Smooth compactifications of certain normic bundles

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Abstract For a finite cyclic Galois extension of fields K/k of degree n and a separable polynomial of degree dn or $dn - 1$, we construct an explicit smooth compactification $X \rightarrow \mathbb{P}_k^1$ of the affine normic bundle X_0 given by

$$N_{K/k}(\mathbf{z}) = P(x) \neq 0,$$

extending the map $X_0 \rightarrow \mathbb{A}_k^1$, where $(\mathbf{z}, x) \mapsto x$. The construction makes no assumption of the characteristic of k , making it a suitable departure point for studying the arithmetic of smooth compactifications of X_0 over global fields of positive characteristic.

Keywords Normic bundles · Hasse principle · Smooth compactifications · Brauer–Manin obstruction

Mathematics Subject Classification 11G35 · 14G05

1 Introduction

For a finite extension K/k of fields and a polynomial $P(x) \in k[x]$, the affine norm hypersurface $X_0 \subset \mathbb{A}_k^{n+1}$ given by

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$$N_{K/k}(\mathbf{z}) = P(x) \neq 0$$

parametrizes the values of $P(x)$ that are norms for K/k .

Suppose that k is a number field. The classical Hasse norm theorem states that if K/k is a cyclic Galois extension and if $P(x)$ is a nonzero constant, then X_0 satisfies the Hasse principle. Although both the Hasse principle and weak approximation fail for more general X_0 , Colliot-Thélène has conjectured that the Brauer–Manin obstruction controls failures of weak approximation on any smooth proper model of X_0 . See [4, Section 1] for a summary of progress towards this conjecture.

The existence of a smooth proper model X of X_0 extending the projection

$$X_0 \rightarrow \mathbb{A}^1, \quad (\mathbf{z}, x) \mapsto x$$

to a map $X \rightarrow \mathbb{P}_k^1$ is especially useful for proving arithmetic results in the direction of Colliot-Thélène’s conjecture, because the map $X \rightarrow \mathbb{P}_k^1$ affords some control over the Brauer group of X . This map can also be used to prove that certain subsets of the number field k are diophantine [3, 7, 8].

The known constructions of $X \rightarrow \mathbb{P}^1$ proceed in two steps. First, one constructs a partial compactification $X' \rightarrow \mathbb{A}_k^1$ (e.g. [2, Section 2] or [3, Proposition 2.2 (i)]). Second, one extends $X' \rightarrow \mathbb{A}_k^1$ to a map $X \rightarrow \mathbb{P}^1$ via Hironaka’s theorem. This second step limits the scope of the construction to fields of characteristic 0.

Our goal in this note is to give an explicit construction of a compactification $X \rightarrow \mathbb{P}_k^1$ convenient for arithmetic applications, under some hypotheses. (For example, the Picard group and Brauer group of such a compactification X are easily computable; see the proofs of [8, Proposition 3.1 and Theorem 3.2].) The construction of X does not impose a restriction on the characteristic of k ; it therefore serves as a starting point for studying the arithmetic of smooth compactifications of X_0 over global fields of positive characteristic.

Theorem 1.1 *Let K/k be a cyclic Galois extension of fields of degree n , and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn - 1$. There exists a smooth proper compactification X of X_0 , fibered over $\mathbb{P}_k^1 = \text{Proj } k[x_0, x_1]$, such that $X \rightarrow \mathbb{P}_k^1$ extends the map $X_0 \rightarrow \mathbb{A}_k^1$. Furthermore, the generic fiber of $X \rightarrow \mathbb{P}_k^1$ is a Severi–Brauer variety, and the degenerate fibers lie over $V(P(x_0/x_1)x_1^{dn})$, and consist of the union of n rational varieties all conjugate under $\text{Gal}(K/k)$.*

1.1 Outline

Our construction of a smooth compactification takes a cue from work of Kang: the generic fiber of our construction is the embedded Severi–Brauer variety in [6].

In Sect. 3.2, we construct a partial compactification $Y_a \rightarrow \text{Spec } R$ of the variety $z_1 \cdots z_n = a \neq 0$ for any k -algebra R with no zero-divisors and any $a \in R \setminus 0$. In Sect. 3.3, we give an explicit open covering of Y_a , which we use in Sect. 3.4 to prove that Y is smooth if and only if $V(a) \subset \text{Spec } R$ is smooth. We describe the geometry of the degenerate fibers of $Y_a \rightarrow \text{Spec } R$ in Sect. 3.5.

In Sect. 4, we construct a K/k -twist of Y_a , $X_{K,a}^0 \rightarrow \operatorname{Spec} R$. Finally, in Sect. 5, we restrict to the case $R = k[x]$ and $a = P(x)$, give a full compactification $X \rightarrow \mathbb{P}_k^1$, and prove Theorem 1.1.

Remark 1.2 Let R be a Dedekind domain, and let $\mathcal{O}$ be a maximal R -order in a central simple $\operatorname{Frac}(R)$ -algebra of rank n^2 . Write $X_{\mathcal{O}} \rightarrow \operatorname{Spec} R$ for the scheme of rank n left ideals of $\mathcal{O}$. In [1] Artin studies the geometry of $X_{\mathcal{O}} \rightarrow \operatorname{Spec} R$ through its functor of points. By taking $R = k[x]$, and an appropriate choice of $\mathcal{O}$, this gives a variety $X_{\mathcal{O}}$ proper over $\mathbb{A}_k^1$, whose generic fiber is the Severi–Brauer variety associated to $X_{0,k(x)}$, and whose degenerate fibers consist of n rational varieties conjugate under $\operatorname{Gal}(K/k)$.

The functor of points description of $X_{\mathcal{O}}$ highlights the connection with central simple algebras, but the embedding $X_{0,k(x)} \hookrightarrow X_{\mathcal{O}}$ is not easily extracted from this description. In contrast, our construction of a compactification of X_0 takes the norm form X_0 as a starting point, and thus the embedding $X_0 \hookrightarrow X$ is readily apparent in the construction.

2 Preliminaries on vectors

Throughout, we fix an integer $n > 1$. By the *weight* of a vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathbb{Z}^n$, we mean the sum $\sum_{m=0}^{n-1} v_m$; we also say $\mathbf{v}$ has *length* n . Let $\mathcal{V}_n$ denote the set of nonnegative integer vectors of weight n and length n . Write $\sigma: \mathcal{V}_n \rightarrow \mathcal{V}_n$ for the *shift operator*

$$\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \mapsto (v_1, v_2, \dots, v_{n-1}, v_0) = \sigma(\mathbf{v}).$$

For any $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we define two nonnegative integers:

$$\begin{aligned} \mu(\mathbf{v}) &= \max_{i \in (0, n]} (i - v_0 - \dots - v_{i-1}), \\ \lambda(\mathbf{v}) &= \mu(\mathbf{v}) + \mu(\sigma(\mathbf{v})) + \dots + \mu(\sigma^{n-1}(\mathbf{v})), \end{aligned}$$

and for any integers i, j with $i \not\equiv j \pmod n$ and $v_j > 0$, we let $\mathbf{v}^{ij} = (v_m + \delta_{i \bmod n, m} - \delta_{j \bmod n, m})_{m=0}^{n-1}$; note that $\mathbf{v}^{ij} \in \mathcal{V}_n$, because $v_j > 0$. We collect a few straightforward relations used frequently below.

Lemma 2.1 *We have the following relations.*

- (a) $\mu(\sigma^s(\mathbf{v})) = \mu(\mathbf{v}) + v_0 + v_1 + \dots + v_{s-1} - s$ for any $\mathbf{v} \in \mathcal{V}_n$ and for any $s \in (0, n]$.
- (b) For any integer r and any vectors $\mathbf{v}_i, \mathbf{w}_i \in \mathcal{V}_n$ with $1 \leq i \leq r$, such that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, we have $\sum_{i=1}^r \lambda(\mathbf{w}_i) - \sum_{i=1}^r \lambda(\mathbf{v}_i) = n(\sum_{i=1}^r \mu(\mathbf{w}_i) - \sum_{i=1}^r \mu(\mathbf{v}_i))$.
- (c) Fix integers $0 \leq r < s < n$ and fix $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ such that $v_r, v_s > 0$. Then $0 \leq \mu(\mathbf{v}^{rs}) + \mu(\mathbf{v}^{sr}) - 2\mu(\mathbf{v}) \leq 1$, and the first inequality is strict if and only if $\mu(\mathbf{v}) = i - \sum_{m=0}^{i-1} v_m = j - \sum_{m=0}^{j-1} v_m$ for some $i \in (r, s]$ and $j \in (0, r] \cup (s, n]$.

Proof Let $\mathbf{w} = \sigma^s(\mathbf{v})$, so that $w_j = v_{j+s}$ if $j < n-s$ and $w_j = v_{j+s-n}$ if $j \geq n-s$. Then $i - w_0 - w_1 - \dots - w_{i-1}$ equals

$$\begin{cases} ((i+s) - v_0 - \cdots - v_{i+s-1}) + v_0 + \cdots + v_{s-1} - s & \text{if } i+s \leq n, \\ ((i+s-n) - v_0 - \cdots - v_{i+s-n-1}) + (n-s) - v_s - \cdots - v_{n-1} & \text{otherwise.} \end{cases}$$

To conclude (a), note that since $\mathbf{v}$ has weight n , we have $v_0 + v_1 + \cdots + v_{s-1} - s = (n-s) - v_s - v_{s+1} - \cdots - v_{n-1}$.

By (a), for any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$ we have

$$\lambda(\mathbf{v}) = n\mu(\mathbf{v}) + \sum_{m=0}^{n-1} ((n-1-m)v_m - m).$$

Using the assumption that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$, the proof of (b) is now a simple manipulation.

It remains to prove (c). Let $\mathbf{w}^- = \mathbf{v}^{rs}$ and $\mathbf{w}^+ = \mathbf{v}^{sr}$. Since $r < s$, by the definition of μ we have

$$\mu(\mathbf{v}) - 1 \leq \mu(\mathbf{w}^-) \leq \mu(\mathbf{v}) \quad \text{and} \quad \mu(\mathbf{v}) \leq \mu(\mathbf{w}^+) \leq \mu(\mathbf{v}) + 1.$$

Furthermore, we have $\mu(\mathbf{w}^-) = \mu(\mathbf{v}) - 1$ if and only if the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0, n]}$ is *only* achieved for $i \in (r, s]$. Similarly, we have $\mu(\mathbf{v}) = \mu(\mathbf{w}^+)$ if and only if the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0, n]}$ is *only* achieved for $i \in (0, r] \cup (s, n]$. $\square$

The following notion is the fundamental book-keeping device in the construction of $X \rightarrow \mathbb{P}_k^1$.

Definition 2.2 A vector $\mathbf{v} = (v_0, v_1, \dots, v_{n-1}) \in \mathcal{V}_n$ is *well-spaced* if

$$v_i > 0 \Rightarrow v_{i+v_i} > 0 \quad \text{and} \quad v_{i+j} = 0, \quad j \in [1, v_i - 1],$$

for all $i \in [0, n)$. Here indices are considered modulo n .

For example, the vector $\mathbf{v} = (0, 3, 0, 0, 2, 0, 4, 0, 0)$ is well-spaced whereas $\mathbf{w} = (0, 3, 0, 0, 2, 4, 0, 0, 0)$ is not. Note that $\sigma(\mathbf{v})$ is well-spaced if and only if $\mathbf{v}$ is well-spaced.

Remark 2.3 We are unaware if well-spaced vectors arise naturally in other fields. It would be interesting to have a conceptual understanding of why these vectors yield useful affine coverings of the varieties under consideration, see Sect. 3.

Lemma 2.4 Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector with $\ell + 1$ nonzero entries indexed by $i_0 < \cdots < i_\ell$. Set $i_{\ell+1} = n + i_0$. Then $\mu(\mathbf{v}) = i_0$, and for any $r, s \in [0, \ell]$ and $j \in (i_r, i_{r+1})$, we have

$$\mu(\mathbf{v}^{j i_r}) = \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor, \quad \mu(\sigma^{i_s}(\mathbf{v}^{j i_r})) = 0,$$

and if $\ell \neq 0$,

$$\mu(\mathbf{v}^{i_r i_{r+1}}) = \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor, \quad \mu(\sigma^{i_s}(\mathbf{v}^{i_r i_{r+1}})) = \delta_{s, r+1 \bmod \ell+1}.$$

Proof For any vector $\mathbf{v} = (v_m)_{m=0}^{n-1} \in \mathcal{V}_n$, the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0, n]}$ is never achieved at $i = j$ where $v_j = 0$. Additionally, since $\mathbf{v}$ is well-spaced, $v_{i_r} = i_{r+1} - i_r$ for all $r \in [0, \ell]$. Hence

$$\mu(\mathbf{v}) = \max_{r \in [0, \ell]} (i_r - v_{i_0} - \cdots - v_{i_{r-1}}) = \max_{r \in [0, \ell]} (i_r - (i_1 - i_0) - \cdots - (i_r - i_{r-1})) = i_0$$

and the maximum of $\{i - v_0 - \cdots - v_{i-1}\}_{i \in (0, n]}$ is achieved at $i = i_r$ for all $r \in [0, \ell]$.

Thus, the formulas for $\mu(\mathbf{v}^{j i_r})$ and $\mu(\mathbf{v}^{i_r i_{r+1}})$ follow from the same argument as in Lemma 2.1 (c). Let $\mathbf{w} = \mathbf{v}^{j i_r}$. Then

$$\begin{aligned} \mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s && \text{by Lemma 2.1 (a),} \\ &= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s && \text{by the formula for } \mu(\mathbf{v}^{j i_r}), \\ &= \mu(\mathbf{v}) - \left\lfloor \frac{j}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \left\lfloor \frac{j}{n} \right\rfloor && \text{by the definition of } \mathbf{v}^{j i_r}, \\ &= \mu(\mathbf{v}) - i_0 && \text{since } v_{i_r} = i_{r+1} - i_r \text{ for } r \in [0, \ell]. \end{aligned}$$

Similarly, if $\mathbf{w} = \mathbf{v}^{i_r i_{r+1}}$, we have

$$\begin{aligned} \mu(\sigma^{i_s}(\mathbf{w})) &= \mu(\mathbf{w}) + w_0 + \cdots + w_{i_s-1} - i_s \\ &= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + w_0 + \cdots + w_{i_s-1} - i_s \\ &= \mu(\mathbf{v}) + \left\lfloor \frac{i_{r+1}}{n} \right\rfloor + v_0 + \cdots + v_{i_s-1} - i_s + \delta_{s, r+1 \bmod \ell+1} - \left\lfloor \frac{i_{r+1}}{n} \right\rfloor \\ &= \mu(\mathbf{v}) - i_0 + \delta_{s, r+1 \bmod \ell+1}. \end{aligned} \quad \square$$

3 The auxiliary bundle $Y \rightarrow \operatorname{Spec} R$

3.1 Notation

Let R be a k -algebra with no zero-divisors. Given a nonzero element $a \in R$, we use the standard notation $D(a)$ to denote the open affine subscheme of $\operatorname{Spec} R$ given by $\operatorname{Spec} R_a$; if R is graded, we let $D_+(a)$ denote the open affine subscheme of $\operatorname{Proj} R$ given by $\operatorname{Spec}(R_a)_0$.

Let $N = \binom{2n-1}{n} - 1$, and fix coordinates on $\mathbb{P}_k^N = \operatorname{Proj} k[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]$. We set $\mathbb{P}_R^N = \mathbb{P}_k^N \times_{\operatorname{Spec} k} \operatorname{Spec} R$.

3.2 Construction of Y_a

For any nonzero $a \in R$, we consider the embedding

$$\iota_a: \text{Proj } R[t_0, \dots, t_{n-1}] \cap D(a) \hookrightarrow \mathbb{P}_R^N \times_{\text{Spec } R} D(a)$$

induced by the map $y_{\mathbf{v}} \mapsto a^{\mu(\mathbf{v})} t_0^{v_0} t_1^{v_1} \cdots t_{n-1}^{v_{n-1}}$. (This is easily seen to be an embedding since it is the composition of the (n) -uple embedding with a scaling of the coordinates by an appropriate power of a .) The image of ι_a is cut out by the equations, see [5, Example 2.6]:

$$a^{\sum_{i=1}^r \mu(\mathbf{w}_i)} \prod_{i=1}^r y_{\mathbf{v}_i} = a^{\sum_{i=1}^r \mu(\mathbf{v}_i)} \prod_{i=1}^r y_{\mathbf{w}_i} \quad (1)$$

for all integers r and all sets of vectors $\mathbf{w}_i, \mathbf{v}_i$, with $1 \leq i \leq r$, such that $\sum_{i=1}^r \mathbf{w}_i = \sum_{i=1}^r \mathbf{v}_i$. Let Y_a be the closure in $\mathbb{P}_R^N$ of the image of ι_a .

Lemma 3.1 *The order n automorphism $\phi: \mathbb{P}_R^N \rightarrow \mathbb{P}_R^N, y_{\mathbf{v}} \mapsto y_{\sigma(\mathbf{v})}$ preserves Y_a .*

Proof Fix an integer r and vectors $\mathbf{v}_i, \mathbf{w}_i$, with $1 \leq i \leq r$, such that $\sum_{i=1}^r \mathbf{v}_i = \sum_{i=1}^r \mathbf{w}_i$. It is clear that $\sum_{i=1}^r \sigma(\mathbf{v}_i) = \sum_{i=1}^r \sigma(\mathbf{w}_i)$. Moreover, by Lemma 2.1 (a),

$$\sum_{i=1}^r \mu(\sigma(\mathbf{v}_i)) - \sum_{i=1}^r \mu(\sigma(\mathbf{w}_i)) = \sum_{i=1}^r \mu(\mathbf{v}_i) - \sum_{i=1}^r \mu(\mathbf{w}_i).$$

Therefore, ϕ preserves the relations (1). $\square$

3.3 An open covering

Proposition 3.2 *The open subvarieties $\{D_+(y_{\mathbf{v}}) \subseteq \mathbb{P}_R^N : \mathbf{v} \in \mathcal{V}_n \text{ well-spaced}\}$ cover Y_a .*

Proof Let $\mathbf{w} = (w_0, w_1, \dots, w_{n-1}) \in \mathcal{V}_n$ be a vector that is not well-spaced. We will show that $D_+(y_{\mathbf{w}}) \cap Y_a \subset D_+(y_{\mathbf{v}})$ for some well-spaced vector $\mathbf{v} \in \mathcal{V}_n$. Let $i_0 < \dots < i_\ell$ be the indices such that $w_{i_j} > 0$, and set $i_{\ell+1} = i_0 + n$.

If $\mu(\mathbf{w}) > i_0 \geq 0$, then there exists $r \in [0, \ell)$ such that $\mu(\mathbf{w}) = i_{r+1} - w_{i_0} - w_{i_1} - \dots - w_{i_r}$. Fix the smallest such r ; then $i_{r+1} - i_r - w_{i_r} > 0$. Since $\mathbf{w}$ has length n and weight n , there exists $r < s \leq \ell$ such that $i_{s+1} - i_s - w_{i_s} < 0$; fix the largest such s . Then by our choice of r and s , if $\mu(\mathbf{w}) = j - v_0 - \dots - v_{j-1}$, we must have $j \in (i_r, i_s]$. Therefore by Lemma 2.1 (c), the defining equations for Y_a include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r i_s}} y_{\mathbf{w}^{i_s i_r}},$$

so $D_+(y_{\mathbf{w}}) \subset D_+(y_{\mathbf{w}^{i_r i_s}})$. After possibly repeating the argument we may assume that $\mu(\mathbf{w}) = i_0$.

If $i_{r+1} - i_r - w_{i_r} = 0$ for all r , then $\mathbf{w}$ is well-spaced. Otherwise, fix the smallest integer r such that $|i_{r+1} - i_r - w_{i_r}| > 0$; since $\mu(\mathbf{w}) = i_0$, we must have $i_{r+1} - i_r - w_{i_r} < 0$. Since $\mathbf{w}$ has length n and weight n , there exists $r < s \leq \ell$ such that $i_{s+1} - i_s - w_{i_s} > 0$; fix the smallest such s . Now by our choice of r and s , if $\mu(\mathbf{w}) = j - v_0 - \cdots - v_{j-1}$, we must have $j \in [0, i_r] \cup (i_s, n)$. Then by the same argument as above, the defining equations for Y_a include the relation

$$y_{\mathbf{w}}^2 = y_{\mathbf{w}^{i_r i_s}} y_{\mathbf{w}^{i_s i_r}}.$$

By replacing $\mathbf{w}$ with $\mathbf{w}^{i_s i_r}$, we reduce the value of $|i_{r+1} - i_r - w_{i_r}|$. Repeating this process we will arrive at a well-spaced vector $\mathbf{v}$ in finitely many steps. $\square$

3.4 Smoothness of Y_a

Proposition 3.3 *Let $\mathbb{A}_R^n = \text{Spec } R[Z_0, \dots, Z_{n-1}]$. Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector with $\ell + 1$ nonzero entries indexed by $i_0 < \cdots < i_\ell$ and set $i_{\ell+1} = i_0 + n$. Then the map*

$$\frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} \mapsto \prod_{\substack{j \in [0, n) \\ v_j = 0}} Z_j^{w_j} \times \prod_{r=0}^{\ell} Z_{i_r}^{\mu(\sigma^{i_{r+1}}(\mathbf{w}))}$$

yields an isomorphism $Y_a \cap D_+(y_{\mathbf{v}}) \cong V(Z_{i_0} \cdots Z_{i_\ell} - a) \subset \mathbb{A}_R^n$. In particular, Y_a is a compactification of the variety in $\mathbb{A}_R^n$ given by $Z_0 Z_1 \cdots Z_{n-1} = a$.

From Propositions 3.2 and 3.3, and the Jacobian criterion we have

Corollary 3.4 *The variety Y_a is smooth if and only if $V(a)$ is smooth in $\text{Spec } R$.*

Proof of Proposition 3.3 Set $i_{-1} = i_\ell - n$. The proof of the proposition differs slightly in the case when $\ell = 0$. To give a unified presentation, if $\ell = 0$ then we set $y_{\mathbf{v}^{i_0 i_{\ell+1}}} = a y_{\mathbf{v}}$. Consider the following functions on $Y_a \cap D_+(y_{\mathbf{v}})$ for $j = i_0, i_0 + 1, \dots, i_0 + n - 1$:

$$g_j = \begin{cases} y_{\mathbf{v}^{j i_r}} y_{\mathbf{v}}^{-1} & \text{if } i_r < j < i_{r+1} \text{ for some } 0 \leq r \leq \ell, \\ y_{\mathbf{v}^{i_r i_{r+1}}} y_{\mathbf{v}}^{-1} & \text{if } j = i_r, 0 \leq r \leq \ell. \end{cases}$$

Lemma 2.4 shows that the map sends $g_j \rightarrow Z_{j \bmod n}$. In addition, Lemma 2.4 together with the relations (1) shows that $g_{i_0} \cdots g_{i_\ell} = a$. Thus, to prove the map is a well-defined isomorphism, we will show that

$$y_{\mathbf{w}} y_{\mathbf{v}}^m = \prod_{\substack{j \in [0, n) \\ v_j = 0}} y_{\mathbf{v}^{j i_r}}^{w_j} \times \prod_{r=0}^{\ell} y_{\mathbf{v}^{i_r i_{r+1}}}^{\mu(\sigma^{i_{r+1}}(\mathbf{w}))},$$

where $m = -1 + \sum_{j, v_j = 0} w_j + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w}))$. By Lemmas 2.1 (a) and 2.4, we have

$$\begin{aligned}
 \sum_{\substack{j \in [0, n) \\ v_j = 0}} w_j \cdot \mu(\mathbf{v}^{j i_r}) + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w})) \mu(\mathbf{v}^{i_r i_{r+1}}) \\
 = (m+1) \mu(\mathbf{v}) - \sum_{j=0}^{i_0-1} w_j + \mu(\sigma^{i_{\ell+1}}(\mathbf{w})) \\
 = (m+1) \mu(\mathbf{v}) + \mu(\mathbf{w}) - i_0 = m \mu(\mathbf{v}) + \mu(\mathbf{w}).
 \end{aligned}$$

Hence, by (1), it suffices to prove that $\mathbf{w} + m\mathbf{v}$ is equal to

$$\mathbf{w}' = \sum_{\substack{j \in [0, n) \\ v_j = 0}} w_j \mathbf{v}^{j i_r} + \sum_{r=0}^{\ell} \mu(\sigma^{i_{r+1}}(\mathbf{w})) \mathbf{v}^{i_r i_{r+1}}.$$

For j such that $v_j = 0$, it is evident that $w'_j = w_j + m v_j$. Further,

$$\begin{aligned}
 w'_{i_s} &= (m+1) v_{i_s} - \sum_{j=i_s+1}^{i_{s+1}-1} w_j + \mu(\sigma^{i_{s+1}}(\mathbf{w})) - \mu(\sigma^{i_s}(\mathbf{w})) \\
 &= (m+1) v_{i_s} - \sum_{j=i_s+1}^{i_{s+1}-1} w_j + \sum_{j=i_s}^{i_{s+1}-1} w_j - i_{s+1} + i_s \quad \text{by Lemma 2.1 (a)} \\
 &= m v_{i_s} + w_{i_s} \quad \text{since } v_{i_s} = i_{s+1} - i_s.
 \end{aligned}$$

□

3.5 The degenerate fibers of $Y_a \rightarrow \text{Spec } R$

Proposition 3.5 *Let $Q \in V(a) \in R$ be a closed point. The fiber $Y_{a,Q}$ consists of n rational $(n-1)$ -dimensional irreducible components which are permuted cyclically by the automorphism ϕ of Lemma 3.1.*

Proof For $i = 0, \dots, n-1$, we define $S_i = Y_{a,Q} \cap V(\langle y_{\mathbf{w}} : \mu(\sigma^{i+1}(\mathbf{w})) > 0 \rangle)$. From the definition, it follows that ϕ acts on the set $\{S_i : 0 \leq i \leq n-1\}$ via the permutation

$$S_0 \mapsto S_{n-1} \mapsto S_{n-2} \mapsto \dots \mapsto S_1 \mapsto S_0.$$

We claim that $Y_{a,Q} = S_0 \cup S_1 \cup \dots \cup S_{n-1}$ and that each S_i is an irreducible rational $(n-1)$ -dimensional variety.

Let $\mathbf{v} \in \mathcal{V}_n$ be a well-spaced vector and let $i_0 < \dots < i_{\ell}$ be the indices of the nonzero entries of $\mathbf{v}$. By Proposition 3.3, $Y_{a,Q} \cap D_+(\mathbf{y}_{\mathbf{v}})$ is isomorphic to a union of $\ell+1$ hyperplanes in $\mathbb{A}_{\mathbf{k}(Q)}^n = \text{Spec } \mathbf{k}(Q)[Z_0, \dots, Z_{n-1}]$. Furthermore, the hyperplane $Z_{i_r} = 0$ is isomorphic to the subvariety

$$V\left(\left\langle \frac{y_{\mathbf{w}}}{y_{\mathbf{v}}} : \mu(\sigma^{i_r+1}(\mathbf{w})) > 0 \right\rangle\right) \subset Y_{a,Q} \cap D_+(y_{\mathbf{v}}).$$

Hence, $Y_{a,Q} \cap D_+(y_{\mathbf{v}})$ is a dense open subset of $S_{i_0} \cup S_{i_1} \cup \cdots \cup S_{i_\ell}$. Since the open subvarieties $\{D_+(y_{\mathbf{v}}) \cap Y_{a,Q} : \mathbf{v} \text{ well-spaced}\}$ cover $Y_{a,Q}$, this completes the proof. $\square$

4 A K/k twist of Y_a

Let K/k be a cyclic Galois extension of degree n , and let $R_K = R \otimes_k K$. Fix a basis $\{\alpha_0, \dots, \alpha_{n-1}\}$ of K as a k -vector space, as well as a generator τ of $\text{Gal}(K/k)$.

Let T be a set of representatives for the orbits of $\mathcal{V}_n$ under the action of the shift operator σ . Consider the K -isomorphism $\psi : \text{Proj } K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \rightarrow \text{Proj } K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]$ determined by

$$\begin{aligned} y_{\mathbf{v}} &\mapsto \alpha_0 z_{\mathbf{v}} + \alpha_1 z_{\sigma(\mathbf{v})} + \cdots + \alpha_{n-1} z_{\sigma^{n-1}(\mathbf{v})} \quad \text{for } \mathbf{v} \in T, \\ y_{\sigma^i(\mathbf{v})} &\mapsto \sum_{j=0}^{n-1} \tau^i(\alpha_j) z_{\sigma^j(\mathbf{v})} \quad \text{for } \mathbf{v} \in T, \quad i = 1, \dots, n-1. \end{aligned}$$

Define $X_{K,a}^0 = \psi_{R_K}^{-1}(Y_a)$. Abusing notation, we write τ for the endomorphism of $\text{Proj } R[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \times_R R_K$ given by $\text{id} \times \tau$. Let $\phi : \mathbb{P}_R^N \rightarrow \mathbb{P}_R^N$ be the automorphism of Lemma 3.1. The following diagram commutes:

$$\begin{array}{ccc} \text{Proj } R_K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] & \xrightarrow{\psi_{R_K}} & \text{Proj } R_K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] \\ \downarrow \tau & & \downarrow \phi_{R_K} \\ \text{Proj } R_K[\{z_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}] & \xrightarrow{\psi_{R_K}} & \text{Proj } R_K[\{y_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}_n\}]. \end{array} \quad (2)$$

By Lemma 3.1, the map ϕ preserves Y_a . Together with the commutativity of the above diagram, this implies that $X_{K,a}^0$ descends to an R -scheme.

5 Proof of Theorem 1.1

Let K/k be a cyclic Galois extension of degree n and let $P(x) \in k[x]$ be a separable polynomial of degree dn or $dn-1$ for some d .

Lemma 5.1 *There exists a smooth projective variety $X = X_{K/k,P(x)} \rightarrow \mathbb{P}_k^1$ such that $X_{\mathbb{A}^1} \cong X_{K,P(x)}^0$ and that $X_{\mathbb{P}^1 \setminus \{0\}} \cong X_{K,P(1/x'),x'^{dn}}^0$, where $x' = 1/x$.*

Proof We will construct X by glueing $Y_{P(x)}$ and $Y_{P(1/x'),x'^{dn}}$ over $\text{Spec } k[x^{\pm 1}]$ and $\text{Spec } k[x^{\pm 1}]$, in a way which is compatible with the map ψ from Sect. 4. Let $y_{\mathbf{v}}$

denote the coordinates on $Y_{P(x)}$ and let y'_v denote the coordinates on $Y_{P(1/x')x'^{dn}}$. By Lemma 2.1 (b), the morphism

$$Y_{P(1/x')x'^{dn}} \times_{\mathbb{A}^1} \operatorname{Spec} k[x', x'^{-1}] \rightarrow Y_{P(x)} \times_{\mathbb{A}^1} \operatorname{Spec} k[x, x^{-1}]$$

where $y_v \mapsto (x')^{d\lambda(v)} y'_v$ and $x \mapsto 1/x'$ is well-defined and is an isomorphism. Since $\lambda(v) = \lambda(\sigma(v))$, this morphism is compatible with ψ and thus gives a glueing of $X_{K, P(x)}^0$ and $X_{K, P(1/x')x'^{dn}}^0$. $\square$

Proposition 5.2 *The variety X is a smooth proper compactification of X_0 , the generic fiber of $X \rightarrow \mathbb{P}^1$ is a Severi–Brauer variety, and the degenerate fibers of $X \rightarrow \mathbb{P}^1$ lie over $V(P(x_0/x_1)x_1^{dn})$ and consist of the union of n rational varieties all conjugate under $\operatorname{Gal}(K/k)$.*

Proof The compatibility (2) together with Proposition 3.3 and Corollary 3.4 implies that

$$(X \times_{\mathbb{P}^1} \mathbb{A}^1) \cap D_+(z_{(1,1,\dots,1)}) \cong X_0,$$

which gives the first claim. The second claim is immediate from the construction of X , and the third claim follows from Proposition 3.5 and the compatibility (2). $\square$

Proposition 5.2 completes the proof of Theorem 1.1.

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