathbb{Z}$ by (d). Take a line l' in a nontrivial fiber ($\simeq \mathbb{P}^{m+1}$) of q. Then we have $\mathcal{E}|_{l'} \simeq \mathcal{O}_{\mathbb{P}^1}(a')^{\oplus m+1}$ for some $a' \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l') = m+1$ and $(-K_X \cdot (l')^{p,0}) \geq m+1$. Thus we have $(m+1)a' = (\det \mathcal{E} \cdot l') = (\det \mathcal{E}_1 \cdot p_*l')$. If $\mathcal{E}_1$ is isomorphic to either of type (b_2) or (b_3), then $(\det \mathcal{E}_1 \cdot p_*l') = (m+1)b+1$. This leads to a contradiction. Hence $\mathcal{E}_1 \simeq \mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m+1}$. In particular X is isomorphic to $\mathbb{P}_{\mathbb{P}^{m+1}}\left(\mathcal{O}^{\oplus m} \oplus \mathcal{O}(1)\right) \times \mathbb{P}^m$.

(c) Let $p' \colon Y = \mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}) \to \mathbb{P}^{m+1}$ be the projection and $q \colon Y \to \mathbb{P}^{m+1}$ be the other contraction morphism. Take any (closed) point $z \in \mathbb{P}^{m+1}$ and any line $l \subset (p')^{-1}(z)(\simeq \mathbb{P}^m) \subset Y$. Then we have $\mathcal{E}|_l \simeq \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m+1}$ for some $a \in \mathbb{Z}$ by (1) and the properties $(-K_Y \cdot l) = m+1$ and $(-K_X \cdot l^{p,0}) \geq m+1$. Moreover, the integer a does not depend on the choices of z and l. Then $\mathcal{E}' = \mathcal{E} \otimes q^*\mathcal{O}_{\mathbb{P}^{m+1}}(-a)$ satisfies $\mathcal{E}'|_{(p')^{-1}(z)} \simeq \mathcal{O}_{\mathbb{P}^m}^{\oplus m+1}$ for any (closed) point $z \in \mathbb{P}^{m+1}$. Thus $\mathcal{E}_1 = p'_*\mathcal{E}'$ is a locally free sheaf on $\mathbb{P}^{m+1}$ of rank $m+1$ and $(p')^*\mathcal{E}_1 \simeq \mathcal{E}'$ holds by the cohomology and base change theorem. Hence $\mathcal{E}_1$ is isomorphic to one of the following:

(c$_1$) $\mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m+1}$,
(c$_1$) $\mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^{m+1}}(b+1)$ or
(c$_1$) $T_{\mathbb{P}^{m+1}} \otimes \mathcal{O}_{\mathbb{P}^{m+1}}(b-1)$

for some $b \in \mathbb{Z}$ by (d). Take a line l' in a fiber ($\simeq \mathbb{P}^m$) of q. Then we have $\mathcal{E}|_{l'} \simeq \mathcal{O}_{\mathbb{P}^1}(a')^{\oplus m+1}$ for some $a' \in \mathbb{Z}$ by the same reason. Thus we have $(m+1)a' = (\det \mathcal{E} \cdot l') = (\det \mathcal{E}_1 \cdot p'_* l')$. If $\mathcal{E}_1$ is isomorphic to either of type (c$_2$) or (c$_3$), then $(\det \mathcal{E}_1 \cdot p'_* l') = (m+1)b+1$. This leads to a contradiction. Hence $\mathcal{E}_1 \simeq \mathcal{O}_{\mathbb{P}^{m+1}}(b)^{\oplus m+1}$. In particular X is isomorphic to $\mathbb{P}_{\mathbb{P}^{m+1}}(T_{\mathbb{P}^{m+1}}) \times \mathbb{P}^m$. □

5 Inductive step

In this section, we prove Conjecture AGM^n_ρ under the conditions that Conjectures $\mathrm{AGM}^{n'}_{\rho-1}$ are true for small n' and there exist special extremal rays for Fano manifolds satisfying the assumptions of Conjecture AGM^n_ρ. We recall a result of Wiśniewski.

Theorem 5.1 (Wiśniewski's inequality [31]) *Let X be a smooth projective variety and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray. Then any nontrivial fiber F of ϕ_R (the contraction morphism associated to R) satisfies the inequality*

$$\dim F \geq \dim X - \dim \mathrm{Exc}(\phi_R) + l(R) - 1.$$

Together with a result due to Höring and Novelli [10], we get the following.

Theorem 5.2 *Let X be a smooth projective variety and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray. If any fiber F of ϕ_R satisfies $\dim F \leq l(R) - 1$, then the morphism $\phi_R \colon X \to X_R$ is a $\mathbb{P}^{l(R)-1}$-fibration.*

Proof For any nontrivial fiber F of ϕ_R, we have $\dim F = l(R) - 1$ and $\dim \mathrm{Exc}(\phi_R) = \dim X$, by Theorem 5.1. Thus we can apply [10, Theorem 1.3]. □

Using this, we get the key proposition in this section.

Proposition 5.3 *Let X be an n-dimensional Fano manifold of the pseudoindex ι. Assume that there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R satisfies $\dim F \leq \iota - 1$.*

(a) *If X satisfies the assumptions of Conjecture GM^n_ρ for some fixed $\rho \geq 2$ and Conjecture $\mathrm{GM}^{n+1-\iota}_{\rho-1}$ is true, then X is isomorphic to $(\mathbb{P}^{\iota-1})^\rho$.*
(b) *If X satisfies the assumptions of Conjecture AGM^n_ρ for some fixed $\rho \geq 2$ and Conjecture $\mathrm{AGM}^{n+1-\iota}_{\rho-1}$ is true, then X is isomorphic to one of in the list of Conjecture AGM^n_ρ.*

Proof The morphism $\phi_R\colon X \to X_R$ is a $\mathbb{P}^{\iota-1}$-fibration by Theorem 5.2. We replace X_R by Y for simplicity. We know that Y is an $(n+1-\iota)$-dimensional Fano manifold with $\rho_Y = \rho_X - 1$ and $\iota_Y \geq \iota_X$ by Lemma 4.2.

(a) We have the inequalities

$$\iota_Y \geq \iota \geq \frac{n+\rho}{\rho} \geq \frac{n+1-\iota+(\rho-1)}{\rho-1}.$$

Thus Y is isomorphic to $(\mathbb{P}^{\iota-1})^{\rho-1}$ since we assume that Conjecture $\mathrm{GM}^{n+1-\iota}_{\rho-1}$ is true. Since Y is rational, the morphism ϕ_R is a projective space bundle by Proposition 4.3. Therefore X is isomorphic to $(\mathbb{P}^{\iota-1})^{\rho}$ by Corollary 4.6.

(b) We have the inequalities

$$\iota_Y \geq \iota \geq \frac{n+\rho-1}{\rho} \geq \frac{n+1-\iota+(\rho-1)-1}{\rho-1}.$$

Thus Y is isomorphic to one of $(\mathbb{P}^{\iota-1})^{\rho-1}$, $\mathbb{Q}^{\iota}\times(\mathbb{P}^{\iota-1})^{\rho-2}$, $\mathbb{P}_{\mathbb{P}^{\iota}}\left(\mathcal{O}^{\oplus\iota-1}\oplus\mathcal{O}(1)\right)\times(\mathbb{P}^{\iota-1})^{\rho-3}$, $\mathbb{P}_{\mathbb{P}^{\iota}}(T_{\mathbb{P}^{\iota}})\times(\mathbb{P}^{\iota-1})^{\rho-3}$ or $\mathbb{P}^{\iota}\times(\mathbb{P}^{\iota-1})^{\rho-2}$ since we assume that Conjecture $\mathrm{AGM}^{n+1-\iota}_{\rho-1}$ is true. Since Y is rational, the morphism ϕ_R is a projective space bundle by Proposition 4.3. Therefore X is isomorphic to one of spaces in the list of Conjecture AGM^{n}_{ρ} by Corollaries 4.6 and 4.7. □

6 Finding a special extremal ray

In this section, we show that Fano manifolds satisfying the assumptions in Conjecture AGM^{n}_{ρ}, $\rho \geq 2$, have an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R is of dimension less than or equal to $\iota_X - 1$ under the assumption that there exist numerically independent unsplit and dominating families of rational curves $V^1, \ldots, V^{\rho-1}$ on X. This is a kind of generalization of Wiśniewski's result [32, Lemma 4].

Theorem 6.1 *Let X be an n-dimensional Fano manifold with $\rho = \rho_X \geq 2$ which satisfies $\iota_X \geq (n+\rho-1)/\rho$. Assume that there exist numerically independent unsplit and dominating families of rational curves $V^1, \ldots, V^{\rho-1}$ on X. Then there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R is of dimension less than or equal to $\iota_X - 1$.*

Proof First, we prove the following assertion.

Claim 6.2 *For any extremal ray $R \subset \mathrm{NE}(X)$ with $R \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$, the contraction morphism $\phi_R\colon X \to X_R$ is either*

(i) *a divisorial contraction and any nontrivial fiber is of dimension ι_X, or*
(ii) *of fiber type and any fiber is of dimension greater than of equal to $\iota_X - 1$.*

Proof Take an arbitrary fiber F of ϕ_R. For any point $x \in F$, we have

$$\dim \mathrm{Locus}(V^1, \ldots, V^{\rho-1})_x \geq \sum_{i=1}^{\rho-1} \big((-K_X \cdot V^i) - 1\big) \geq (\iota_X - 1)(\rho - 1)$$

by Lemma 2.7. Since $\mathrm{N}_1\big(\mathrm{Locus}(V^1, \ldots, V^{\rho-1})_x, X\big) = \sum_{i=1}^{\rho-1} \mathbb{R}[V^i]$, by Lemma 2.9 (b), and $\mathrm{N}_1(F, X) = \mathbb{R}R$, we have $\dim\big(F \cap \mathrm{Locus}(V^1, \ldots, V^{\rho-1})_x\big) = 0$. Hence

$$\dim F \leq n - \dim \mathrm{Locus}(V^1, \ldots, V^{\rho-1})_x \leq n - (\iota_X - 1)(\rho - 1) \leq \iota_X.$$

Moreover, we have

$$\dim F \geq n - \dim \mathrm{Exc}(\phi_R) + l(R) - 1 \geq \iota_X - 1$$

by Theorem 5.1. Hence the assertion of claim follows. ∎

Next, we prove the following assertion.

Claim 6.3 *Take arbitrary distinct extremal rays $R, R' \subset \mathrm{NE}(X)$ with $R \not\subset \sum_{i=1}^{\rho-1} \mathbb{R}[V^i]$. Assume that any fiber F' of $\phi_{R'}$ intersects some fiber F of ϕ_R. Then the morphism $\phi_{R'}$ also satisfies either property* (i) *or* (ii) *in Claim 6.2. Moreover, the following holds:*

(i) *If ϕ_R is a divisorial contraction, then $\phi_{R'}$ is of fiber type and any fiber of $\phi_{R'}$ is of dimension less than or equal to $\iota_X - 1$.*
(ii) *If $\phi_{R'}$ is a divisorial contraction, then any fiber of ϕ_R that intersects some fiber of $\phi_{R'}$ is of dimension less than or equal to $\iota_X - 1$.*

Proof We can assume that $\mathrm{N}_1(X) = \mathbb{R}R + \mathbb{R}R' + \sum_{i=1}^{\rho-2} \mathbb{R}[V^i]$ by renumbering $V^1, \ldots, V^{\rho-1}$. Then we have $\mathrm{N}_1\big(\mathrm{Locus}(V^1, \ldots, V^{\rho-2})_F, X\big) = \mathbb{R}R + \sum_{i=1}^{\rho-2} \mathbb{R}[V^i]$ by Lemma 2.9 (b) and

$$\begin{aligned}\dim \mathrm{Locus}(V^1, \ldots, V^{\rho-2})_F &\geq \dim F + \sum_{i=1}^{\rho-2} \big((-K_X \cdot V^i) - 1\big) \\ &\geq \dim F + (\rho - 2)(\iota_X - 1) \geq n - \iota_X\end{aligned}$$

holds by Lemma 2.7 and Claim 6.2. Moreover, if ϕ_R is a divisorial contraction, then we have $\dim \mathrm{Locus}(V^1, \ldots, V^{\rho-2})_F \geq n + 1 - \iota_X$ since $\dim F = \iota_X$. Since $\mathrm{N}_1(F', X) = \mathbb{R}R'$, we have $\dim\big(F' \cap \mathrm{Locus}(V^1, \ldots, V^{\rho-2})_F\big) = 0$. Thus $\dim F' \leq n - \dim \mathrm{Locus}(V^1, \ldots, V^{\rho-2})_F \leq \iota_X$. If ϕ_R is a divisorial contraction, then $\dim F' \leq \iota_X - 1$. Moreover, $\dim F' \geq n - \dim \mathrm{Exc}(\phi_{R'}) + l(R') - 1 \geq \iota_X - 1$ by Theorem 5.1. If $\phi_{R'}$ is a divisorial contraction, then $\dim F' \geq \iota_X$. Therefore the assertion of claim follows. ∎

Assume that there exists an extremal ray $R \subset \mathrm{NE}(X)$ with $R \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$ such that the contraction morphism ϕ_R is a divisorial contraction. Set $E = \mathrm{Exc}(\phi_R)$. Then there exists an extremal ray $R' \subset \mathrm{NE}(X)$ with $R' \neq R$ such that $(E \cdot R') > 0$ since $\mathrm{NE}(X)$ is spanned by finite number of extremal rays. Then any fiber F' of $\phi_{R'}$ intersects E. Thus $\dim F' \leq \iota_X - 1$ by Claim 6.3 (i).

Hence we can assume that any extremal ray $R \subset \mathrm{NE}(X)$ with $R \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$ satisfies that the contraction morphism ϕ_R is of fiber type. We fix an extremal ray $R \subset \mathrm{NE}(X)$ with $R \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$. Then any extremal ray $R' \subset \mathrm{NE}(X)$ with $R' \neq R$ satisfies either property (i) or (ii) in Claim 6.2 by Claim 6.3.

Assume that there exists an extremal ray $R' \subset \mathrm{NE}(X)$ with $R' \neq R$ such that the contraction morphism $\phi_{R'}$ is a divisorial contraction. Set $E' = \mathrm{Exc}(\phi_{R'})$. Then there exists an extremal ray $R'' \subset \mathrm{NE}(X)$ with $R'' \neq R'$ such that $(E' \cdot R'') > 0$. If $R'' \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$, then any fiber of the morphism $\phi_{R''}$ is of dimension less than or equal to $\iota_X - 1$ by Claim 6.3 (ii). Thus we can assume that $R'' \subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$. In particular, ρ must be greater than or equal to three. We can assume that $\mathrm{N}_1(X) = \mathbb{R}R + \mathbb{R}R' + \mathbb{R}R'' + \sum_{i=1}^{\rho-3}\mathbb{R}[V^i]$ by renumbering $V^1, \ldots, V^{\rho-1}$ since $R \not\subset \sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$ and two distinct extremal rays R' and R'' are in $\sum_{i=1}^{\rho-1}\mathbb{R}[V^i]$. Take any fiber F'' of $\phi_{R''}$. Then we can take a fiber F' of $\phi_{R'}$ such that $F' \cap F'' \neq \emptyset$ since $(E' \cdot R'') > 0$ holds. Then $\mathrm{N}_1\big(\phi_R^{-1}(\phi_R(F')), X\big) = \mathbb{R}R + \mathbb{R}R'$ and

$$\dim \phi_R^{-1}(\phi_R(F')) \geq \iota_X - 1 + \dim \phi_R(F') = \iota_X - 1 + \dim F' = 2\iota_X - 1$$

since any fiber of ϕ_R is of dimension greater than or equal to $\iota_X - 1$ and the restriction morphism $\phi_R|_{F'}\colon F' \to \phi_R(F')$ is a finite morphism. Moreover, we have

$$\mathrm{N}_1\big(\mathrm{Locus}(V^1, \ldots, V^{\rho-3})_{\phi_R^{-1}(\phi_R(F'))}, X\big) = \mathbb{R}R + \mathbb{R}R' + \sum_{i=1}^{\rho-3}\mathbb{R}[V^i],$$

$$\begin{aligned}\dim \mathrm{Locus}(V^1, \ldots, V^{\rho-3})_{\phi_R^{-1}(\phi_R(F'))} &\geq \dim \phi_R^{-1}(\phi_R(F')) + \sum_{i=1}^{\rho-3}\big((-K_X \cdot V^i) - 1\big)\\ &\geq n + 1 - \iota_X\end{aligned}$$

by Lemmas 2.7 and 2.9 (b). Thus $\dim\big(F'' \cap \mathrm{Locus}(V^1, \ldots, V^{\rho-3})_{\phi_R^{-1}(\phi_R(F'))}\big) = 0$. Therefore $\dim F'' \leq n - \dim \mathrm{Locus}(V^1, \ldots, V^{\rho-3})_{\phi_R^{-1}(\phi_R(F'))} \leq \iota_X - 1$ for any fiber F'' of $\phi_{R''}$.

Hence we can assume that any extremal ray $R_1 \subset \mathrm{NE}(X)$ satisfies that the contraction morphism ϕ_{R_1} is of fiber type. For any fiber F_1 of ϕ_{R_1}, we have $\dim F_1 \geq \iota_X - 1$ by Theorem 5.1. We can assume that there exists an extremal ray $R_1 \subset \mathrm{NE}(X)$ and a fiber F_1 of ϕ_{R_1} such that the dimension of F_1 is greater than or equal to ι_X. Take any $(\rho-1)$-dimensional extremal face $S \subset \mathrm{NE}(X)$ such that $R_1 \subset S$ and let $\phi_S\colon X \to X_S$ be the contraction morphism of S. Then there exists a fiber F_S of ϕ_S such that $\dim F_S \geq n + 1 - \iota_X$. Indeed, let $x_S \in X_S$ be the image of $F_1 \subset X$. Then $\dim \phi_S^{-1}(x_S) \geq \iota_X + (\rho - 2)(\iota_X - 1) \geq n + 1 - \iota_X$. We also take an extremal ray $R_0 \subset \mathrm{NE}(X)$ such that $R_0 \cap S = 0$. Then for any fiber F_0 of ϕ_{R_0}, we have $\dim(F_0 \cap F_S) = 0$. Therefore $\dim F_0 \leq n - \dim F_S \leq \iota_X - 1$.

Consequently, we complete the proof of Theorem 6.1. □

As a corollary, we get the following result.

Corollary 6.4 *Let X be an n-dimensional Fano manifold satisfying the assumptions of Conjecture* AGM^n_ρ *for some $\rho \geq 2$. Let $V^1, \ldots, V^k$ be families of rational curves on X as in Construction 3.5. If V^i are unsplit for all $1 \leq i \leq k$, then there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R is of dimension less than or equal to $\iota_X - 1$.*

Proof We know that $k = \rho_X$ by Proposition 3.3. If $k \geq \rho+1$, then $n \geq \sum_{i=1}^{k}\big(\dim X - \dim \mathrm{Locus}(V^i) + (-K_X \cdot V^i) - 1\big) \geq k(\iota_X - 1) \geq (\rho+1)(\iota_X - 1)$ by Lemma 3.6. We note that $\iota_X \geq 2$ and $\iota_X \rho + 1 - \rho \geq n$. Thus $\iota_X = 2, n = k = \rho + 1$ and $V^1, \ldots, V^n$ are numerically independent dominating and unsplit family of rational curves such that $(-K_X \cdot V^i) = 2$ for any $1 \leq i \leq n$. Therefore X is isomorphic to $(\mathbb{P}^1)^n$ by [25, Theorem 1.1].

Hence we can assume that $\rho_X = k = \rho$. Then

$$n \geq \sum_{i=1}^{k}\big(\dim X - \dim \mathrm{Locus}(V^i) + (-K_X \cdot V^i) - 1\big) \geq \rho(\iota_X - 1) \geq n - 1$$

by Lemma 3.6. Thus at least $\rho - 1$ of families in $\{V^1, \ldots, V^\rho\}$ are dominating families of rational curves. Therefore we can apply Theorem 6.1. □

7 Proof of Proposition 1.8

In this section, we prove Proposition 1.8. First, we consider Proposition 1.8 (a).

Proposition 7.1 *Let X be an n-dimensional Fano manifold with $\rho_X \geq 2$ and $\iota_X \geq (n+1)/2$. Then there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R satisfies* $\dim F \leq \iota_X - 1$.

Proof Take families $V^1, \ldots, V^k$ of rational curves on X as in Construction 3.5. By Corollary 6.4, it is enough to show that all of $V^1, \ldots, V^k$ are unsplit. Assume that there exists a non-unsplit family, say V^j. Then $(-K_X \cdot V^j) \geq 2\iota_X$. Thus $j = k = 1$, $(-K_X \cdot V^1) = 2\iota_X$ and $n = 2\iota_X - 1$ by Lemma 3.6. However, since $3\iota_X > (-K_X \cdot V^1) > n + 1 - \iota_X = \iota_X$, we have $\rho_X = 1$ by Theorem 3.4 (b). This leads to a contradiction. Therefore all of $V^1, \ldots, V^k$ are unsplit families. □

Proposition 7.2 *Let X be an n-dimensional Fano manifold with $\rho_X \geq 3$ and $\iota_X \geq (n+2)/3$. Then there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that any fiber F of ϕ_R satisfies* $\dim F \leq \iota_X - 1$.

Proof Take families $V^1, \ldots, V^k$ of rational curves on X as in Construction 3.5. By Corollary 6.4, it is enough to show that all of $V^1, \ldots, V^k$ are unsplit.

Assume that there exists a non-unsplit family, say V^j. Then $(-K_X\cdot V^j)\geq 2\iota_X$. If $k=1$, then we have $3\iota_X > 3\iota_X-1\geq n+1\geq (-K_X\cdot V^1)\geq 2\iota_X\geq n+2-\iota_X > n+1-\iota_X$. Thus $\rho_X=1$ by Theorem 3.4 (b). This leads to a contradiction. Hence $k\geq 2$. By Lemma 3.6, we have

$$\begin{aligned} n &\geq \sum_{i=1}^{k}\bigl(n-\dim \mathrm{Locus}(V^i)+(-K_X\cdot V^i)-1\bigr)\\ &\geq (2\iota_X-1)+(k-1)(\iota_X-1)\geq 3\iota_X-2\geq n. \end{aligned}$$

Hence we get $k=2$, $n=3\iota_X-2$, both V^1 and V^2 are dominating families, $(-K_X\cdot V^j)=2\iota_X$ and $(-K_X\cdot V^i)=\iota_X$ hold, where $\{i,j\}=\{1,2\}$. Thus for general $x\in X$, we have

$$\begin{aligned} \dim \mathrm{Locus}(V^j,V^i)_x &\geq (-K_X\cdot V^j)-1+(-K_X\cdot V^i)-1=n,\\ \mathrm{N}_1\bigl(\mathrm{Locus}(V^j,V^i)_x,X\bigr) &= \mathbb{R}[V^j]+\mathbb{R}[V^i] \end{aligned}$$

by Lemmas 2.7 and 2.9 (b). Thus $\rho_X=2$. This leads to a contradiction. Therefore all of $V^1,\dots,V^k$ are unsplit families. □

By Propositions 7.1, 7.2 and 5.3 (b), we have proved Proposition 1.8 (a).

Next, we consider Proposition 1.8 (b). By [1,11,12,20,21,23,24,28], Propositions 7.1 and 7.2, it is enough to study five-dimensional Fano manifolds X with $\iota_X=2$ and $\rho_X=4$.

Proposition 7.3 *Let X be a five-dimensional Fano manifold with $\iota_X=2$ and $\rho_X=4$. Then X is isomorphic to one of $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))\times(\mathbb{P}^1)^2$, $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})\times(\mathbb{P}^1)^2$ or $\mathbb{P}^2\times(\mathbb{P}^1)^3$.*

Proof Take families $V^1,\dots,V^k$ of rational curves on X as in Construction 3.5. We note that Conjecture AGM_3^4 is true by [20,21] and Proposition 1.8 (a). Thus by Corollary 6.4 and Proposition 5.3 (b) it is enough to show that all of $V^1,\dots,V^k$ are unsplit.

Assume that V^j is non-unsplit for some $1\leq j\leq k$. Such V^j is unique and $k\leq 3$ holds due to the inequalities

$$5\geq\sum_{i=1}^{k}\bigl(5-\dim \mathrm{Locus}(V^i)+(-K_X\cdot V^i)-1\bigr)\geq (2\cdot 2-1)+(k-1)(2-1)$$

in Lemma 3.6. Moreover, we know that $j=1$ by Lemma 3.7.

Assume $k=3$. Then we have $(-K_X\cdot V^1)=4$, $(-K_X\cdot V^i)=2$ and V^i is a dominating family for $i=2,3$ due to the inequalities

$$5\geq\sum_{i=1}^{3}\bigl(5-\dim \mathrm{Locus}(V^i)+(-K_X\cdot V^i)-1\bigr)\geq (2\cdot 2-1)+2(2-1)=5$$

in Lemma 3.7. This leads to a contradiction since V^1 is a minimal dominating family. Thus $k\leq 2$.

Assume $k = 2$. We repeat the proof in [24, Theorem 5]. We have either $\dim \mathrm{Locus}(V^2) = 4$ and $(-K_X \cdot V^1) = 4$ and $(-K_X \cdot V^2) = 2$, or $\dim \mathrm{Locus}(V^2) = 5$ and $(-K_X \cdot V^1) \geq 4 > 3 \geq (-K_X \cdot V^2)$ due to the inequalities

$$5 \geq \sum_{i=1}^{2} \bigl(5 - \dim \mathrm{Locus}(V^i) + (-K_X \cdot V^i) - 1\bigr) \geq (2 \cdot 2 - 1) + (2 - 1) = 4$$

in Lemma 3.6. If $\dim \mathrm{Locus}(V^2) = 5$, then this leads to a contradiction since V^1 is a minimal dominating family. We can assume that $\dim \mathrm{Locus}(V^2) = 4$. For a general $x \in X$, we have $\mathrm{Locus}(V^1_x) = (\pi^1)^{-1}(\pi^1(x))$ by Lemma 3.6, where $\pi^1 : X \dashrightarrow Z^1$ is the $\mathrm{rc}(\mathcal{V}^1)$-fibration. Thus $\mathrm{Locus}(V^1_x) \cap \mathrm{Locus}(V^2) \neq \emptyset$ since V^2 is a horizontal dominating family with respect to π^1. Hence we have

$$\dim \mathrm{Locus}(V^1, V^2)_x \geq 4,$$
$$\mathrm{N}_1\bigl(\mathrm{Locus}(V^1, V^2)_x, X\bigr) = \mathbb{R}[V^1] + \mathbb{R}[V^2]$$

by Lemmas 2.7 and 2.9 (b). Therefore $\rho_X \leq 3$ by [5, Theorem 1.2]. This leads to a contradiction.

Assume $k = 1$. If $\dim \mathrm{Locus}(V^1_x) \geq 4$ for a general $x \in X$, then $\rho_X \leq 2$ by [5, Theorem 1.2]. Hence $\dim \mathrm{Locus}(V^1_x) \leq 3$ for a general $x \in X$. Then $(-K_X \cdot V^1) = 4$ by Proposition 2.5. We have $3\iota_X = 6 > 4 = (-K_X \cdot V^1) = \dim X + 1 - \iota_X$. Thus $\rho_X \leq 3$ by Theorem 3.4 (c). This leads to a contradiction. □

As a consequence, we have proved Proposition 1.8 (b).

8 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. By [14], [30, Theorem B], [32, Theorem], [24, Theorem 3], [23, Theorem 5.1] and Proposition 1.8 (b), it is enough to show the following.

Theorem 8.1 *Set $r \geq 3$. If X is a $(3r - 2)$-dimensional Fano manifold with $r_X = r$ and $\rho_X = 3$, then X is isomorphic to one of $\mathbb{Q}^r \times (\mathbb{P}^{r-1})^2$, $\mathbb{P}_{\mathbb{P}^r}\bigl(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1)\bigr) \times \mathbb{P}^{r-1}$ or $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}) \times \mathbb{P}^{r-1}$.*

Proof By [24, Theorem 3], we have $\iota_X = r_X = r$. By Proposition 7.2 and Theorem 5.2, there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that the associated contraction morphism $\phi_R : X \to Y$ is a $\mathbb{P}^{r-1}$-fibration. The variety Y is a $(2r-1)$-dimensional Fano manifold with $\iota_Y \geq r$ and $\rho_Y = 2$ by Lemma 4.2. By [24, Theorem 3], we have $\iota_Y = r$. By Proposition 7.1 and Theorem 5.2, there exists an extremal ray $S \subset \mathrm{NE}(Y)$ such that the associated contraction morphism $\phi_S : Y \to Z$ is a $\mathbb{P}^{r-1}$-fibration.

Claim 8.2 *The variety Z is isomorphic to either $\mathbb{P}^r$ or $\mathbb{Q}^r$.*

Proof Set $\pi = \phi_S \circ \phi_R : X \to Z$. Let $R' \subset \mathrm{NE}(X)$ be the extremal ray such that the morphism π corresponds to the extremal face $R + R' \subset \mathrm{NE}(X)$. Choose any extremal ray $R'' \subset \mathrm{NE}(X)$ with $R'' \neq R, R'$. Then any nontrivial fiber F of $\phi_{R''}\colon X \to X_{R''}$ satisfies $\dim F \leq r$ since $\pi|_F\colon F \to Z$ is a finite morphism. On the other hand, by Theorem 5.1,

$$\dim F \geq \dim X - \dim \mathrm{Exc}(\phi_{R''}) + l(R'') - 1.$$

Thus $l(R'') = r$ and there are three possibilities:

(a) $\phi_{R''}$ is a divisorial contraction and any fiber F of $\phi_{R''}$ satisfies $\dim F = r$.
(b) $\phi_{R''}$ is of fiber type and any fiber F of $\phi_{R''}$ satisfies $\dim F = r$.
(c) $\phi_{R''}$ is of fiber type and a general fiber F of $\phi_{R''}$ satisfies $\dim F = r - 1$.

We consider the case (a). Then $F \simeq \mathbb{P}^r$ by [2, Theorem 4.1 (iii)]. Thus $Z \simeq \mathbb{P}^r$ by [26, Theorem 1]. We consider the case (b). Then a general fiber F is isomorphic to $\mathbb{Q}^r$ by [14]. Thus $Z \simeq \mathbb{P}^r$ or $\mathbb{Q}^r$ by [7]. We consider the case (c). Set

$$B = \left\{x \in X_{R''} : \dim \phi_{R''}^{-1}(x) \geq r\right\}.$$

Since $\mathrm{codim}_X\, \phi_{R''}^{-1}(B) \geq 2$, by [15, Proposition II.3.7, Theorems IV.3.10 and V.2.13] we can take a general (complete) very free rational curve C on $X \setminus \phi_{R''}^{-1}(B)$ such that $C'' = \phi_{R''}(C)$ is not a point. By [10, Theorem 1.3], $\phi_{R''}^{-1}(x)$ is scheme-theoretically isomorphic to $\mathbb{P}^{r-1}$ for any $x \in C''$. Let $\nu\colon \mathbb{P}^1 \to C'' \hookrightarrow X_{R''}$ be the normalization morphism and set $T = X \times_{X_{R''}} \mathbb{P}^1$ as in Definition 4.4. Since $T \to \mathbb{P}^1$ is a $\mathbb{P}^{r-1}$-fibration, T is a toric variety. For any fiber F'' of $T \to \mathbb{P}^1$, the morphism $\pi : X \to Z$ restricted to the image of F'' is a finite morphism. Since C is general, the morphism $T \to Z$ is surjective. Therefore $Z \simeq \mathbb{P}^r$ by [26, Theorem 1]. ■

By using Proposition 4.3 and Corollary 4.7 twice of each, we get the possibilities of the structures of Y and X. Thus we get the assertion. □

As a consequence, we complete the proof of Theorem 1.6.

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