A pre-Lie algebra associated to a linear endomorphism and related algebraic structures

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Abstract We construct a functor T from the category of endomorphisms of vector spaces to the category of Com-PreLie algebras. For any endomorphism f of a vector space V , we describe the enveloping algebra of the pre-Lie algebra $T(V, f)$, the dual Hopf algebra and the associated group of characters. For $f = \text{Id}_V$, we find the algebra of formal diffeomorphisms, seen as a subalgebra of the Connes–Kreimer Hopf algebra of rooted trees in the context of QFT; for other well-chosen nilpotent f , we obtain the groups of Fliess operators in Control Theory. An algebraic study of these Com-PreLie Hopf algebras is carried out: gradations, generation, subobject generated by V , etc.

Keywords Pre-Lie algebras · Com-PreLie Hopf algebras · Fliess operators · Faà di Bruno Hopf algebra

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1 Introduction

The Hopf algebra H of coordinates of the group of Fliess operators G , used to study the feedback in Control Theory, is described in [11]. This Hopf algebra is studied in the one-dimensional case in [10], and it is shown it is a right-sided Hopf algebra in the sense of [15]. Consequently, its dual Lie algebra $\mathfrak{g}$, which is the Lie algebra of G , inherits a right pre-Lie algebra structure. It is proved that $\mathfrak{g}$ is in fact a Hopf Com-PreLie algebra, that is to say:

1. $\mathfrak{g}$ has a commutative product $\sqcup$ and a coproduct Δ , making it a commutative Hopf algebra.
2. $\mathfrak{g}$ has a nonassociative product $\bullet$ satisfying the (right) pre-Lie axiom: for all $a, b, c \in \mathfrak{g}$, $a \bullet (b \bullet c) - (a \bullet b) \bullet c = a \bullet (c \bullet b) - (a \bullet c) \bullet b$. The Lie bracket of $\mathfrak{g}$ is the antisymmetrization of $\bullet$.
3. For all $a, b, c \in \mathfrak{g}$, $(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b)$.
4. For all $a, b \in T(V)$, $\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \sqcup b^{(2)}$, with Sweedler's notations.

Our aim in this text is to give a generalization of the construction of $\mathfrak{g}$ and its relative H and G , and to study some general properties of this construction. Let us take any linear endomorphism f of a vector space V . We inductively define a pre-Lie product $\bullet$ on the shuffle Hopf algebra $(T(V), \sqcup, \Delta)$, making it a Com-PreLie Hopf algebra denoted by $T(V, f)$ (Definition 2.1 and Theorem 2.2). For example, if $x_1, x_2, x_3 \in V$ and $w \in T(V)$, then

$$\begin{aligned} x_1 \bullet w &= f(x_1)w, \\ x_1 x_2 \bullet w &= x_1 f(x_2)w + f(x_1)(x_2 \sqcup w), \\ x_1 x_2 x_3 \bullet w &= x_1 x_2 f(x_3)w + x_1 f(x_2)(x_3 \sqcup w) + f(x_1)(x_2 x_3 \sqcup w). \end{aligned}$$

Here are some examples:

1. Take V be one-dimensional and $f = \text{Id}$. Taking a basis (x) of V , we obtain a pre-Lie product on $\mathbb{K}[x]$ given by

$$x^i \bullet x^j = \binom{i+j}{i-1} x^{i+j}.$$

We shall prove that $\mathbb{K}[x]$ admits a basis $(y_n)_{n \geq 1}$ such that for all $i, j \geq 0$,

$$[y_i, y_j] = (i - j) y_{i+j}.$$

Hence, $\mathbb{K}[x]_+ = \text{Vect}(y_i : i \geq 1)$ is isomorphic, as a Lie algebra, to the Faà di Bruno Lie algebra [7–9]. This case is studied in Sect. 4.1.

2. Take $V = \text{Vect}(x_0, x_1)$ and $f : V \rightarrow V$, with matrix in the basis (x_0, x_1) given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The Com-PreLie Hopf algebra $T(V, f)$ is precisely the one obtained from the group of Fliess operators in dimension 1. In dimension $n \geq 2$, the group of Fliess operators admits a decomposition into a direct product of n subgroups $G_1, \dots, G_n$. The group G_i is obtained from the Com-PreLie Hopf algebra $T(V_i, f_i)$, where $V_i = \text{Vect}(x_0, \dots, x_n)$, and f_i is defined by $f_i(v_j) = \delta_{ij}v_0$. These cases are studied in Sect. 4.2.

For any linear endomorphism f , the Com-PreLie Hopf algebra $T(V, f)$ is graded by the length of words. We study the existence of other gradations of $T(V)$ as a pre-Lie algebra in Proposition 2.11. This holds for the pre-Lie algebra of Fliess operators. In this case, the dimension of the homogeneous parts of the gradation are given by the Fibonacci polynomials, Theorem 4.8, and consequently $K\langle x_1, \dots, x_n \rangle$ is a minimal space of generators of $T(V_i, f_i)$. In the one-dimensional case, we get in this way a result which was already proved in [10], where a presentation of the pre-Lie algebra of Fliess operators in dimension one is described.

According to [18, 19], we can give a description of the enveloping algebra $\mathcal{U}(T(V, f))$ in Theorems 2.13 and 2.14. We describe the dual Hopf algebra in Theorem 2.16 and its group of characters in Theorem 2.17; this group is in a certain sense the exponentiation of the Lie algebra $T(V, f)$. Because of the usual problems of duality of an infinite-dimensional Hopf algebra, we have to restrict here to the case where f is locally nilpotent. This holds in the Fliess operators case, and we indeed recover the group of Fliess operators in this way.

Using a generic case, we give systems of generators of $T(V, f)$ as a pre-Lie algebra and as a Com-PreLie algebra. We also study the pre-Lie and Com-PreLie subalgebra of $T(V, f)$ generated by a subspace W in the next section. We use for this a description of free Com-PreLie algebras in terms of partitioned trees [10]. First, the generic case proves that the pre-Lie subalgebra generated by V can be free, Theorem 3.18 and its Corollary 3.20; we give in Theorem 3.21 and Corollary 3.22 equivalent conditions for $T_+(V, f)$ to be generated as a pre-Lie or as a Com-PreLie algebra by a subspace of V in terms of the codimension of $\text{Im } f$. Applied to the Faà di Bruno case, this gives by duality an injection of the Faà di Bruno Hopf algebra into the Connes–Kreimer Hopf algebra of rooted trees, and we get in this way the subalgebra of formal diffeomorphisms defined in [3, 4].

This paper is organized in the following way. The first section groups all the algebraic structures associated to the linear endomorphism V : the Com-PreLie algebra $T(V, f)$, its enveloping algebra, the dual Hopf algebra and the associated group of characters. We also study the gradations of $T(V, f)$ here. Generation by V and morphisms from free Com-PreLie algebras to $T(V, f)$ are studied in the second section; it uses the notion of admissible words, whose study is reported to the appendix. The last section deals with the two main examples, the diagonalizable case, which leads to the Faà di Bruno Lie algebra, and the Fliess operators cases.

Notation We denote by $\mathbb{K}$ a commutative field of characteristic zero. All the objects (algebra, coalgebras, pre-Lie algebras, etc.) in this text will be taken over $\mathbb{K}$.

2 Algebraic structures associated to a linear endomorphism

Notation 1. Let V be a $\mathbb{K}$ -vector field. We denote by $T(V)$ the free associative, unitary algebra generated over $\mathbb{K}$ by V :

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}.$$

As a vector space, it is generated by the set of words with letters in V ; the unit is the empty word $\emptyset$ and the product is the concatenation of words. The augmentation ideal of $T(V)$ is

$$T_+(V) = \bigoplus_{n \geq 1} V^{\otimes n}.$$

2. We denote by $\text{Sh}(k, l)$ the set of (k, l) -shuffles, that is to say permutations $\sigma \in \mathfrak{S}_{k+l}$ such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+l)$.
3. $T(V)$ is a commutative dendriform algebra (or Zinbiel algebra [5, 6, 13, 14, 16]), with the half-shuffle product $<$: if $k, l \geq 1$ and $x_1, \dots, x_k, y_1, \dots, y_l \in V$, then

$$x_1 \dots x_k < y_1 \dots y_l = \sum_{\substack{\sigma \in \text{Sh}(k, l) \\ \sigma^{-1}(1)=1}} \sigma \cdot x_1 \dots x_k y_1 \dots y_l;$$

the n -th symmetric group acts on words of length n is by permutation of the letters. By convention, if w is a nonempty word, $\emptyset < w = 0$ and $w < \emptyset = w$; note that $\emptyset < \emptyset$ is not defined.

4. The induced commutative, associative product is the usual shuffle product $\sqcup$ [21]:

$$\begin{aligned} x_1 \dots x_k \sqcup y_1 \dots y_l &= x_1 \dots x_k < y_1 \dots y_l + y_1 \dots y_l < x_1 \dots x_k \\ &= \sum_{\sigma \in \text{Sh}(k, l)} \sigma \cdot x_1 \dots x_k y_1 \dots y_l. \end{aligned}$$

For all $x, y \in V$ and $u, v \in T(V)$,

$$xu < v = x(u \sqcup v), \quad xu \sqcup yv = x(u \sqcup yv) + y(xu \sqcup v).$$

Examples If $a, b, c, d \in V$, then

$$\begin{aligned} a < bcd &= abcd, & a \sqcup bcd &= abcd + bacd + bcad + bcda, \\ ab < cd &= abcd + acbd + acdb, & ab \sqcup cd &= abcd + acbd + acdb + cabd \\ & & & \quad + cadb + cdab, \\ abc < d &= abcd + abdc + adbc, & abc \sqcup d &= abcd + abdc + adbc + dabc. \end{aligned}$$

2.1 Construction of $T(V, f)$

We now fix a vector space V and a linear endomorphism $f: V \rightarrow V$.

Definition 2.1 We define a bilinear product $\bullet$ on $T(V)$ by induction on the length of words in the following way: for all $x \in V$ and $w, w' \in T(V)$, put

$$\begin{cases} \emptyset \bullet w' = 0, \\ xw \bullet w' = x(w \bullet w') + f(x)(w \sqcup w'). \end{cases}$$

Examples If $x_1, x_2, x_3 \in V$ and $w \in T(V)$, then

$$\begin{aligned} x_1 \bullet w &= f(x_1)w, \\ x_1 x_2 \bullet w &= x_1 f(x_2)w + f(x_1)(x_2 \sqcup w), \\ x_1 x_2 x_3 \bullet w &= x_1 x_2 f(x_3)w + x_1 f(x_2)(x_3 \sqcup w) + f(x_1)(x_2 x_3 \sqcup w). \end{aligned}$$

Theorem 2.2 1. For all $a, b, c \in T(V)$, $(a \sqcup b) \bullet c = (a \bullet c) \sqcup b + a \sqcup (b \bullet c)$.
 2. For all words $a, b, c \in A$ with $(a, b) \neq (\emptyset, \emptyset)$, $(a \prec b) \bullet c = (a \bullet c) \prec b + a \prec (b \bullet c)$.
 3. For all $a, b, c \in T(V)$, $(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b)$.
 4. For all $a, b \in T(V)$, $\Delta(a \bullet b) = a^{(1)} \otimes a^{(2)} \bullet b + a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \sqcup b^{(2)}$.

By points 1 and 3, $(T(V), \sqcup, \bullet)$ is a Com-PreLie algebra [17]. This structure is denoted by $T(V, f)$.

Proof 1. We assume that a, b, c are words. If $a = \emptyset$, then

$$(\emptyset \sqcup b) \bullet c = b \bullet c = (\emptyset \bullet c) \sqcup b + \emptyset \sqcup (b \bullet c).$$

If $b = \emptyset$, then $(a \sqcup \emptyset) \bullet c = a \bullet c = (a \bullet c) \sqcup \emptyset + a \sqcup (\emptyset \bullet c)$. We now assume that $a \neq \emptyset$ and $b \neq \emptyset$. We prove the result by induction on $n = \lg(a) + \lg(b)$. This is obvious if $n \leq 1$. Let us assume the result at rank $n - 1$. We put $a = xu$ and $b = yv$. Then

$$\begin{aligned} (a \sqcup b) \bullet c &= (x(u \sqcup yv)) \bullet c + (y(xu \sqcup v)) \bullet c \\ &= x((u \sqcup yv) \bullet c) + f(x)(u \sqcup yv \sqcup c) + y((xu \sqcup v) \bullet c) \\ &\quad + f(y)(xu \sqcup v \sqcup c) \\ &= x((u \bullet c) \sqcup yv) + x(u \sqcup (yv \bullet c)) + y((xu \bullet c) \sqcup b) \\ &\quad + y(xu \sqcup (v \bullet c)) + f(x)(u \sqcup yv \sqcup c) \\ &\quad + f(y)(xu \sqcup v \sqcup c) \\ &= x((u \bullet c) \sqcup yv) + x(u \sqcup y(v \bullet c)) + x(u \sqcup f(y)(v \sqcup c)) \\ &\quad + y(x(u \bullet c) \sqcup b) + y(f(x)(u \sqcup c) \sqcup b) \\ &\quad + y(xu \sqcup (v \bullet c)) + f(x)(u \sqcup yv \sqcup c) \\ &\quad + f(y)(xu \sqcup v \sqcup c) \\ &= x((u \bullet c) \sqcup yv) + y(x(u \bullet c) \sqcup b) + x(u \sqcup y(v \bullet c)) \end{aligned}$$

$$\begin{aligned}
& + y(xu \sqcup (v \bullet c)) + x(u \sqcup f(y)(v \sqcup c)) \\
& + f(y)(xu \sqcup v \sqcup c) + y(f(x)(u \sqcup c) \sqcup b) \\
& \quad + f(x)(u \sqcup yv \sqcup c) \\
& = (x(u \bullet c)) \sqcup yv + xu \sqcup y(v \bullet c) + xu \sqcup f(y)(v \sqcup c) \\
& \quad + f(x)(u \sqcup c) \sqcup yv \\
& = ((xu) \bullet c) \sqcup yv + xu \sqcup (yv \bullet c) \\
& = (a \bullet c) \sqcup b + a \sqcup (b \bullet c).
\end{aligned}$$

2. If $a = \emptyset$, $(\emptyset \prec b) \bullet c = 0 = (\emptyset \bullet b) \prec c + \emptyset \prec (b \bullet c)$. If $a = xu$, then

$$\begin{aligned}
(a \prec b) \bullet c &= (x(u \sqcup b)) \bullet c \\
&= x((u \sqcup b) \bullet c) + f(x)(u \sqcup b \sqcup c) \\
&= x((u \bullet c) \sqcup b) + x(u \sqcup (b \bullet c)) + f(x)(u \sqcup b \sqcup c) \\
&= x((u \bullet c) \sqcup b) + f(x)(u \sqcup c \sqcup b) + x(u \sqcup (b \bullet c)) \\
&= (x(u \bullet c) + f(x)(u \sqcup c)) \prec b + x(u \sqcup (b \bullet c)) \\
&= (xu \bullet c) \prec b + xu \prec (b \bullet c) \\
&= (a \bullet c) \prec b + a \prec (b \bullet c).
\end{aligned}$$

3. We assume that a, b, c are words and we proceed by induction on $\lg(a)$. If $a = \emptyset$, then

$$(\emptyset \bullet b) \bullet c - \emptyset \bullet (b \bullet c) = 0.$$

If $a = xu$, then

$$\begin{aligned}
(a \bullet b) \bullet c - a \bullet (b \bullet c) &= (x(u \bullet b)) \bullet c + (f(x)(u \sqcup b)) \bullet c - x(u \bullet (b \bullet c)) \\
&\quad - f(x)(u \sqcup (b \bullet c)) \\
&= x((u \bullet b) \bullet c) + f(x)((u \bullet b) \sqcup c) + f(x)((u \sqcup b) \bullet c) \\
&\quad + f^2(x)(u \sqcup b \sqcup c) - x(u \bullet (b \bullet c)) \\
&\quad - f(x)(u \sqcup (b \bullet c)) \\
&= x((u \bullet b) \bullet c - u \bullet (b \bullet c)) + f(x)((u \bullet b) \sqcup c + (u \bullet c) \sqcup b) \\
&\quad + f^2(x)(u \sqcup b \sqcup c).
\end{aligned}$$

The second and third terms are obviously symmetric in b and c . By the induction hypothesis applied to u , the first term is also symmetric in b and c , so the pre-Lie relation is satisfied for a, b, c .

4. Let us assume that a and b are words. We proceed by induction on the length of a . If $a = \emptyset$, then

$$\Delta(\emptyset \bullet b) = \emptyset \otimes \emptyset \bullet b + \emptyset \bullet b^{(1)} \otimes \emptyset \sqcup b^{(2)} = 0.$$

If the length of a is ≥ 1 , we put $a = xu$. Applying the induction hypothesis to u , we have

$$\begin{aligned}\Delta(a \bullet b) &= \Delta(x(u \bullet b) + f(x)(u \sqcup b)) \\ &= xu^{(1)} \otimes u^{(2)} \bullet b + x(u^{(1)} \bullet b) \otimes u^{(2)} \sqcup b^{(2)} + \emptyset \otimes x(u \bullet b) \\ &\quad + f(x)(u^{(1)} \sqcup b^{(2)}) \otimes u^{(2)} \sqcup b^{(2)} + \emptyset \otimes f(x)(u \sqcup b) \\ &= xu^{(1)} \otimes u^{(2)} \bullet b + (xu^{(1)}) \bullet b^{(1)} \otimes u^{(2)} \sqcup b^{(2)} + \emptyset \otimes (xu) \bullet b; \\ \\ a^{(1)} \otimes a^{(2)} \bullet b &= xu^{(1)} \otimes u^{(2)} \bullet b + \emptyset \otimes (xu) \bullet b, \\ a^{(1)} \bullet b^{(1)} \otimes a^{(2)} \sqcup b^{(2)} &= (xu^{(1)}) \bullet b^{(1)} \otimes u^{(2)} \sqcup b^{(2)} + \emptyset \bullet b^{(1)} \otimes (xu) \sqcup b \\ &= (xu^{(1)}) \bullet b^{(1)} \otimes u^{(2)} \sqcup b^{(2)}.\end{aligned}$$

Hence, the result holds for all words a and b . $\square$

Let us now give a closed formula for the pre-Lie product of two words.

Definition 2.3 Let $k, l \in \mathbb{N}$. For all $\sigma \in \mathfrak{S}_{k+l}$, we put

$$m_k(\sigma) = \max \{i \in \{1, \dots, k\} : \sigma(1) = 1, \dots, \sigma(i) = i\},$$

with the convention $m_k(\sigma) = 0$ if there is no $i \in \{1, \dots, k\}$ such that $\sigma(i) = i$.

Proposition 2.4 Let $k, l \in \mathbb{N}$ and let $x_1, \dots, x_{k+l} \in V$. Then

$$\begin{aligned}x_1 \dots x_k \bullet x_{k+1} \dots x_{k+l} \\ = \sum_{\sigma \in \text{Sh}(k, l)} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l-i)} (\sigma.x_1 \dots x_{k+l}).\end{aligned}$$

Proof By induction on k . If $k = 0$, then $x_1 \dots x_k = \emptyset$. For all $\sigma \in \text{Sh}(0, l)$, $m_0(\sigma) = 0$ by convention, so

$$\sum_{\sigma \in \text{Sh}(k, l)} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l-i)} (\sigma.x_1 \dots x_{k+l}) = 0 = \emptyset \bullet x_{k+1} \dots x_{k+l}.$$

Let us assume the result at rank $k - 1$.

$$\begin{aligned}x_1 \dots x_k \bullet x_{k+1} \dots x_{k+l} \\ = x_1(x_2 \dots x_k \bullet x_{k+1} \dots x_{k+l}) + f(x_1)(x_2 \dots x_k \sqcup x_{k+1} \dots x_{k+l}) \\ = \sum_{\tau \in \text{Sh}(k-1, l)} \sum_{i=1}^{m_{k-1}(\tau)} \text{Id} \otimes \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l)} (x_1 \tau.(x_2 \dots x_{k+l}))\end{aligned}$$

$$\begin{aligned}
& + \sum_{\tau \in \text{Sh}(k-1, l)} f \otimes \text{Id}^{k+l-1}(x_1 \tau.(x_2 \dots x_{k+l})) \\
& = \sum_{\substack{\sigma \in \text{Sh}(k, l) \\ \sigma(1)=1}} \sum_{i=2}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l)} (\sigma.(x_1 \dots x_{k+l})) \\
& \quad + \sum_{\substack{\sigma \in \text{Sh}(k, l) \\ \sigma(1)=1}} f \otimes \text{Id}^{k+l-1} (\sigma.(x_1 \dots x_{k+l})) \\
& = \sum_{\substack{\sigma \in \text{Sh}(k, l) \\ \sigma(1)=1}} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l)} (\sigma.(x_1 \dots x_{k+l})) \\
& = \sum_{\sigma \in \text{Sh}(k, l)} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(k+l)} (\sigma.(x_1 \dots x_{k+l})),
\end{aligned}$$

so the result holds for all k . $\square$

Remark 2.5 In particular, if $x_1, \dots, x_k \in V$, then

$$x_1 \dots x_k \bullet \emptyset = \sum_{i=1}^k x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_k.$$

The pre-Lie product $\bullet$ is generally nonassociative, as proved in the following proposition.

Proposition 2.6 *The following conditions are equivalent:*

1. $\bullet$ is trivial.
2. $\bullet$ is associative.
3. $f = 0$.

Proof $1 \Rightarrow 2$. Obvious.

$2 \Rightarrow 3$. Let $x \in V$. Then

$$(x \bullet \emptyset) \bullet \emptyset - x \bullet (\emptyset \bullet \emptyset) = f(x) \bullet \emptyset - 0 = f^2(x),$$

so $f^2 = 0$. Moreover,

$$\begin{aligned}
(xx \bullet x) \bullet \emptyset - xx \bullet (x \bullet \emptyset) &= (2f(x)xx + xf(x)x) \bullet \emptyset - xx \bullet f(x) \\
&= 2f^2(x)xx + 2f(x)f(x)x + 2f(x)xf(x) \\
&\quad + f(x)f(x)x + xf^2(x)x + xf(x)f(x) \\
&\quad - f(x)f(x)x - f(x)xf(x) - xf(x)f(x) \\
&= 2f(x)f(x)x + f(x)xf(x).
\end{aligned}$$

Let us assume that $f(x) \neq 0$. There exists $g \in V^*$ such that $g(f(x)) = 1$.

$$(g \otimes \text{Id} \otimes g)((xx \bullet x) \bullet \emptyset - xx \bullet (x \bullet \emptyset)) = 2f(x)g(x) + x = 0,$$

so there exists $\lambda \in \mathbb{K}$ such that $x = \lambda f(x)$. Then $f(x) = \lambda f^2(x) = 0$: this is a contradiction. Finally, $f(x) = 0$.

$3 \Rightarrow 1$. This is a direct consequence of Proposition 2.4. $\square$

Corollary 2.7 *We have*

$$T(V, f) \bullet T(V, f) = \bigoplus_{n \geq 1} \sum_{i=1}^n T(V, f)^{\otimes(i-1)} \otimes \text{Im } f \otimes T(V, f)^{\otimes(n-i)}.$$

Consequently, $T(V, f) \bullet T(V, f) \subseteq T_+(V, f)$.

Proof We put

$$W = \bigoplus_{n \geq 1} \sum_{i=1}^n T(V, f)^{\otimes(i-1)} \otimes \text{Im } f \otimes T(V, f)^{\otimes(n-i)}.$$

Proposition 2.4 directly implies that $T(V, f) \bullet T(V, f) \subseteq W$. Let

$$w = x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_n \in T(V, f)^{\otimes(i-1)} \otimes \text{Im } f \otimes T(V, f)^{\otimes(n-i)},$$

for $1 \leq i \leq n$. Let us prove that $w \in T(V, f) \bullet T(V, f)$ by induction on i . If $i = 1$, then

$$x_1 \bullet x_2 \dots x_n = f(x_1) x_2 \dots x_n + 0 \in T(V, f) \bullet T(V, f).$$

Let us assume the result at all ranks $j < i$. So

$$\begin{aligned} & x_1 \dots x_i \bullet x_{i+1} \dots x_n - x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_n \\ &= \sum_{\substack{\sigma \in \text{Sh}(i, n-i) \\ \sigma \neq \text{Id}}} \sum_{j=1}^{m_i(\sigma)} \text{Id}^{\otimes(j-1)} \otimes f \otimes \text{Id}^{\otimes(n-j)} (\sigma.x_1 \dots x_n) \\ &\quad + \sum_{j=1}^i \text{Id}^{\otimes(j-1)} \otimes f \otimes \text{Id}^{\otimes(n-j)} (x_1 \dots x_n) - x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_n \\ &= \sum_{\substack{\sigma \in \text{Sh}(i, n-i) \\ \sigma \neq \text{Id}}} \sum_{j=1}^{m_i(\sigma)} \text{Id}^{\otimes(j-1)} \otimes f \otimes \text{Id}^{\otimes(n-j)} (\sigma.x_1 \dots x_n) \\ &\quad + \sum_{j=1}^{i-1} \text{Id}^{\otimes(j-1)} \otimes f \otimes \text{Id}^{\otimes(n-j)} (\sigma.x_1 \dots x_n). \end{aligned}$$

If $\sigma \in \text{Sh}(i, n-i)$ is different from the identity of $\mathfrak{S}_n$, then $m_i(\sigma) < i$. By the induction hypothesis, $x_1 \dots x_i \bullet x_{i+1} \dots x_n - x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_n \in T(V, f) \bullet T(V, f)$, so $x_1 \dots x_{i-1} f(x_i) x_{i+1} \dots x_n \in T(V, f) \bullet T(V, f)$. Finally, $W = T(V, f) \bullet T(V, f)$. $\square$

Notice that $\mathbb{K}\emptyset$ is a trivial pre-Lie subalgebra of $T(V, f)$, and $T_+(V, f)$ is a pre-Lie ideal of $T(V, f)$. Hence

Proposition 2.8 *As a Lie algebra, $T(V, f) = T_+(V, f) \rtimes \mathbb{K}\emptyset$. The right action of $\emptyset$ on $T(V, f)$ is given by*

$$x_1 \dots x_k \bullet \emptyset = \sum_{i=1}^k x_1 \dots f(x_i) \dots x_k.$$

Proof It remains to compute the bracket $[x_1 \dots x_k, \emptyset]$. First

$$[x_1 \dots x_k, \emptyset] = x_1 \dots x_k \bullet \emptyset - \emptyset \bullet x_1 \dots x_k = x_1 \dots x_k \bullet \emptyset.$$

The result comes then from Proposition 2.4. $\square$

2.2 Functoriality

Lemma 2.9 *Let W be a subspace of V . We put*

$$I = \bigoplus_{n=1}^{\infty} \sum_{i=1}^n V_1^{\otimes(i-1)} \otimes W \otimes V_1^{\otimes(n-i)}.$$

Then I is a Com-PreLie ideal of $T(V, f)$ if and only if $f(W) \subseteq W$. Moreover, if it is the case and if f is surjective, then I is generated by W as a Com-PreLie ideal.

Proof I is clearly an ideal for the shuffle product. Let us assume that it is an ideal for $\bullet$. For all $w \in W$, $w \in I$, so $w \bullet \emptyset = f(w) \in I \cap V = W$: W is stable under f . Conversely, if $f(W) \subseteq W$, by Proposition 2.4, I is an ideal for $\bullet$.

Let us assume that f is surjective. We denote by J the Com-PreLie ideal of $T(V, f)$ generated by W . As I is a Com-PreLie ideal of $T(V, f)$ which contains W , $J \subseteq I$. Let $w = x_1 \dots x_n \in V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}$, and let us prove that $w \in J$ by induction on n . If $n = 1$, then $w \in W \subseteq J$. Let us assume the result at rank $n-1$. If $i \geq 2$, then $x_2 \dots x_n \in J$. As f is surjective, there exists $y_1 \in V$ such that $f_1(y_1) = x_1$. Then

$$y_1 \bullet x_2 \dots x_n = f(y_1) x_2 \dots x_n = w \in J.$$

If $i = 1$, then $x_1 \in J$ and

$$x_1 \sqcup x_2 \dots x_n = x_1 \dots x_n + \sum_{i=2}^n x_2 \dots x_i x_1 x_{i+1} \dots x_n \in J.$$

We already proved that $x_2 \dots x_i x_1 x_{i+1} \dots x_n \in J$ for all $2 \leq i \leq n$, so $w \in J$. As a conclusion, $I \subseteq J$. $\square$

Proposition 2.10 *Let V_1, V_2 be two vector spaces and $f_1: V_1 \rightarrow V_1, f_2: V_2 \rightarrow V_2$ be two endomorphisms. If $f: V_1 \rightarrow V_2$ satisfies $f \circ f_1 = f_2 \circ f$, then the following map is a morphism of Com-PreLie algebras:*

$$F: \begin{cases} T(V_1) \rightarrow T(V_2) \\ x_1 \dots x_n \rightarrow f(x_1) \dots f(x_n). \end{cases}$$

Moreover, the image of this morphism is $T(\text{Im } f, (f_2)_{|\text{Im } f})$ and its kernel is

$$\text{Ker } F = \bigoplus_{n=1}^{\infty} \sum_{i=1}^n V_1^{\otimes(i-1)} \otimes \text{Ker } f \otimes V_1^{\otimes(n-i)}.$$

If f_1 is surjective, $\text{Ker } F$ is generated, as a Com-PreLie ideal, by $\text{Ker } f$.

Proof The map F is clearly an algebra morphism from $(T(V_1), \sqcup)$ to $(T(V_2), \sqcup)$. Let $x_1, \dots, x_{k+l} \in V_1$. By Proposition 2.4,

$$\begin{aligned} & F(x_1 \dots x_k \bullet x_{k+1} \dots x_{k+l}) \\ &= \sum_{\sigma \in \text{Sh}(k,l)} \sum_{i=1}^{m_k(\sigma)} f^{\otimes(i-1)} \otimes (f \circ f_1) \otimes f^{\otimes(k+l-i)} (\sigma.x_1 \dots x_{k+l}) \\ &= \sum_{\sigma \in \text{Sh}(k,l)} \sum_{i=1}^{m_k(\sigma)} f^{\otimes(i-1)} \otimes (f_2 \circ f) \otimes f^{\otimes(k+l-i)} (\sigma.x_1 \dots x_{k+l}) \\ &= \sum_{\sigma \in \text{Sh}(k,l)} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f_2 \otimes \text{Id}^{\otimes(k+l-i)} (\sigma.f(x_1) \dots f(x_{k+l})) \\ &= F(x_1 \dots x_k) \bullet F(x_{k+1} \dots x_{k+l}). \end{aligned}$$

So F is a pre-Lie algebra morphism. By Lemma 2.9, if f_1 is surjective, then $\text{Ker } F$ is generated by $\text{Ker } f$. $\square$

2.3 Gradations of $T(V, f)$

By Proposition 2.4, the Com-PreLie algebra $T(V, f)$ is graded by the length of words. This is not a connected gradation of the pre-Lie algebra $T(V, f)$, as $\emptyset$ is homogeneous of degree 0. Let us now define other gradations, when V itself is graded.

Proposition 2.11 *Let us assume that V admits a gradation $(V_n)_{n \geq 0}$ such that f is homogeneous of degree $N \in \mathbb{N}$. For all $n \geq 0$, we put*

$$T(V, f)_n = \bigoplus_{k=1}^{n-N} \bigoplus_{i_1 + \dots + i_k = n-N} V_{i_1} \otimes \dots \otimes V_{i_k}.$$

Then $(T(V, f)_n)_{n \geq 0}$ is a gradation of the pre-Lie algebra $T(V, f)$.

Proof Let $u = x_1 \dots x_k$ and $v = y_1 \dots y_l$ be two words of $T(V, f)$ with homogeneous letters. Then u is homogeneous of degree $\deg u = \deg x_1 + \dots + \deg x_k + N$ and v is homogeneous of degree $\deg v = \deg y_1 + \dots + \deg y_l + N$. Let us prove that $u \bullet v$ is homogeneous of degree $\deg u + \deg v$. We proceed by induction on k . If $k = 0$, then $u = \emptyset$, $u \bullet v = 0$ and the result is obvious. If $k \geq 1$, then

$$u \bullet v = x_1(x_2 \dots x_k \bullet v) + f(x_1)(x_2 \dots x_k \sqcup v).$$

By the induction hypothesis, $x_1 \dots x_k \bullet v$ is homogeneous of degree $\deg x_2 + \dots + \deg x_k + N + \deg v$, so $x_1(x_2 \dots x_k \bullet v)$ is homogeneous of degree $\deg x_1 + \dots + \deg x_k + N + \deg v = \deg u + \deg v$. Moreover, $f(x_1)$ is homogeneous of degree $\deg x_1 + N$ and $x_2 \dots x_k \sqcup v$ is homogeneous of degree $\deg x_2 + \dots + \deg x_k + \deg y_1 + \dots + \deg y_l + N$, so $f(x_1)(x_2 \dots x_k \sqcup v)$ is homogeneous of degree $\deg x_1 + N + \deg x_2 + \dots + \deg x_k + \deg y_1 + \dots + \deg y_l + N = \deg u + \deg v$. Finally, $u \bullet v$ is homogeneous of degree $\deg u + \deg v$. $\square$

Under the hypothesis of Proposition 2.11, $T(V, f)$ is a bigraded pre-Lie algebra, with

$$T(V, f)_{nk} = \bigoplus_{i_1 + \dots + i_k = n-N} V_{i_1} \otimes \dots \otimes V_{i_k}, \quad k, n \geq 0.$$

If the gradation of V is finite-dimensional, then the bigradation of $T(V, f)$ is finite-dimensional. We put

$$F_V(X) = \sum_{n=0}^{\infty} \dim V_n X^n, \quad F_{T(V, f)}(X, Y) = \sum_{n, k \geq 0} \dim T(V, f)_{nk} X^n Y^k.$$

We obtain

Corollary 2.12 1. *Under the hypothesis of Proposition 2.11,*

$$F_{T(V, f)}(X, Y) = \frac{X^N}{1 - F_V(X)Y}.$$

2. *Under the hypothesis of Proposition 2.11, if for all $i \geq 0$, V_i is finite-dimensional and $V_0 = (0)$, then for all $n \geq 0$, $T(V, f)_n$ is finite-dimensional and*

$$\sum_{n \geq 0} \dim T(V, f)_n X^n = \frac{X^N}{1 - F_V(X)}.$$

Moreover, if $N \geq 1$, then $T(V, f)_0 = (0)$.

Proof 1. Indeed, putting $p_i = \dim V_i$ for all i , we have

$$\begin{aligned} F_{T(V, f)}(X, Y) &= \sum_{k, n \geq 0} \sum_{i_1 + \dots + i_k = n - N} p_{i_1} \dots p_{i_k} X^n Y^k \\ &= \sum_{k \geq 0} \sum_{i_1 \dots i_k} p_{i_1} \dots p_{i_k} X^{i_1 + \dots + i_k + N} Y^k \\ &= X^N \sum_{k \geq 0} \sum_{i_1 \dots i_k} p_{i_1} X^{i_1} Y \dots p_{i_k} X^{i_k} Y = \frac{X^N}{1 - F_V(X)Y}. \end{aligned}$$

2. Take $Y = 1$ in the first point. □

2.4 Enveloping algebra of $T(V, f)$

We use here the Oudom–Guin construction of the enveloping algebra on $S(T(V, f))$ [18, 19]. In order to avoid confusions between the product of $S(V)$ and the concatenation of words, we shall denote by $\times$ the product of $S(T(V, f))$ and by 1 its unit. We extend the pre-Lie product into a product from $S(T(V, f)) \otimes S(T(V, f))$ into $S(T(V, f))$ in the following way:

1. If $u_1, \dots, u_k \in T(V, f)$, $(u_1 \times \dots \times u_k) \bullet 1 = u_1 \times \dots \times u_k$.
2. If $u, u_1, \dots, u_k \in T(V, f)$,

$$u \bullet (u_1 \times \dots \times u_k) = (u \bullet (u_1 \times \dots \times u_{k-1})) \bullet u_k - u \bullet ((u_1 \times \dots \times u_{k-1}) \bullet u_k).$$

3. If $x, y, z \in S(T(V, f))$, $(x \times y) \bullet z = (x \bullet z^{(1)}) \times (y \bullet z^{(2)})$, where $\Delta(z) = z^{(1)} \otimes z^{(2)}$ is the usual coproduct of $S(T(V, f))$, defined by

$$\Delta(u_1 \times \dots \times u_k) = (u_1 \otimes 1 + 1 \otimes u_1) \times \dots \times (u_k \otimes 1 + 1 \otimes u_k),$$

for all $u_1, \dots, u_k \in T(V, f)$.

In particular, if $u_1, \dots, u_k, u \in T(V, f)$, then

$$(u_1 \times \dots \times u_k) \bullet u = \sum_{i=1}^k u_1 \times \dots \times (u_i \bullet u) \times \dots \times u_k.$$

Denoting by $\star$ the associative product induced on $S(T(V, f))$ by $\bullet$, for all $x, y \in S(T(V, f))$, we have

$$x \star y = (x \bullet y^{(1)}) \times y^{(2)}.$$

Notation Let $w_1, \dots, w_k \in T(V, f)$. For all $I = \{i_1, \dots, i_p\} \subseteq \{1, \dots, k\}$, we put

$$w_I = w_{i_1} \times \dots \times w_{i_p}, \quad w_I^{\sqcup} = w_{i_1} \sqcup \dots \sqcup w_{i_p}.$$

With these notations, the coproduct of $S(T(V, f))$ is given by

$$\Delta(w_1 \times \cdots \times w_k) = \sum_{I \subseteq \{1, \dots, k\}} w_I \otimes w_{\bar{I}}.$$

Theorem 2.13 *Let $w_1, \dots, w_k \in T(V, f)$, $k \geq 0$, $x \in V$, and $w \in T(V, f)$. Then*

1. *If $k \geq 1$, $\emptyset \bullet (w_1 \times \cdots \times w_k) = 0$.*
2. *$xw \bullet (w_1 \times \cdots \times w_k) = \sum_{I \subseteq \{1, \dots, k\}} f^{k-|I|}(x)((w \bullet w_I) \sqcup w_{\bar{I}}^{\sqcup})$.*
3. *$\emptyset \star (w_1 \times \cdots \times w_k) = \emptyset \times w_1 \times \cdots \times w_k$.*
4. *$xw \star (w_1 \times \cdots \times w_k) = \sum_{I \sqcup J \sqcup K = \{1, \dots, k\}} (f^{|J|}(x)((w \bullet w_I) \sqcup w_J^{\sqcup})) \times w_K$.*

Proof 1. We proceed by induction on k . If $k = 1$, $\emptyset \bullet w_1 = 0$. Let us assume the result at rank k_1 . Then

$$\begin{aligned} \emptyset \bullet w_1 \times \cdots \times w_k &= (\emptyset \bullet w_1 \times \cdots \times w_{k-1}) \bullet w_k \\ &\quad - \sum_{i=1}^{k-1} \emptyset \bullet (w_1 \times \cdots \times (w_i \bullet w_k) \times \cdots \times w_{k-1}). \end{aligned}$$

By the induction hypothesis, both terms of this sum are equal to zero.

2. If $k = 0$, then

$$\sum_{I \subseteq \emptyset} f^{0-|I|}(x)((w \bullet w_I) \sqcup w_{\bar{I}}^{\sqcup}) = x((w \bullet 1) \sqcup \emptyset) = xw = xw \bullet 1.$$

If $k = 1$, then

$$\begin{aligned} \sum_{I \subseteq \{1\}} f^{k-|I|}(x)((w \bullet w_I) \sqcup w_{\bar{I}}^{\sqcup}) &= f(x)((w \bullet 1) \sqcup w_1) + x((w \bullet w_1) \sqcup \emptyset) \\ &= f(x)(w \sqcup w_1) + x(w \bullet w_1) = xw \bullet w_1. \end{aligned}$$

Let us assume the result at rank $k - 1$. For all $I \subseteq \{1, \dots, k - 1\}$, we denote by $\tilde{I}$ its complement in $\{1, \dots, k - 1\}$ and by $\bar{I} = \tilde{I} \cup \{k\}$ its complement in $\{1, \dots, k\}$. Then

$$\begin{aligned} &xw \bullet (w_1 \times \cdots \times w_k) \\ &= (xw \bullet (w_1 \times \cdots \times w_{k-1})) \bullet w_k - \sum_{i=1}^{k-1} xw(w_1 \times \cdots \times (w_i \bullet w_k) \times \cdots \times w_{k-1}) \\ &= \sum_{I \subseteq \{1, \dots, k-1\}} (f^{k-1-|I|}(x)((w \bullet w_I) \sqcup w_{\bar{I}}^{\sqcup})) \bullet w_k \\ &\quad - \sum_{i=1}^{k-1} \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x)((w \bullet w'_I) \sqcup w_{\bar{I}}^{\sqcup}) \\ &= \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x)((w \bullet w_I) \bullet w_k) \sqcup w_{\bar{I}}^{\sqcup} \\ &\quad + \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x)((w \bullet w_I) \sqcup (w_{\bar{I}}^{\sqcup} \bullet w_k)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-|I|}(x) ((w \bullet w_I) \sqcup (w_I^{\sqcup} \sqcup w_k)) \\
& - \sum_{I \subseteq \{1, \dots, k-1\}} \sum_{i \in I} f^{k-1-|I|}(x) ((w \bullet w'_I) \sqcup w_I^{\sqcup}) \\
& - \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x) ((w \bullet w_I) \sqcup (w_I^{\sqcup} \bullet w_k)) \\
= & \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x) (((w \bullet w_I) \bullet w_k) \sqcup w_I^{\sqcup}) \\
& - \sum_{I \subseteq \{1, \dots, k-1\}} \sum_{i \in I} f^{k-1-|I|}(x) ((w \bullet w'_I) \sqcup w_I^{\sqcup}) \\
& + \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-|I|}(x) ((w \bullet w_I) \sqcup (w_I^{\sqcup} \sqcup w_k)) \\
= & \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-1-|I|}(x) ((w \bullet w_{I \cup \{k\}}) \sqcup w_I^{\sqcup}) \\
& + \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-|I|}(x) ((w \bullet w_I) \sqcup w_I^{\sqcup}) \\
= & \sum_{\substack{J \subseteq \{1, \dots, k\} \\ k \in I}} f^{k-|J|}(x) ((w \bullet w_J) \sqcup w_J^{\sqcup}) \\
& + \sum_{I \subseteq \{1, \dots, k-1\}} f^{k-|I|}(x) ((w \bullet w_I) \sqcup w_I^{\sqcup}) \\
= & \sum_{I \subseteq \{1, \dots, k\}} f^{k-|I|}(x) ((w \bullet w_I) \sqcup w_I^{\sqcup}),
\end{aligned}$$

where $w'_j = w_j$ if $j \neq i$ and $w'_i = w_i \bullet w_k$.

3 and 4. For all $u \in T(V, f)$,

$$u \star (w_1 \times \dots \times w_k) = \sum_{I \subseteq \{1, \dots, k\}} (u \bullet w_I) \times w_{\bar{I}}.$$

The results come then directly from points 1 and 2. $\square$

Theorem 2.14 *Let $x_1, \dots, x_i \in V$, $w_1, \dots, w_k \in T(V, f)$, $i \geq 1$, and $k \geq 0$. If $w = x_1 \dots x_i$, then*

$$\begin{aligned}
& w \bullet (w_1 \times \dots \times w_k) \\
= & \sum_{I_1 \sqcup \dots \sqcup I_i = \{1, \dots, k\}} f^{|I_1|}(x_1) (f^{|I_2|}(x_2) (\dots (f^{|I_i|}(x_i) w_{I_i}^{\sqcup}) \sqcup w_{I_{i-1}}^{\sqcup}) \dots) \sqcup w_{I_1}^{\sqcup}),
\end{aligned}$$

$$\begin{aligned}
& w \star (w_1 \times \dots \times w_k) \\
= & \sum_{I_1 \sqcup \dots \sqcup I_{i+1} = \{1, \dots, k\}} f^{|I_1|}(x_1) (f^{|I_2|}(x_2) (\dots (f^{|I_i|}(x_i) w_{I_i}^{\sqcup}) \sqcup w_{I_{i-1}}^{\sqcup}) \dots) \sqcup w_{I_1}^{\sqcup}) \times w_{I_{i+1}}.
\end{aligned}$$

Proof By induction on i . If $i = 1$, then

$$\begin{aligned} x_1 \bullet (w_1 \times \cdots \times w_k) &= \sum_{I \subseteq \{1, \dots, k\}} f^{k-|I|}(x_1) ((\emptyset \bullet w_I) \sqcup w_I^{\sqcup}) \\ &= f^k(x_1) ((\emptyset \bullet 1) \sqcup w_1 \sqcup \cdots \sqcup w_k) + 0, \end{aligned}$$

so the result holds if $k = 1$. Let us assume the result at rank $i - 1$. Then

$$x_1 \dots x_i \bullet (w_1 \times \cdots \times w_k) = \sum_{I_1 \sqcup J = \{1, \dots, k\}} f^{|I_1|}(x_1) ((x_2 \dots x_i \bullet w_J) \sqcup w_{I_1}^{\sqcup}).$$

The induction hypothesis applied to $x_2 \dots x_i \bullet w_J$ gives the result. $\square$

Remark 2.15 In particular, $x_1 \bullet (w_1 \times \cdots \times w_k) = f^k(x_1)(w_1 \sqcup \cdots \sqcup w_k)$.

2.5 Dual Hopf algebra

By duality, if f is locally nilpotent, the symmetric algebra $S(T(V, f))$, with its usual product $\times$, inherits a Hopf algebra structure. Let us describe its coproduct Δ , dual to the product $\star$ on $S(T(V^*, f^*))$.

Theorem 2.16 *Let us assume that f is locally nilpotent. We define a coproduct $\tilde{\Delta}$ on $S(T(V, f))$ in the following way:*

1. $\tilde{\Delta}(\emptyset) = \emptyset \otimes 1$.
2. If $x \in V$ and $u \in T(V, f)$, put

$$\tilde{\Delta}(xu) = \sum_{i \geq 0} (\theta_{f^i(x)} \otimes \text{Id}) \circ (\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}^{\otimes i}) \circ \Delta_{\sqcup}^{(i+1)}(u).$$

We used the iterated products $m^{(j)}: S(T(V, f))^{\otimes j} \rightarrow S(T(V, f))$ and the iterated coproducts $\Delta_{\sqcup}^{(j)}: T(V, f) \rightarrow T(V, f)^{\otimes j}$.

For all $w \in T(V, f)$, we put $\Delta(w) = \tilde{\Delta}(w) + 1 \otimes w$. With this coproduct, $S(T(V, f))$ is a Hopf algebra.

Proof We denote by g the transpose of f . As $S(T(V, f))$ is the dual of the enveloping algebra of a right pre-Lie algebra, for all $w \in T(V, f)$,

$$\Delta(w) - 1 \otimes w \in T(V, f) \otimes S(T(V, f)).$$

We let $\tilde{\Delta}(w) = \Delta(w) - 1 \otimes w$ for all $w \in T(V, f)$. Let $v, v_1, \dots, v_k$ be words in $T(V^*, f^*)$. If $v \neq \emptyset$, then $v \bullet v_1 \times \cdots \times v_k \in T_+(V^*, f^*)$, so

$$\begin{aligned} \langle v \otimes v_1 \times \cdots \times v_k, \tilde{\Delta}(\emptyset) \rangle &= \langle v \bullet (v_1 \times \cdots \times v_k), \emptyset \rangle \\ &= 0 = \langle v \otimes v_1 \times \cdots \times v_k, \emptyset \otimes 1 \rangle. \end{aligned}$$

If $v = \emptyset$, then

$$\begin{aligned}\langle \emptyset \otimes v_1 \times \cdots \times v_k, \tilde{\Delta}(\emptyset) \rangle &= \langle \emptyset \bullet (v_1 \times \cdots \times v_k), \emptyset \rangle \\ &= \delta_{k0} = \langle \emptyset \otimes v_1 \times \cdots \times v_k, \emptyset \otimes 1 \rangle.\end{aligned}$$

So $\tilde{\Delta}(\emptyset) = \emptyset \otimes 1$. Let $v_1, \dots, v_k \in T(V^*, f^*)$. Then

$$\begin{aligned}\langle \emptyset \otimes v_1 \times \cdots \times v_k, \tilde{\Delta}(xu) \rangle &= \langle \emptyset \bullet (v_1 \times \cdots \times v_k), xu \rangle = \delta_{k0} \langle \emptyset, xu \rangle = 0 \\ &= \left\langle \emptyset \otimes v_1 \times \cdots \times v_k, \sum_{i \geq 0} (\theta_{f^i(x)} \otimes \text{Id}) \circ (\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}^{\otimes i}) \circ \Delta_{\sqcup}^{(i+1)}(u) \right\rangle.\end{aligned}$$

Let $y \in V^*$ and $v, v_1, \dots, v_k \in T(V^*, f^*)$. Then

$$\begin{aligned}\langle yv \otimes v_1 \times \cdots \times v_k, \tilde{\Delta}(xu) \rangle &= \langle yv \bullet (v_1 \times \cdots \times v_k), xu \rangle \\ &= \sum_{I \subseteq \{1, \dots, k\}} \langle g^{k-|I|}(y), x \rangle \langle (v \bullet v_I) \sqcup v_{\bar{I}}, u \rangle \\ &= \sum_{I \subseteq \{1, \dots, k\}} \langle y, f^{k-|I|}(x) \rangle \left\langle v \otimes v_I \otimes \Delta^{(|\bar{I}|)}(w_{\bar{I}}), (\tilde{\Delta} \otimes \text{Id}^{\otimes |I|}) \circ \Delta_{\sqcup}^{(|\bar{I}|+1)}(u) \right\rangle \\ &= \sum_{i=0}^k \langle y, f^{k-|I|}(x) \rangle \left\langle v \otimes v_1 \times \cdots \times v_k, (\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}^{\otimes i}) \circ \Delta_{\sqcup}^{(i+1)}(u) \right\rangle \\ &= \sum_{i \geq 0} \left\langle uv \otimes v_1 \times \cdots \times v_k, (\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}^{\otimes i}) \circ \Delta_{\sqcup}^{(i+1)}(u) \right\rangle.\end{aligned}$$

This implies the announced result. $\square$

Examples Let $x_1, x_2 \in V$.

$$\begin{aligned}\tilde{\Delta}(\emptyset) &= \emptyset \otimes 1, \\ \tilde{\Delta}(x_1) &= \sum_{i \geq 0} f^i(x_1) \otimes \emptyset^{\times i}, \\ \tilde{\Delta}(x_1 x_2) &= \sum_{i, j \geq 0} f^i(x_1) f^j(x_2) \otimes \emptyset^{\times (i+j)} + \sum_{i \geq 0} i f^i(x_1) \otimes x_2 \times \emptyset^{\times (i-1)}.\end{aligned}$$

In particular, the second line shows that the condition of local nilpotence of f is necessary in order to obtain a coproduct with values in $S(T(V, f)) \otimes S(T(V, f))$.

2.6 Group of characters

Let us now describe the character group G associated to f . Any character of $S(T(V^*, f^*))$ is the extension of a linear map from $T(V^*, f^*)$ to $\mathbb{K}$, so, as a set,

we identify G and $\prod_{n=0}^{\infty} (V^{**})^{\otimes n}$. The composition of G is denoted by $\diamond$. We shall consider the following subset of G' :

$$\overline{T}(V) = \prod_{n=0}^{\infty} V^{\otimes n}.$$

We give G' and $\overline{T}(V)$ their usual ultrametric topology.

We shall prove that this is a subgroup of G . Note that $G = \overline{T}(V)$ if and only if V is finite-dimensional.

Theorem 2.17 *We inductively define a continuous composition $\tilde{\diamond}$ on $\overline{T}(V)$: for all $v \in \overline{T}(V)$,*

1. $\emptyset \tilde{\diamond} v = \emptyset$.
2. if $x \in V$ and $u \in G$, $xu \tilde{\diamond} v = \sum_{i \geq 0} f^{(i)}(x) ((u \tilde{\diamond} v) \sqcup v^{\sqcup i})$.

Then for all $u, v \in G$, $u \diamond v = u \tilde{\diamond} v + v$.

Proof We define a composition $\tilde{\triangleleft}$ on $\overline{T}(V)$ by $u \tilde{\triangleleft} v(w) = (u \otimes v) \circ \tilde{\Delta}(w)$ for all $w \in T(V, f)$. Then, for all $w \in T(V, f)$,

$$(u \diamond v)(w) = (u \otimes v) \circ \Delta(w) = (u \otimes v) \circ \tilde{\Delta}(w) + u(1)v(w) = (u \tilde{\triangleleft} v)(w) + v(w).$$

So $u \diamond v = u \tilde{\triangleleft} v + v$.

Let $w \in T(V, f)$. Then $(\emptyset \tilde{\triangleleft} v)(w) = (\emptyset \otimes v) \circ \tilde{\Delta}(w)$. If $w \neq \emptyset$, we observed that $\tilde{\Delta}(w) \in T_+(V, f) \otimes T(V, f)$, so $(\emptyset \otimes v) \circ \tilde{\Delta}(w) = 0$. Hence, $\emptyset \tilde{\triangleleft} v = \lambda \emptyset$ for a particular λ . If $w = \emptyset$, we obtain $(\emptyset \tilde{\triangleleft} v)(w) = \emptyset(\emptyset)v(1) = 1$, so $\lambda = 1$.

Let us compute $xu \tilde{\triangleleft} v$. First, $xu \tilde{\triangleleft} v(\emptyset) = (xu \otimes v)(\emptyset \otimes 1) = 0$. Let us take $y \in V^*$ and $w \in T(V^*, f^*)$. Then

$$\begin{aligned} xu \tilde{\triangleleft} v(yw) &= \sum_{i \geq 0} (xu \otimes v) \left((\theta_{g^i(y)} \otimes \text{Id}) \circ (\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}) \circ \Delta_{\sqcup}^{(i+1)}(w) \right) \\ &= \sum_{i \geq 0} g^i(y)(x)(u \otimes v) \left((\text{Id} \otimes m^{(i+1)}) \circ (\tilde{\Delta} \otimes \text{Id}) \circ \Delta_{\sqcup}^{(i+1)}(w) \right) \\ &= \sum_{i \geq 0} y(f^i(x))(u \otimes v^{\otimes(i+1)}) \left((\tilde{\Delta} \otimes \text{Id}) \circ \Delta_{\sqcup}^{(i+1)}(w) \right) \\ &= \sum_{i \geq 0} y(f^i(x))(u \tilde{\triangleleft} v \otimes v^{\otimes i}) \left(\Delta_{\sqcup}^{(i+1)}(w) \right) \\ &= \sum_{i \geq 0} y(f^i(x))((u \tilde{\triangleleft} v) \sqcup v^{\sqcup i})(w) \\ &= \sum_{i \geq 0} f^i(x)((u \tilde{\triangleleft} v) \sqcup v^{\sqcup i})(yw). \end{aligned}$$

So the composition $\tilde{\triangleleft}$ defined in this proof is the composition $\tilde{\diamond}$ defined in Theorem 2.17. Moreover, an easy induction on the length proves that for all words u and all $w \in \overline{T}(V)$, $u \tilde{\diamond} w \in \overline{T}(V)$. By linearity and continuity, $\overline{T}(V) \tilde{\diamond} \overline{T}(V) \subseteq \overline{T}(V)$. $\square$

Now, by induction on k we get

Corollary 2.18 *For all $x_1, \dots, x_k \in V$ and $v \in G$, we have*

$$\begin{aligned} & x_1 \dots x_k \diamond v \\ &= \sum_{i_1, \dots, i_k \geq 0} f^{i_1}(x_1)(f^{i_2}(x_2)(\dots(f^{i_k}(x_k)v^{\sqcup i_k}) \sqcup v^{\sqcup i_{k-1}}) \dots) \sqcup v^{\sqcup i_1}) + v. \end{aligned}$$

3 From free Com-PreLie algebras to $T(V, f)$

3.1 Free Com-PreLie algebras

Let us recall the construction of free Com-Prelie algebras [10].

Definition 3.1 1. A *partitioned forest* is a pair (F, I) such that:

- F is a rooted forest (the edges of F being oriented from the roots to the leaves).
- I is a partition of the vertices of F with the following condition: if x, y are two vertices of F which are in the same part of I , then either they are both roots, or they have the same direct ascendant.

The parts of the partition are called *blocks*.

2. We shall say that a partitioned forest is a *partitioned tree* if all the roots are in the same block.
3. Let $\mathcal{D}$ be a set. A *partitioned tree decorated by $\mathcal{D}$* is a pair (t, d) , where t is a partitioned tree and d is a map from the set of vertices of t into $\mathcal{D}$. For any vertex x of t , $d(x)$ is called the *decoration* of x .
4. The set of isoclasses of partitioned trees will be denoted by $\mathcal{PT}$. For any set $\mathcal{D}$, the set of isoclasses of partitioned trees decorated by $\mathcal{D}$ will be denoted by $\mathcal{PT}(\mathcal{D})$.

Examples We represent partitioned trees by the Hasse graph of the underlying rooted forest, the blocks of cardinality ≥ 2 being represented by horizontal edges of different colors. Here are the partitioned trees with ≤ 4 vertices:

$$\begin{aligned} & \bullet, \downarrow, \text{---}; \quad \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \text{---} = \text{---}, \text{---}; \quad \downarrow, \downarrow = \downarrow, \downarrow, \downarrow, \downarrow = \downarrow, \downarrow = \downarrow, \\ & \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \text{---} = \text{---}, \downarrow, \downarrow = \downarrow, \downarrow, \downarrow = \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \text{---} = \text{---} = \text{---}. \end{aligned}$$

Definition 3.2 Let $t = (t, I)$ and $t' = (t', J) \in \mathcal{PT}$.

1. Let s be a vertex of t' . The partitioned tree $t \bullet_s t'$ is defined as follows:
 - As a rooted forest, $t \bullet_s t'$ is obtained by grafting all the roots of t' on the vertex s of t .
 - We put $I = \{I_1, \dots, I_k\}$ and $J = \{J_1, \dots, J_l\}$. The partition of the vertices of this rooted forest is $I \sqcup J = \{I_1, \dots, I_k, J_1, \dots, J_l\}$.
2. The partitioned tree $t \sqcup t'$ is defined as follows:
 - As a rooted forest, $t \sqcup t'$ is tt' .

- We put $I = \{I_1, \dots, I_k\}$ and $J = \{J_1, \dots, J_l\}$ and we assume that the set of roots of t is I_1 and the set of roots of t' is J_1 . The partition of the vertices of $t \sqcup t'$ is $\{I_1 \sqcup J_1, I_2, \dots, I_k, J_1, \dots, J_l\}$.

Examples 1. Here are the three possible graftings $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet_s \bullet$, $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$, $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$ and $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$.

2. Here are the two possible graftings $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet_s \bullet$: $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$ and $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$.

3. $\begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \sqcup \bullet = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} \bullet$.

These operations are similarly defined for decorated partitioned trees.

The free nonunitary Com-PreLie algebra is described in [10].

Proposition 3.3 *Let $\mathcal{D}$ be a set. The free nonunitary Com-PreLie algebra $\text{ComPrelie}(\mathcal{D})$ generated by $\mathcal{D}$ is the vector space generated by $\mathcal{PT}(\mathcal{D})$. We extend $\sqcup$ by bilinearity and the pre-Lie product $\bullet$ is defined on in the following way: if $t, t' \in \mathcal{PT}(\mathcal{D})$, put*

$$t \bullet t' = \sum_{s \in V(t)} t \bullet_s t'.$$

It satisfies the following universal property: if A is a Com-PreLie algebra and $a_d \in \mathcal{D}$ for all $d \in \mathcal{D}$, there exists a unique Com-PreLie algebra morphism $\phi: \text{ComPrelie}(\mathcal{D}) \rightarrow A$ such that $\phi(\bullet_d) = a_d$ for all $d \in \mathcal{D}$.

Rooted trees are identified with partitioned rooted trees such that any block has cardinality one. The free pre-Lie algebra $\text{Prelie}(\mathcal{D})$, which is based on decorated rooted trees [2], is seen as a pre-Lie subalgebra of $\text{ComPrelie}(\mathcal{D})$.

3.2 A generic case

Definition 3.4 1. Let $\mathcal{D}$ be a set. The set of biletters in $\mathcal{D}$ is $\mathbb{N} \times \mathcal{D}$. A billetter will be denoted by $\binom{k}{d}$, with $k \in \mathbb{N}$ and $d \in \mathcal{D}$. A biword in $\mathcal{D}$ is a word in biletters in $\mathcal{D}$.

2. Let $V_{\mathcal{D}}$ be the space generated by the biletters in $\mathcal{D}$. We define a map $f_{\mathcal{D}}: V_{\mathcal{D}} \rightarrow V_{\mathcal{D}}$ by $f_{\mathcal{D}}(\binom{k}{d}) = \binom{k+1}{d}$ for all $k \geq 0$ and $d \in \mathcal{D}$.

3. The pre-Lie subalgebra $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ generated by the biletters $\binom{0}{d}$, $d \in \mathcal{D}$, is denoted by $g_{\mathcal{D}}$.

The pair $(V_{\mathcal{D}}, f_{\mathcal{D}})$ satisfies a universal property

Proposition 3.5 *Let V be a vector space, $g: V \rightarrow V$ a linear map, and for all $d \in \mathcal{D}$, $x_d \in V$. There exists a unique linear map $F: V_{\mathcal{D}} \rightarrow V$ such that $F \circ f_{\mathcal{D}} = g \circ F$ and $F(\binom{0}{d}) = x_d$ for all $d \in \mathcal{D}$.*

This map is $F(\binom{k}{d}) = g^k(x_d)$.

3.3 Generation of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$

We now assume that $\mathcal{D}$ is a totally ordered set. Consequently, the set of biletters $\binom{0}{d}$, $d \in \mathcal{D}$, is also totally ordered.

Proposition 3.6 1. *The pre-Lie algebra $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is generated by the set of words in biletters $\binom{0}{d}$, $d \in \mathcal{D}$. Moreover, this set is a minimal set of generators.*
 2. *The Com-PreLie algebra $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is generated by $\emptyset$ and the set of Lyndon words in biletters $\binom{0}{d}$, $d \in \mathcal{D}$. Moreover, this set is a minimal set of generators.*

Proof 1. We denote by W the subspace of $V_{\mathcal{D}}$ generated by the biletters $\binom{0}{d}$, $d \in \mathcal{D}$. Let A be the pre-Lie subalgebra of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ generated by $T(W)$. Let $w = x_1 \dots x_n$ be a word in biletters $\binom{i}{d}$, $i \geq 0$, $d \in \mathcal{D}$. If all its biletters are of the form $\binom{0}{d}$, then it belongs to $T(W) \subseteq A$. Let us now assume that it contains at least one billetter $\binom{i}{d}$, with $i \geq 1$. We prove that $w \in A$ by induction on n . If $n = 1$, we put $x_1 = \binom{i}{d}$, and

$$\binom{0}{d} \bullet (\emptyset^{\times i}) = \binom{i}{d} \in A.$$

So the result is true at rank 1. Let us assume it at all ranks $< n$, $n \geq 2$. Let k be the smallest integer such that $x_k = \binom{i}{d}$, with $i \geq 1$. We proceed by induction on k . If $k = 1$, then by the induction hypothesis on n , $x_2, \dots, x_n \in A$, and by Theorem 2.14,

$$\binom{0}{d} \bullet (\emptyset^{\times(i-1)} \times x_2 \times \dots \times x_n) = \binom{i}{d} x_2 \dots x_n = x_1 \dots x_n \in A.$$

Let us assume the result at all ranks $< k$, $k \geq 2$. We already know that $x_1 \dots x_{k-1} \binom{0}{d}$, and $x_{k+1} \dots x_n \in A$. By Theorem 2.14,

$$\begin{aligned} x_1 \dots x_{k-1} \binom{0}{d} \bullet (\emptyset^{\times(i-1)} \times x_{k+1} \times \dots \times x_n) \\ = x_1 \dots x_n + \text{words of length } n \text{ satisfying} \\ \text{the induction hypothesis on } k \in A. \end{aligned}$$

So $x_1 \dots x_n \in A$. Finally, $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is generated by $T(W)$ as a pre-Lie algebra.

As $\text{Im } f_{\mathcal{D}} \cap W = (0)$, by Corollary 2.7,

$$T(W) \oplus (T(V_{\mathcal{D}}, f_{\mathcal{D}}) \bullet T(V_{\mathcal{D}}, f_{\mathcal{D}})) = T(V_{\mathcal{D}}, f_{\mathcal{D}}).$$

So $T(W)$ is a minimal subspace of generators of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$.

2. Let W' be the subspace of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ generated by $\emptyset$ and the Lyndon words in biletters $\binom{0}{d}$. We denote by B the Com-PreLie subalgebra of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ generated by W . Then B contains the subalgebra generated by W' (for the product $\sqcup$), that is to say the shuffle algebra $T(W)$. By the first point, $B = T(V_{\mathcal{D}}, f_{\mathcal{D}})$.

Let us prove now the minimality of W' . First, Lyndon words are a minimal system of generators of the shuffle algebra $T(W)$. Moreover, $T(V_{\mathcal{D}}, f_{\mathcal{D}}) \bullet T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is contained in the set of words in biletters $\binom{i}{d}$, so

$$W' \cap ((T_+(V_{\mathcal{D}}, f_{\mathcal{D}}) \sqcup T_+(V_{\mathcal{D}}, f_{\mathcal{D}})) + (T(V_{\mathcal{D}}, f_{\mathcal{D}}) \bullet T(V_{\mathcal{D}}, f_{\mathcal{D}}))) = (0).$$

Consequently, W' is a minimal subspace of generators of $(T(V_{\mathcal{D}}, f_{\mathcal{D}}), \sqcup, \bullet)$. $\square$

Corollary 3.7 *Let V be a vector space and let $g: V \rightarrow V$. Let W be a subspace of V such that*

$$V = \sum_{n=0}^{\infty} g^n(W).$$

Then $T(V, g)$ is generated, as a pre-Lie algebra, by $T(W)$. We give W a totally ordered basis $(w_d)_{d \in \mathcal{D}}$. Then $T(V, g)$ is generated, as a Com-PreLie algebra, by $\emptyset$ and the set of Lyndon words in letters $w_d, d \in \mathcal{D}$.

Proof Let $f: V_{\mathcal{D}} \rightarrow V$ be the unique map such that $f \circ f_{\mathcal{D}} = g \circ f$, and let $F: T(V_{\mathcal{D}}, f_{\mathcal{D}}) \rightarrow T(V, g)$ be obtained by functoriality. We obtain

$$\text{Im } f = \sum_{n=0}^{\infty} g^n(W) = V,$$

so $\text{Im } F = T(\text{Im } f) = T(V, g)$. As $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is generated, as a pre-Lie algebra, by the words in biletters $\binom{0}{d}$, $T(V, g)$ is generated by their images, that is to say words in W .

As $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is generated, as a Com-PreLie algebra, by $\emptyset$ and the Lyndon words in biletters $\binom{0}{d}, d \in \mathcal{D}$, $T(V, g)$ is generated by their images, that is to say $\emptyset$ and the Lyndon words in letters $w_d, d \in \mathcal{D}$. $\square$

Remark 3.8 1. If W is 1-dimensional, that is to say if (V, g) is cyclic, then $T(V, g)$ is generated, as a Com-PreLie algebra, by $\emptyset$ and any nonzero element of W .

2. These sets of generators are not necessarily minimal. For example, if g is surjective, we shall prove in Lemma 4.6 that $\emptyset$ and W generate $T(V, g)$, even if V is not one-dimensional.

3.4 A morphism from partitioned trees to words in biletters

Our aim is now to prove that $\mathfrak{g}_{\mathcal{D}}$, the pre-Lie subalgebra of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ generated by the biletters $\binom{0}{d}, d \in \mathcal{D}$, is free. We shall use the following morphism.

Definition 3.9 We denote by Φ_{CPL} the Com-PreLie algebra morphism defined by

$$\Phi_{\text{CPL}}: \begin{cases} \text{ComPreLie}(\mathcal{D}) \rightarrow T(V_{\mathcal{D}}, f_{\mathcal{D}}) \\ \bullet_d \rightarrow \binom{0}{d}. \end{cases}$$

Its restriction to $\text{Prelie}(\mathcal{D})$ is denoted by Φ_{PL} . Note that $\mathfrak{g}_{\mathcal{D}} = \Phi_{\text{PL}}(\text{ComPrelie}(\mathcal{D}))$.

Examples Let $a, b, c, d \in \mathcal{D}$.

$$\begin{aligned}\Phi_{\text{CPL}}(\mathfrak{I}_a^b) &= \binom{10}{ab}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^b \xrightarrow{c}) &= \binom{100}{abc} + \binom{100}{acb} + \binom{010}{cab}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^b \vee_a^c) &= \binom{200}{abc} + \binom{200}{acb}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^c \mathfrak{I}_a^b) &= \binom{110}{abc}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^c \vee_a^d) &= \binom{3000}{abcd} + \binom{3000}{abdc} + \binom{3000}{acbd} + \binom{3000}{acdb} + \binom{3000}{adbc} + \binom{3000}{adcb}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^c \vee_a^d \mathfrak{I}_a^b) &= \binom{2100}{abcd} + \binom{2100}{abdc} + \binom{2010}{adbc}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^d \vee_a^c \mathfrak{I}_a^b) &= \binom{1200}{abcd} + \binom{1200}{abdc}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^c \mathfrak{I}_a^b \xrightarrow{d}) &= \binom{1010}{acbd} + \binom{1100}{abcd} + \binom{1100}{abdc} + \binom{1100}{bacd} + \binom{1100}{badc} + \binom{1010}{bdac}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^c \vee_a^d \mathfrak{I}_a^b) &= \binom{1100}{abcd} + \binom{1100}{abdc} + \binom{1010}{adbc}, \\ \Phi_{\text{CPL}}(\mathfrak{I}_a^d \mathfrak{I}_a^c \mathfrak{I}_a^b) &= \binom{1110}{abcd}.\end{aligned}$$

Remark 3.10 Φ_{CPL} is not injective: for any $d \in \mathcal{D}$, $\Phi_{\text{CPL}}(\mathfrak{I}_a^d \mathfrak{I}_a^d - 2\mathfrak{I}_a^d \vee_a^d \mathfrak{I}_a^d) = 0$. However, we will prove that the restriction Φ_{PL} is injective, Theorem 3.18.

By the universal property of $V_{\mathcal{D}}$ and the factoriability of the construction of $T(V, f)$ we have

Theorem 3.11 *Let V be a vector space, $f: V \rightarrow V$ a linear map and $x_d \in V$ for all $d \in \mathcal{D}$. The following map is a Com-PreLie algebra morphism:*

$$\varphi_{(V, f)}: \begin{cases} T(V_{\mathcal{D}}, f_{\mathcal{D}}) \rightarrow T(V, f) \\ \begin{pmatrix} a_1 \dots a_n \\ d_1 \dots d_n \end{pmatrix} \rightarrow f^{a_1}(x_{d_1}) \dots f^{a_n}(x_{d_n}). \end{cases}$$

Moreover, denoting by W the smallest subspace of V stable under f and containing all x_d , the image of $\varphi_{(V, f)}$ is $T(W)$.

Definition 3.12 Let t be a partitioned tree decorated by $\mathcal{D}$.

1. The number of vertices of t is denoted by $|t|$.
2. Let s be a vertex of t .
 - The *fertility* of s is the number of blocks B such that there is an edge from s to any vertex of B . It is denoted by $\text{fert}(s)$.
 - The decoration of s is denoted by $d(s)$.
3. A *linear extension* of t is a bijection $\sigma: \{1, \dots, |t|\} \rightarrow \text{Vert}(t)$ such that if x is a child of y , then $\sigma^{-1}(x) > \sigma^{-1}(y)$ (the edges of t being oriented from the roots to the leaves). The set of linear extensions of t is denoted by $L(t)$.

Let us now give a direct description of Φ_{CPL} .

Proposition 3.13 *For all partitioned tree decorated by $\mathcal{D}$,*

$$\Phi_{\text{CPL}}(t) = \sum_{\sigma \in L(t)} \begin{pmatrix} \text{fert}(\sigma(1)) & \dots & \text{fert}(\sigma(|t|)) \\ d(\sigma(1)) & \dots & d(\sigma(|t|)) \end{pmatrix}.$$

Proof By induction on $n = |t|$. It is obvious if $n = 0$ or $n = 1$. Let us assume the result at all ranks $< n$. Two cases can occur.

First case. Let us assume that t has several roots. We can write $t = t_1 \sqcup t_2$, with t_1, t_2 two partitioned trees with $< n$ vertices. Let us denote by k and l the number of vertices of t_1 and t_2 . Then

$$L(t) = \bigsqcup_{\sigma \in \text{Sh}(k, l)} (L(t_1) \otimes L(t_2)) \circ \sigma^{-1}.$$

Hence, by the induction hypothesis applied to t_1 and t_2 ,

$$\begin{aligned} \Phi_{\text{CPL}}(t) &= \Phi_{\text{CPL}}(t_1) \sqcup \Phi_{\text{CPL}}(t_2) \\ &= \sum_{\substack{\sigma_1 \in L(t_1) \\ \sigma_2 \in L(t_2)}} \begin{pmatrix} \text{fert}(\sigma_1(1)) & \dots & \text{fert}(\sigma_1(k)) \\ d(\sigma_1(1)) & \dots & d(\sigma_1(k)) \end{pmatrix} \sqcup \begin{pmatrix} \text{fert}(\sigma_2(1)) & \dots & \text{fert}(\sigma_2(l)) \\ d(\sigma_2(1)) & \dots & d(\sigma_2(l)) \end{pmatrix} \\ &= \sum_{\sigma \in \text{Sh}(k, l)} \sum_{\substack{\sigma_1 \in L(t_1) \\ \sigma_2 \in L(t_2)}} \begin{pmatrix} \text{fert}((\sigma_1 \otimes \sigma_2) \circ \sigma^{-1}(1)) & \dots & \text{fert}((\sigma_1 \otimes \sigma_2) \circ \sigma^{-1}(k+l)) \\ d((\sigma_1 \otimes \sigma_2) \circ \sigma^{-1}(1)) & \dots & d((\sigma_1 \otimes \sigma_2) \circ \sigma^{-1}(k+l)) \end{pmatrix} \\ &= \sum_{\sigma \in L(\sigma)} \begin{pmatrix} \text{fert}(\sigma(1)) & \dots & \text{fert}(\sigma(n)) \\ d(\sigma(1)) & \dots & d(\sigma(n)) \end{pmatrix}. \end{aligned}$$

Second case. Let us assume that t has a single root. Let $t' = t_1 \sqcup \dots \sqcup t_k$ be the partitioned tree obtained by deleting the root of t , where $t_1, \dots, t_k$ are partitioned trees with a single root. If d is the decoration of the root of t , then $t = \bullet_d \bullet t_1 \times \dots \times t_k$. We put $w_i = \Phi_{\text{CPL}}(t_i)$ for all i . Then, by Theorem 2.13,

$$\begin{aligned}
\Phi_{\text{CPL}}(t) &= \Phi_{\text{CPL}}(\bullet_d) \bullet w_1 \times \cdots \times w_k = \binom{0}{d} \bullet w_1 \times \cdots \times w_k \\
&= \sum_{I \subseteq \{1, \dots, k\}} \binom{k - |I|}{d} ((\emptyset \bullet w_I) \sqcup w_{\overline{I}}) = 0 + \binom{k}{d} (\emptyset \sqcup w_{\overline{\emptyset}}) \\
&= \binom{k}{d} (w_1 \sqcup \cdots \sqcup w_k) = \binom{k}{d} (\Phi_{\text{CPL}}(G)) \\
&= \sum_{\tau \in L(t')} \binom{k}{d} \begin{pmatrix} \text{fert}(\tau(1)) & \cdots & \text{fert}(\tau(n-1)) \\ d & d(\tau(1)) & \cdots & d(\tau(n-1)) \end{pmatrix}.
\end{aligned}$$

Moreover, for all $\sigma \in L(t)$, $\sigma(1)$ is the root of t . Hence

$$L(t) = (1) \otimes L(t').$$

Finally,

$$\Phi_{\text{CPL}}(t) = \sum_{\tau \in L(t)} \binom{k}{d} \begin{pmatrix} \text{fert}((1) \otimes \tau(1)) & \cdots & \text{fert}((1) \otimes \tau(n)) \\ d((1) \otimes \tau(1)) & \cdots & d((1) \otimes \tau(n)) \end{pmatrix}.$$

So the result holds for all partitioned trees t . $\square$

Remark 3.14 If t is a partitioned tree with k blocks, then

$$\sum_{i=1}^{|t|} \text{fert}(\sigma(i)) = \sum_{x \in V(t)} \# \{\text{blocks of direct descendants of } x\} = k - 1 \leq |t| - 1.$$

Moreover, this is an equality if and only if t is a rooted tree. Hence, for all partitioned trees of degree n , $\Phi_{\text{CPL}}(t)$ is a sum of biwords of length n , such that the sum of its upper letters is $\leq n - 1$. Consequently, Φ_{CPL} is not surjective.

3.5 Freeness of $\mathfrak{g}_{\mathcal{D}}$

We now prove that the restriction of Φ_{CPL} to the free pre-Lie algebra $\text{Prelie}(\mathcal{D})$ is injective. We shall use certain families of biwords.

Definition 3.15 Let $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$ be a biword.

1. We shall say that w is *admissible* if:
 - For all $1 \leq i \leq n$, $a_i + \cdots + a_n \leq n - i$.
 - $a_1 + \cdots + a_n = n - 1$.
2. We shall say that w is σ -*admissible* if it can be written as $w = w_1 \dots w_k$, with $w_1, \dots, w_k$ admissible.

Examples 1. Here are the possible upper words for admissible words of length ≤ 4 :

$$0; \quad 10; \quad 200, 110; \quad 3000, 2100, 1200, 2010, 1110.$$

2. Here are the possible upper words for σ -admissible words of length ≤ 4 :
- 0; 00, 10; 000, 100, 200, 010, 110;
 0000, 1000, 2000, 3000, 0100, 1100, 2100,
 0200, 1200, 0010, 1010, 2010, 0110, 1110.

Remark 3.16 If $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$ is admissible and $n \geq 1$, then $a_n \leq n - n = 0$, so $a_n = 0$.

- Lemma 3.17** 1. Let $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$ be a σ -admissible biword. Then it can be uniquely written as $w = w_1 \dots w_k$, with $w_1, \dots, w_k$ admissible. Moreover, $a_1 + \dots + a_n = n - k$.
2. Let $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$ be a σ -admissible word. Then for all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n - i$.
3. Let $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$ be a word such that for all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n - i$. We put $k = n - (a_1 + \dots + a_n)$. Then $k \geq 1$ and for $d \in \mathcal{D}$, $\binom{ka_1 \dots a_n}{dd_1 \dots d_n}$ is admissible.

Proof 1. Let $w = w_1 \dots w_k = w'_1 \dots w'_l$ be two decompositions of w in admissible words. Then

$$\begin{aligned} a_1 + \dots + a_n &= \sum_{i=1}^k \text{sum of the letters of } w_i \\ &= \sum_{i=1}^k (\lg(w_i) - 1) = \lg(w) - k = n - k. \end{aligned}$$

Similarly, $a_1 + \dots + a_n = n - l$, so $k = l$. Let us assume that w_1 is formed by the first i letters of w , w'_1 is formed by the first j letters of w , with $i < j$. As w_1 and w'_1 are admissible, $a_1 + \dots + a_j = j - 1$ and $a_1 + \dots + a_i = i - 1$, so $a_{i+1} + \dots + a_j = (j - 1) - (i - 1) = j - i$. As w'_1 is admissible, $a_{i+1} + \dots + a_j \leq j - i - 1$: this is a contradiction. So $i = j$, and $w_1 = w'_1$; hence, $w_2 \dots w_k = w'_2 \dots w'_k$. Iterating the process, $w_2 = w'_2, \dots, w_k = w'_k$.

2. We put $w = w_1 \dots w_k$, where $w_1, \dots, w_k$ are admissible. We put, for all i , $w_i = \binom{a_{i1} \dots a_{il_i}}{d_{i1} \dots d_{il_i}}$. For all i, j , we have

$$\begin{aligned} a_{ij} + \dots + a_{il_i} + a_{i+1,1} + \dots + a_{kl_k} &\leq l_i - j + l_{i+1} - 1 + \dots + l_k - 1 \\ &\leq l_i - j + l_{i+1} + \dots + l_l. \end{aligned}$$

Hence, if $w = \binom{a_1 \dots a_n}{d_1 \dots d_n}$, for all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n - i$.

3. We put $b_1 \dots b_{n+1} = ka_1 \dots a_n$. If $2 \leq i \leq n + 1$, then

$$b_i + \dots + b_{n+1} = a_{i-1} + \dots + a_n \leq n - (i - 1) \leq n + 1 - i.$$

Moreover, $b_1 + \dots + b_{n+1} = k + a_1 + \dots + a_n = n = (n + 1) - 1$. So $\binom{b_1 \dots b_{n+1}}{d_1 \dots d_{n+1}}$ is admissible. $\square$

Theorem 3.18 *The map $\Phi_{\text{PL}} : \text{Prelie}(\mathcal{D}) \rightarrow T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is injective.*

Proof We shall consider the subalgebra (for the product $\sqcup$) generated by $\text{Prelie}(\mathcal{D})$ in $\text{ComPrelie}(\mathcal{D})$. We denote it by $\mathcal{H}_{\mathcal{D}}$. A basis of $\mathcal{H}_{\mathcal{D}}$ is given by the set of partitioned trees decorated by $\mathcal{D}$ such that any block has cardinality 1, except the block containing the roots. Forgetting the blocks, these special partitioned trees are identified with rooted forests. Here are for example rooted forests with ≤ 4 vertices:

Let $\mathcal{W}_n$ be the set of σ -admissible words of length n and let $\mathcal{W}$ be the union of $\mathcal{W}_n$. We choose a total order on $\mathcal{W}$ such that:

1. If $u \in \mathcal{W}_m$ and $v \in \mathcal{W}_n$ with $m > n$, then $u > v$.
2. For all $n \geq 0$ and all $d \in \mathcal{D}$ the following map is increasing:

$$\left\{ \begin{array}{l} \mathcal{W}_n \rightarrow \mathcal{W}_{n+1} \\ \begin{pmatrix} a_1 \dots a_n \\ d_1 \dots d_n \end{pmatrix} \rightarrow \begin{pmatrix} n - a_1 - \dots - a_n & a_1 \dots a_n \\ d & d_1 \dots d_n \end{pmatrix} \end{array} \right.$$

Note that this map takes its values in the set of admissible words, by Lemma 3.17.

3. If $u = u_1 \dots u_k$ and $v = v_1 \dots v_l$ are two elements of $\mathcal{W}_n$ decomposed into admissible words, and if for a particular index i , $u_k = v_l$, $u_{k-1} = v_{l-1}$, $\dots$, $u_{k-i+1} = v_{k-i+1}$ and $u_{k-i} > v_{l-i}$, then $u > v$.

For any decorated rooted forest F (still identified with a partitioned tree by putting all the roots in the same block), we inductively define a biword w_F in the following way:

- $w_1 = \emptyset$.
- If F is not a tree, we put $F = t_1 \dots t_k$, indexed so that $w_{t_1} \leq \dots \leq w_{t_k}$. Then $w_F = w_{t_1} \dots w_{t_k}$.
- If F is a tree, let d be the decoration of its root and let $t_1 \dots t_k$ be the forest obtained by deleting the root of F , its trees being indexed so that $w_{t_1} \leq \dots \leq w_{t_k}$. Then $w_F = \binom{k}{d} w_{t_1} \dots w_{t_k}$.

First step. Let us prove that for all rooted forest F :

1. The coefficient of w_F in $\Phi_{\text{CPL}}(F)$ is not zero.
2. w_F is σ -admissible. Moreover, if F is a tree, w_F is admissible.

We proceed by induction on the number n of vertices of F . If $n = 0$, this is obvious. Let us assume the result at all ranks $< k$. If F is not a tree, we put $F = t_1 \dots t_k$. By the induction hypothesis, $w_{t_1}, \dots, w_{t_k}$ are admissible, so $w_{t_1} \dots w_{t_k}$ is σ -admissible. For all $1 \leq i \leq k$, there exists a linear extension σ_i of t_i such that

$$w_{t_i} = \begin{pmatrix} \text{fert}(\sigma_i(1)) \dots \text{fert}(\sigma_i(n_i)) \\ d(\sigma_i(1)) \dots d(\sigma_i(n_i)) \end{pmatrix}.$$

Then $\sigma = \sigma_1 \otimes \dots \otimes \sigma_k$ is a linear extension of F and

$$\begin{pmatrix} \text{fert}(\sigma(1)) \dots \text{fert}(\sigma(n)) \\ d(\sigma(1)) \dots d(\sigma(n)) \end{pmatrix} = w_{t_1} \dots w_{t_k},$$

so this biword appears in $\Phi_{\text{CPL}}(F)$. If F is a tree, keeping the notations of the definition of w_F , by the induction hypothesis, $w_{t_1}, \dots, w_{t_k}$ are admissible. By Lemma 3.17, $\binom{k}{d} w_{t_1} \dots w_{t_k}$ is admissible. For all $1 \leq i \leq k$, there exists a linear extension σ_i of t_i such that

$$w_{t_i} = \left(\begin{array}{c} \text{fert}(\sigma_i(1)) \dots \text{fert}(\sigma_i(n_i)) \\ d(\sigma_i(1)) \dots d(\sigma_i(n_i)) \end{array} \right).$$

Then $\sigma = (1) \otimes \sigma_1 \otimes \dots \otimes \sigma_k$ is a linear extension of F and

$$\left(\begin{array}{c} \text{fert}(\sigma(1)) \dots \text{fert}(\sigma(n)) \\ d(\sigma(1)) \dots d(\sigma(n)) \end{array} \right) = \binom{k}{d} w_{t_1} \dots w_{t_k},$$

so this biword appears in $\Phi_{\text{CPL}}(F)$.

Second step. Let F, G be two forests such that $w_F = w_G$. As $w = w_F = w_G$ appears in $\Phi_{\text{CPL}}(F)$ and $\Phi_{\text{CPL}}(G)$, then the number of vertices of F and G is the length n of w . Let us prove that $F = G$ by induction on n . It is obvious if $n = 1$. Let us assume the result at all ranks $< n$. Let us put $F = t_1 \dots t_k$, and $G = t'_1 \dots t'_l$. As the biwords w_{t_i} are admissible, by Lemma 3.17, the sum of letters of w is $n - k$; similarly, this sum is $n - l$, so $k = l$. If $k \geq 2$, by the unicity of the decomposition of w into admissible biwords, $w_{t_i} = w_{t'_i}$ for all i . By the induction hypothesis, $t_i = t'_i$ for all i , so $F = G$. If $k = 1$, considering the first biletter of w , the roots of F and G have the same decoration and the same fertility. Considering the biword obtained by deleting the first biletter of w , if F' and G' are the forests obtained by deleting the roots of F and G , the induction hypothesis implies that $F' = G'$; hence, $F = G$.

Third step. For all forests F , we put

$$w'_F = \max \{ w : \text{the coefficient of } w \text{ in } \Phi_{\text{CPL}}(F) \text{ is not } 0 \text{ and } w \text{ is admissible} \}.$$

By the preceding observations, w'_F exists and $w'_F \geq w_F$. Let us prove that $w'_F = w_F$ for any forest F . We proceed by induction on the number n of vertices of F . It is obvious if $F = 1$. Let us assume the result at all ranks $< n$.

We put $w'_F = w_1 \dots w_k$, with $w_1 \dots w_k$ admissible, and f the linear extension of F corresponding to w'_F . For all $1 \leq i \leq k$, let X_i be the subset of the vertices of F corresponding to the biletters of w_i . If $x \in X_i$ and $y \in X_j$ is a child of x in F , then as f is a linear extension of F , $j \geq i$. So for all $1 \leq i \leq k$, $F_i = X_i \sqcup \dots \sqcup X_k$ is an ideal of F , and $w_i \dots w_k$ appears in $\Phi_{\text{CPL}}(F_i)$. By Lemma 3.17, the sum of the first letters of $w_i \dots w_k$ is $|F_i| - (k - i + 1) = |F_i| - \lg(F_i)$, so F_i is a forest of $k - i + 1$ trees.

Let us assume there exists an index i such that F_i is not the disjoint union of a tree formed by the vertices in X_i and F_{i+1} . We choose i maximal; for all $j > i$, let t_j be the tree formed by the vertices of X_j such that $F_{i+1} = t_{i+1} \dots t_k$. As F_i is a forest of length $k - i + 1$, and X_i does not form a subtree of F_i , X_i forms a forest $s_1 \dots s_l$ with $l > 1$, and at least one of s_p , say s_l for example, has for a child a root of one of t_j . Let t be the subtree of F containing s_l and t_j . Then a word of the form $\dots w_t w_{t_{j+1}} \dots w_{t_k}$ appears in $\Phi_{\text{CPL}}(F)$. As $|t| > |t_j|$, by the first condition

on the order on admissible words, $w_{t_{j+1}} > w_{t_j}$. By the third condition on the order, $\dots w_t w_{t_{j+1}} \dots w_{t_k} > w_{t_1} \dots w_{t_k} = w'_F$: this contradicts the maximality of w'_F . We obtain that $F = t_1 \dots t_k$, the vertices of t_i corresponding via f to the billetters of w_i for all i . Consequently, $w_i \leq w'_{t_i} = w_{t_i}$ for all i .

Let us assume that $i < j$ and $w_i > w_j$. Shuffling the billetters corresponding to the vertices of t_i and t_j , $w_1 \dots w_j \dots w_i \dots w_k$ also appears in $\Phi_{\text{CPL}}(F)$ and is strictly greater than w'_F , by the third condition on the order: a contradiction. So $w_1 \leq \dots \leq w_k$.

Let us assume that $k \geq 2$. We put $F = t'_1 \dots t'_k$, with $w_{t'_1} \leq \dots \leq w_{t'_k}$, such that $w_F = w_{t'_1} \dots w_{t'_k}$. By the induction hypothesis, for all i , $w_{t'_i} = w'_{t'_i}$. In particular, $w_k \leq w'_{t'_k} \leq w_{t'_k}$, as t_k is one of t'_j . If $w_k < w_{t'_k}$, then $w_1 \dots w_k < w_F$ by the third condition of the total order, and this contradicts the maximality of w'_F . So $w_k = w'_{t'_k}$. Moreover, $w_1 \dots w_{t'_k}$ also appears in $\Phi_{\text{CPL}}(F)$; by maximality of w'_F , necessarily $w'_{t'_k} \leq w_k$, so $w'_{t'_k} = w_k$. By the second step, $t_k = t'_k$. We then obtain that $t_1 \dots t_{k-1} = t'_1 \dots t'_{k-1}$ and that $w'_{t_1 \dots t_{k-1}} = w_1 \dots w_{k-1}$. By the induction hypothesis, $w_1 \dots w_{k-1} = w'_{t'_1} \dots w'_{t'_{k-1}}$, so $w'_F = w_F$.

Let us assume that $k = 1$. Let d the decoration of the root of F , l its fertility and G be the forest obtained by deleting the root of F . Then $\Phi_{\text{CPL}}(F) = \binom{l}{d} \Phi_{\text{CPL}}(G)$. By the second condition on the total order on σ -admissible biwords, $w'_F = \binom{l}{d} w'_G$. By the induction hypothesis, $w'_G = w_G$, so $w'_F = w_F$.

Last step. Let $x = \sum a_F F$ be a nonzero element of $\mathcal{H}_{\mathcal{D}}$. As $F \rightarrow w_F$ is injective by the second step, let us totally order the set of decorated rooted forests such that $F < G$ if and only if $w_F < w_G$. Let F_0 be the maximal forest such that $a_F \neq 0$. If $F < F_0$, then $w_F < w_{F_0}$; as $w_F = w'_F$, w_{F_0} does not appear in $\Phi_{\text{CPL}}(F)$. If $F > F_0$, $a_F = 0$. So the coefficient of w_F in $\Phi_{\text{CPL}}(F)$ is a_{F_0} times the coefficient of w_{F_0} in $\Phi_{\text{CPL}}(F_0)$, which is not zero. So $\Phi_{\text{CPL}}(x) \neq 0$: Φ_{CPL} is injective.

Remark 3.19 1. We even prove that the restriction of Φ_{CPL} to $\mathcal{H}_{\mathcal{D}}$ is injective.

2. Although it contains a free pre-Lie subalgebra, $T_+(V_{\mathcal{D}}, f_{\mathcal{D}})$ is not free.

Let us assume it is free. As the length of words gives a connected gradation of $T_+(V_{\mathcal{D}}, f_{\mathcal{D}})$, it is freely generated by any graded complement of $T_+(V_{\mathcal{D}}, f_{\mathcal{D}}) \bullet T_+(V_{\mathcal{D}}, f_{\mathcal{D}})$. Let us choose $d \in \mathcal{D}$. Then it is not difficult to prove that we can choose a complement of $T_+(V_{\mathcal{D}}, f_{\mathcal{D}}) \bullet T_+(V_{\mathcal{D}}, f_{\mathcal{D}})$ which contains $\binom{0}{d}$, $\binom{1}{d}$ and $\binom{00}{dd}$, which implies that the pre-Lie subalgebra generated by these three elements is free. Hence, the following pre-Lie morphism is injective:

$$\psi: \begin{cases} \text{Prelie}(\{0, 1, 00\}) \rightarrow T_+(V_{\mathcal{D}}, f_{\mathcal{D}}) \\ \bullet_0 \rightarrow \binom{0}{d} \\ \bullet_1 \rightarrow \binom{1}{d} \\ \bullet_{00} \rightarrow \binom{00}{dd}. \end{cases}$$

But $\psi\left(\binom{0}{d} \bullet \binom{0}{d} - 2 \binom{00}{dd}\right) = 0$: this is a contradiction. So $T_+(V_{\mathcal{D}}, f_{\mathcal{D}})$ is not free.

Corollary 3.20 *The pre-Lie subalgebra $\mathfrak{g}_{\mathcal{D}}$ of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$ is freely generated by the billetters $\binom{0}{d}$, $d \in \mathcal{D}$.*

We shall give more results on the morphism Φ_{CPL} and the admissible words in the appendix.

3.6 Prelie subalgebra generated by V

We here consider the pre-Lie subalgebra of V generated by V . We denote by $\text{Prelie}(V)$ the free pre-Lie algebra generated by V . This is the space of rooted trees decorated by V , each vertex being linear in its decoration. The pre-Lie product is also given by grafting. We shall also consider the free Com-PreLie algebra generated by V . This is the space of partitioned trees decorated by V , each vertex being linear in its decoration. We consider the two morphisms

$$\phi_{\text{PL}} : \begin{cases} \text{Prelie}(V) \rightarrow T_+(V) \\ \bullet_v \rightarrow v, \end{cases} \quad \phi_{\text{CPL}} : \begin{cases} \text{ComPrelie}(V) \rightarrow T_+(V) \\ \bullet_v \rightarrow v. \end{cases}$$

Note that ϕ_{PL} is the restriction of ϕ_{CPL} on $\text{Prelie}(V) \subseteq \text{ComPrelie}(V)$.

Combining Theorem 3.11 and Proposition 3.13, $\phi_{\text{CPL}} = \varphi_{(V, f)} \circ \Phi_{\text{CPL}}$. Hence, for all partitioned tree t decorated by V , we have

$$\phi_{\text{CPL}}(t) = \sum_{\sigma \in L(t)} f^{\text{fert}(\sigma(1))}(d(\sigma(1))) \dots f^{\text{fert}(\sigma(|t|))}(d(\sigma(|t|))).$$

Theorem 3.21 1. ϕ_{PL} is surjective if and only if $f(V) = V$.

2. ϕ_{CPL} is surjective if and only if the codimension of $f(V)$ in V is ≤ 1 .

Proof 1. $\Rightarrow$. The length of words is a gradation of $T(V, f)$; moreover, ϕ_{CPL} is homogeneous for this gradation. Hence

$$\begin{aligned} \text{Im } \phi_{\text{PL}} \cap V^{\otimes 2} &= \text{Vect}(\phi_{\text{PL}}(\downarrow_v^w) : v, w \in V) \\ &= \text{Vect}(f(v)w : v, w \in V) = f(V) \otimes V. \end{aligned}$$

If ϕ_{PL} is surjective, then $f(V) \otimes V = V \otimes V$, so $f(V) = V$.

1. $\Leftarrow$. Let $v_1 \dots v_n \in V^{\otimes n}$. As $f(V) = V$, for all $1 \leq i \leq n-1$, there exists $w_i \in V$ such that $f(w_i) = v_i$. Let t be the ladder with n vertices, decorated from the root to the leaf by $w_1, \dots, w_{n-1}, v_n$. Then $L(t)$ is reduced to a single element, and $\phi_{\text{PL}}(t) = f(w_1) \dots f(w_{n-1})v_n = v_1 \dots v_n$. So ϕ_{PL} is surjective.

2. $\Rightarrow$. By homogeneity of ϕ_{CPL} ,

$$\begin{aligned} \text{Im } \phi_{\text{CPL}} \cap V^{\otimes 2} &= \text{Vect}(\phi_{\text{CPL}}(\downarrow_v^w), \phi_{\text{CPL}}(v \rightarrow w) : v, w \in V) \\ &= \text{Vect}(f(v)w, vw + wv : v, w \in V). \end{aligned}$$

Let us assume that the codimension of $f(V)$ in V is ≥ 2 . Let $x, y \in V$, linearly independent, such that $f(V) \cap \text{Vect}(x, y) = (0)$. There exists a linear map $g: V \rightarrow V$ such that $g(f(V)) = (0)$ and $g(x) = x, g(y) = y$. As ϕ_{CPL} is surjective, we can write

$$xy = \sum_{i \in I} f(v_i)w_i + \sum_{j \in J} (v_j w_j + w_j v_j).$$

Hence

$$\begin{aligned} xy &= g(x)g(y) = \sum_{i \in I} g(f(v_i))g(w_i) + \sum_{j \in J} (g(v_j)g(w_j) + g(w_j)g(v_j)) \\ &= 0 + \sum_{j \in J} (g(v_j)g(w_j) + g(w_j)g(v_j)) = \sum_{j \in J} (g(w_j)g(v_j) + g(v_j)g(w_j)) \\ &= g(y)g(x) = yx. \end{aligned}$$

So $xy = yx$, and consequently x and y are linearly independent: this is a contradiction. So the codimension of $f(V)$ in V is ≤ 1 .

2. $\Leftarrow$. If $f(V) = V$, then by the first point, ϕ_{PL} , and consequently ϕ_{CPL} , is surjective. Let us assume that the codimension of $f(V)$ in V is 1. Let us choose $x \in V$ such that $V = f(V) \oplus \text{Vect}(x)$. In order to prove that ϕ_{CPL} is surjective, it is enough to prove that $v_1 \dots v_n \in \text{Im } \phi_{\text{CPL}}$ for all $n \geq 1$ and $v_1, \dots, v_n \in \{x\} \cup f(V)$. We proceed by induction on n . If $n = 1$, then $v_1 = \phi_{\text{CPL}}(\bullet_{v_1})$. Let us assume the result at rank $n - 1$. Let k be the greatest integer such that $v_1 = \dots = v_k = x$, with the convention that $k = 0$ if $v_1 \neq x$. We proceed by induction on k . If $k = 0$, let us put $v_1 = f(w_1)$. By the induction on n , there exists $P \in \text{ComPrelie}(V)$ such that $\phi_{\text{CPL}}(P) = v_2 \dots v_n$. Then

$$\phi_{\text{CPL}}(\bullet_{w_1} \bullet X) = w_1 \bullet v_2 \dots v_n = f(w_1)v_2 \dots v_n = v_1 \dots v_n.$$

Let us assume the result at rank $k - 1$, with $k \geq 1$. By the induction hypothesis on n , there exists $X \in \text{ComPrelie}(V)$ such that $\phi_{\text{CPL}}(X) = x^{k-1}v_{k+1} \dots v_n$. Then

$$\begin{aligned} \phi_{\text{CPL}}(\bullet_x \sqcup X) &= x \sqcup x^{k-1}v_{k+1} \dots v_n \\ &= kx^k v_{k+1} \dots v_n + \sum_{i=k+1}^n x^{k-1}v_{k+1} \dots v_i x v_{i+1} \dots v_n. \end{aligned}$$

By the induction hypothesis on k , for all $k + 1 \leq i \leq n$, $x^{k-1}v_{k+1} \dots v_i x v_{i+1} \dots v_n \in \text{Im } \phi_{\text{CPL}}$. So $kx^k v_{k+1} \dots v_n \in \text{Im } \phi_{\text{CPL}}$. As the characteristic of the base field is zero, $v_1 \dots v_n = x^k v_{k+1} \dots v_n \in \text{Im } \phi_{\text{CPL}}$. So ϕ_{CPL} is surjective.

Corollary 3.22 1. *The subspace V generates the pre-Lie algebra $T_+(V)$ if and only if $f(V) = V$. The subspace V generates the Com-PreLie algebra $T_+(V)$ if and only if the codimension of $f(V)$ in V is ≤ 1 .*

2. Let W be a subspace of V . We put

$$\overline{W} = \sum_{k=0}^{\infty} f^k(W),$$

that is to say the smallest subspace of V stable by f which contains W .

- 2a. $K\emptyset \oplus W$ generates the pre-Lie algebra $T(V, f)$ if and only if $\overline{W} = V$ and $f(V) = V$.
 2b. $K\emptyset \oplus W$ generates the Com-PreLie algebra $T(V, f)$ if and only if $\overline{W} = V$ and the codimension of $f(V)$ in V is ≤ 1 .

Proof 1. It is a direct consequence of Theorem 3.21.

2a. We denote by A the pre-Lie subalgebra of $T(V, f)$ generated by $K\emptyset \oplus W$ and by A' the pre-Lie subalgebra of $T(V, f)$ generated by $\overline{W}$. Let us prove that $A = K\emptyset \oplus A'$. First, for all $w_1, \dots, w_k \in \overline{W}$,

$$w_1 \dots w_k \bullet \emptyset = \sum_{i=1}^k w_1 \dots w_{i-1} f(w_i) w_{i+1} \dots w_k \in A',$$

as $\overline{W}$ is stable under f . As $\emptyset \circ w = 0$ for any $w \in A'$, $K\emptyset \oplus A'$ is a pre-Lie subalgebra which contains W , so it contains A . For any $x \in W$ and any $k \geq 0$, we have

$$f^k(x) = (\dots ((x \bullet \emptyset) \bullet \emptyset) \dots) \bullet \emptyset \in A,$$

so $\overline{W} \subseteq A$, which implies $A' \subseteq A$ and $K\emptyset \oplus A' \subseteq A$.

$\Rightarrow$. If $A = T(V, f)$, then $A' = T(V, f)$. Comparing the homogeneous component of degree 1, $\overline{W} = V$. By the first point, $f(V) = V$.

$\Leftarrow$. In this case, $A' = T_+(V)$, so $A = K\emptyset \oplus T_+(V) = T(V)$.

Point 2b is proved similarly. $\square$

4 Examples

4.1 The diagonalizable case

We here assume that the endomorphism f is diagonalizable. Let $(x_d)_{d \in \mathcal{D}}$ be a basis of V such that $f(x_d) = \lambda_d x_d$ for all $d \in \mathcal{D}$. We now construct the pre-Lie algebra $T_+(V, f)$ as the dual of a Hopf subalgebra of the Connes–Kreimer Hopf algebra $\mathcal{H}_{CK}^{\mathcal{D}}$ of rooted trees decorated by $\mathcal{D}$ [1, 3]. As an algebra, it is the free commutative, associative algebra generated by the set of rooted trees decorated by $\mathcal{D}$. Its coproduct is given by admissible cuts: for any tree t ,

$$\Delta(t) = \sum_{\substack{c \text{ admissible} \\ \text{cut of } t}} R^c(t) \otimes P^c(t).$$

For example,

$$\begin{aligned} \Delta\left({}^b\mathbb{V}_a^c\right) &= {}^b\mathbb{V}_a^c \otimes 1 + 1 \otimes {}^b\mathbb{V}_a^c + \mathbb{I}_a^c \otimes \bullet_d + {}^b\mathbb{V}_a^c \otimes \bullet_d \\ &\quad + \mathbb{I}_a^b \otimes \mathbb{I}_c^d + \mathbb{I}_a^c \otimes \bullet_b \bullet_d + \bullet_a \otimes \bullet_b \mathbb{I}_c^d. \end{aligned}$$

The enveloping algebra of $\text{Prelie}(\mathcal{D})$ and $\mathcal{H}_{\text{CK}}^{\mathcal{D}}$ are in duality by the pairing [12, 20]

$$\langle \cdot, \cdot \rangle: \begin{cases} \mathcal{U}(\text{Prelie}(\mathcal{D})) \times \mathcal{H}_{\text{CK}}^{\mathcal{D}} \rightarrow \mathbb{K} \\ (F, G) \rightarrow \delta_F G s_F, \end{cases}$$

where s_F is the number of symmetries of the forest F . As the characteristic of $\mathbb{K}$ is zero, this pairing is nondegenerate. Consequently, the subspace of $\mathcal{H}_{\text{CK}}^{\mathcal{D}}$ generated by the set of rooted trees inherits a pre-Lie cobracket defined by $\delta = (\pi \otimes \pi) \circ \Delta$, where π is defined by

$$\pi(F) = \begin{cases} F & \text{if } F \text{ is tree,} \\ 0 & \text{if } F \text{ is a forest which is not a tree.} \end{cases}$$

Following [3, 4], for all $d \in \mathcal{D}$, we define a linear map $N_d: \mathcal{H}_{\text{CK}}^{\mathcal{D}} \rightarrow \mathcal{H}_{\text{CK}}^{\mathcal{D}}$ by

$$N_d(F) = \sum_{s \in V(F)} \lambda_{d(s)} F \bullet_s \bullet_d,$$

for any forest F . We shall also need the following map:

$$\phi_\lambda: \begin{cases} \mathcal{H}_{\text{CK}}^{\mathcal{D}} \rightarrow \mathcal{H}_{\text{CK}}^{\mathcal{D}} \\ F \rightarrow \sum_{s \in V(F)} \lambda_{d(s)} F. \end{cases}$$

Proposition 4.1 *For all $d \in \mathcal{D}$,*

1. *For all $x, y \in \mathcal{H}_{\text{CK}}^{\mathcal{D}}$, $N_d(xy) = N_d(x)y + xN_d(y)$.*
2. *$N_d \circ \pi = \pi \circ N_d$.*
3. *For all $x \in \mathcal{H}_{\text{CK}}^{\mathcal{D}}$,*

$$\Delta \circ N_d(x) = (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \Delta(x) + (1 \otimes \bullet_d) \cdot (\phi_\lambda \otimes \text{Id}) \circ \Delta(x).$$

3. *If x is a linear span of trees, $\delta \circ N_d(x) = (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \delta(x) + \phi_\lambda(x) \otimes \bullet_d$.*

Proof 1. This is obvious if x and y are forests, as $V(xy) = V(x) \sqcup V(y)$.

2. If F is a tree, then $N_d(F)$ is a linear span of trees, so $N_d \circ \pi(F) = N_d(F) = \pi \circ N_d(F)$. If F is a forest which is not a tree, then $N_d(F)$ is a linear span of forests which are not trees, so $N_d \circ \pi(F) = 0 = \pi \circ N_d(F)$.

3. Let F be a forest, and $s \in V(F)$. We denote by l the leaf decorated by d grafted onto F to obtain $G = F \circ_s \bullet_d$. There are the following three types of admissible cuts of G :

- Admissible cuts c such that $s \in V(P^c(G))$. Then $R^c(G) = R^{c|F}(F)$ and $P^c(G) = R^{c|F}(F) \bullet_s \bullet_d$.
- Admissible cuts c such that $s, \ell \in V(R^c(G))$. Then $R^c(G) = R^{c|F}(F) \bullet_s \bullet_d$ and $P^c(G) = R^{c|F}(F)$.
- Admissible cuts c such that $s \in V(R^c(G))$ and $\ell \in V(P^c(G))$. Then $R^c(G) = R^{c|F}(F)$ and $P^c(G) = \bullet_d R^{c|F}(F)$.

Summing all these possibilities, we obtain

$$\begin{aligned}
 \Delta(N_d(F)) &= \sum_{\substack{c \text{ admissible} \\ \text{cut of } F}} \sum_{s \in V(P^c(F))} \lambda_{d(s)} R^c(F) \otimes P^c(F) \bullet_s \bullet_d \\
 &= \sum_{\substack{c \text{ admissible} \\ \text{cut of } F}} \sum_{s \in V(R^c(F))} \lambda_{d(s)} R^c(F) \bullet_s \bullet_d \otimes P^c(F) \\
 &= \sum_{\substack{c \text{ admissible} \\ \text{cut of } F}} \sum_{s \in V(R^c(F))} \lambda_{d(s)} R^c(F) \otimes \bullet_d P^c(F) \\
 &= \sum_{\substack{c \text{ admissible} \\ \text{cut of } F}} N_d(R^c(F)) \otimes P^c(F) + R^c(F) \otimes N_d(P^c(F)) \\
 &\quad + \phi_\lambda(R^c(F)) \otimes \bullet_d P^c(F) \\
 &= (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \Delta(F) + (1 \otimes \bullet_d) \cdot (\phi_\lambda \circ \text{Id}) \circ \Delta(F).
 \end{aligned}$$

4. Let t be a tree. We put $\Delta(t) = t \otimes 1 + \sum t' \otimes t''$, where $\sum t' \otimes t''$ is a sum of tensors such that the left part is a forest and the right part is a tree. Then

$$\begin{aligned}
 (\pi \otimes \pi)((1 \otimes \bullet_d) \cdot (\phi_\lambda \otimes \text{Id}) \circ \Delta(t)) &= (\pi \otimes \pi)\left(\phi_\lambda(t) \otimes \bullet_d + \sum \phi_\lambda(t') \otimes t'' \bullet_d\right) \\
 &= \phi_\lambda(t) \otimes \bullet_d.
 \end{aligned}$$

Hence

$$\begin{aligned}
 \delta \circ N_d(t) &= (\pi \otimes \pi) \circ \Delta \circ N_d(t) \\
 &= (\pi \otimes \pi) \circ (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \Delta(t) + (\pi \otimes \pi)((1 \otimes \bullet_d) \cdot (\phi_\lambda \otimes \text{Id}) \circ \Delta(t)) \\
 &= (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ (\pi \otimes \pi) \circ \Delta(t) + \phi_\lambda(t) \otimes \bullet_d \\
 &= (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \delta(t) + \phi_\lambda(t) \otimes \bullet_d,
 \end{aligned}$$

which proves the last point. $\square$

Definition 4.2 Let $d_1, \dots, d_n \in \mathcal{D}$, $n \geq 1$. We inductively defined $t_{d_1 \dots d_n}$ in the following way:

1. If $n = 1$, $t_{d_1} = \bullet_{d_1}$.
2. If $n \geq 2$, $t_{d_1 \dots d_n} = N_{d_n}(t_{d_1 \dots d_{n-1}})$.

We denote by $\mathcal{H}_\lambda$ the subalgebra of $\mathcal{H}_{CK}^{\mathcal{D}}$ generated by the elements $t_{d_1 \dots d_n}$.

Examples If $a, b, c, d \in \mathcal{D}$, then

$$\begin{aligned} t_a &= \bullet_a, \\ t_{ab} &= \lambda_a \begin{array}{c} \bullet \\ \downarrow \\ b \end{array}, \\ t_{abc} &= \lambda_a^{2b} \begin{array}{c} \bullet \\ \downarrow \\ a \end{array} \begin{array}{c} \bullet \\ \downarrow \\ c \end{array} + \lambda_a \lambda_b \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} \begin{array}{c} \bullet \\ \downarrow \\ c \end{array}, \\ t_{abcd} &= \lambda_a^{3b} \begin{array}{c} \bullet \\ \downarrow \\ a \end{array} \begin{array}{c} \bullet \\ \downarrow \\ d \end{array} + \lambda_a^2 \lambda_b \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} \begin{array}{c} \bullet \\ \downarrow \\ c \end{array} + \lambda_a^2 \lambda_c \begin{array}{c} \bullet \\ \downarrow \\ c \end{array} \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} + \lambda_a^2 \lambda_b \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} \begin{array}{c} \bullet \\ \downarrow \\ d \end{array} + \lambda_a \lambda_b^2 \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} \begin{array}{c} \bullet \\ \downarrow \\ a \end{array} \begin{array}{c} \bullet \\ \downarrow \\ d \end{array} + \lambda_a \lambda_b \lambda_c \begin{array}{c} \bullet \\ \downarrow \\ c \end{array} \begin{array}{c} \bullet \\ \downarrow \\ b \end{array} \begin{array}{c} \bullet \\ \downarrow \\ a \end{array}. \end{aligned}$$

Proposition 4.3 $\mathcal{H}_\lambda$ is stable under N_d for all $d \in \mathcal{D}$ and under ϕ_λ , and is a Hopf subalgebra of $\mathcal{H}_{CK}^{\mathcal{D}}$.

Proof We consider

$$A = \{x \in \mathcal{H}_\lambda : \phi_\lambda(x), N_d(x) \in \mathcal{H}_\lambda \text{ for all } d \in \mathcal{D}\}.$$

As N_d and ϕ_λ are derivations, this is a subalgebra of $\mathcal{H}_\lambda$. For all nonempty words $w = d_1 \dots d_n$, $N_d(t_w) = t_{wd}$ and $\phi_\lambda(t_w) = (\lambda_{d_1} + \dots + \lambda_{d_n})t_w$, so A contains the generators of $\mathcal{H}_\lambda$: $A = \mathcal{H}_\lambda$, which proves the first point. Let us prove that $\Delta(t_w) \in \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda$ for all nonempty words w by induction on the length n . If $n = 1$, then $t_w = \bullet_d$ for a particular d , so $\Delta(t_w) = t_w \otimes 1 + 1 \otimes t_w \in \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda$. Let us assume the result at rank $n - 1$. We put $w = w'd$, with $d \in \mathcal{D}$. By the induction hypothesis, $\Delta(t_{w'}) \in \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda$. Then

$$\begin{aligned} \Delta(t_w) &= \Delta \circ N_d(t_{w'}) \\ &= (N_d \otimes \text{Id} + \text{Id} \otimes N_d) \circ \Delta(t_{w'}) + (1 \otimes \bullet_d) \cdot (\phi_\lambda \otimes \text{Id}) \circ \Delta(t_{w'}). \end{aligned}$$

As $\mathcal{H}_\lambda$ is stable under N_d and ϕ_λ and contains $\bullet_d$, this belongs to $\mathcal{H}_\lambda \otimes \mathcal{H}_\lambda$. So $\mathcal{H}_\lambda$ is a Hopf subalgebra of $\mathcal{H}_{CK}^{\mathcal{D}}$. $\square$

As its generators are linear spans of trees, $\mathcal{H}_\lambda$ is stable under π . As it is also stable under Δ , the vector space generated by the elements t_w is stable under δ , so is a pre-Lie coalgebra. Let us give a formula for this pre-Lie cobracket.

Proposition 4.4 Let $d_1, \dots, d_n \in \mathcal{D}$. For all $J = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, with $i_1 < \dots < i_k$, we put

- $w_J = d_{i_1} \dots d_{i_k}$.
- $m(J) = \max \{i : \{1, \dots, i\} \subseteq J\}$, with the convention $m(J) = 0$ if $1 \notin J$.

Then

$$\delta(t_{d_1 \dots d_n}) = \sum_{I \subsetneq \{1, \dots, n\}} \sum_{i=1}^{m(I)} \lambda_i t_{w_I} \otimes t_{w_{I^c}}.$$

Proof By induction on n . It is obvious if $n = 1$, as $\delta(t_{d_1}) = 0$. Let us assume the result at rank $n - 1$.

$$\begin{aligned} \delta(t_{d_1, \dots, d_n}) &= \delta \circ N_{d_n}(t_{d_1, \dots, d_{n-1}}) \\ &= (N_{d_n} \otimes \text{Id} + \text{Id} \otimes N_{d_n}) \circ \delta(t_{d_1, \dots, d_{n-1}}) + \phi_\lambda(t_{d_1, \dots, d_{n-1}}) \otimes \bullet_{d_n} \\ &= \sum_{I \subsetneq \{1, \dots, n-1\}} \left(\sum_{i=1}^{m(I)} \lambda_{d_i} \right) (N_{d_n} \otimes \text{Id} + \text{Id} \otimes N_{d_n}) (t_{w_I} \otimes t_{w_{\{1, \dots, n-1\} \setminus I}}) \\ &\quad + \sum_{i=1}^{n-1} \lambda_{d_i} t_{d_1, \dots, d_{n-1}} \otimes t_{d_n} \\ &= \sum_{I \subsetneq \{1, \dots, n-1\}} \sum_{i=1}^{m(I)} \lambda_{d_i} (t_{w_{I \cup \{n\}}} \otimes t_{w_{\{1, \dots, n-1\} \setminus I}} + t_{w_I} \otimes t_{w_{\{1, \dots, n\} \setminus I}}) \\ &\quad + \sum_{i=1}^{n-1} \lambda_{d_i} t_{d_1, \dots, d_{n-1}} \otimes t_{d_n} \\ &= \sum_{\substack{J \subsetneq \{1, \dots, n\} \\ n \in J}} \sum_{i=1}^{m(J)} \lambda_{d_i} t_{w_J} \otimes t_{w_{J^c}} + \sum_{J \subsetneq \{1, \dots, n-1\}} \sum_{i=1}^{m(J)} \lambda_{d_i} t_{w_J} \otimes t_{w_{J^c}} \\ &\quad + \sum_{i=1}^{n-1} \lambda_{d_i} t_{d_1, \dots, d_{n-1}} \otimes t_{d_n} \\ &= \sum_{J \subsetneq \{1, \dots, n\}} \sum_{i=1}^{m(J)} \lambda_{d_i} t_{w_J} \otimes t_{w_{J^c}}. \end{aligned}$$

So the result holds for all n . $\square$

Let us assume that all λ_d are nonzero. For all words $w = d_1 \dots d_n$, in t_w appears only one ladder, that is to say a tree with no ramification: its vertices, from the root to the unique leaf, are decorated by $d_1, \dots, d_n$, and its coefficient is $\lambda_{d_1} \dots \lambda_{d_{n-1}}$. So the elements t_w are linearly independent. Dually, identifying the dual basis $(t_{d_1 \dots d_n}^*)_{d_1, \dots, d_n \in \mathcal{D}}$ of the graded dual of the pre-Lie coalgebra of the generators of $\mathcal{H}_\lambda$ and the basis $(x_{d_1} \dots x_{d_n})_{d_1, \dots, d_n \in \mathcal{D}}$, we obtain a pre-Lie algebra structure on the space $T(V)$. If $d_1, \dots, d_{k+l} \in \mathcal{D}$, then

$$\begin{aligned}
x_{d_1} \dots x_{d_k} \bullet x_{d_{k+1}} \dots x_{d_l} &= \sum_{\sigma \in \text{Sh}(k,l)} \sum_{i=1}^{m_k(\sigma)} \lambda_{d_i} (\sigma.(x_{d_1} \dots x_{d_{k+l}})) \\
&= \sum_{\sigma \in \text{Sh}(k,l)} \sum_{i=1}^{m_k(\sigma)} \text{Id}^{\otimes(i-1)} \otimes f \otimes \text{Id}^{\otimes(n-i)} (\sigma.(x_{d_1} \dots x_{d_{k+l}})),
\end{aligned}$$

as if $i \leq m_k(\sigma)$, $\sigma^{-1}(i) = i$ and $f(x_{d_{\sigma^{-1}(i)}}) = \lambda_{d_i} x_{d_i}$. Hence, we obtain the pre-Lie algebra $(T(V, f), \circ)$.

If x is an eigenvector of f with eigenvalue λ , the preceding formula gives

$$x^k \bullet x^l = \lambda \binom{k+l}{k} x^{k+l}.$$

Let us assume that $\lambda \neq 0$. For all $k \geq 0$, we put $y_k = (k+1)!x^k/\lambda$. As the characteristic of $\mathbb{K}$ is zero, $(y_k)_{k \geq 0}$ is a basis of $\mathbb{K}[x]$. We obtain

$$y_k \bullet y_l = \frac{k(k+1)}{k+l+1} y_{k+l},$$

so $[y_k, y_l] = (k-l)y_{k+l}$. Hence, if $\lambda \neq 0$, $\mathbb{K}[x]_+ \subseteq T(V, f)$ is isomorphic, as a Lie algebra, to the Faà di Bruno Lie algebra [7–9].

Remark 4.5 In particular, if V is one-dimensional and $f = \text{Id}_V$, we can work with a single decoration and delete it everywhere. The dual of the enveloping algebra of $T(V, f)$ is the subalgebra of $\mathcal{H}_{\text{CK}}$ generated by the elements t_n inductively defined by $t_1 = \bullet$ and $t_{n+1} = N(t_n)$,

$$\begin{aligned}
t_1 &= \bullet, \\
t_2 &= \downarrow, \\
t_3 &= \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \\
t_4 &= \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}, \\
t_5 &= \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + 6 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + 4 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + 4 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + 3 \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \downarrow \\ \bullet \end{array}.
\end{aligned}$$

This is the Connes–Moscovici subalgebra of $\mathcal{H}_{\text{CK}}$ [3, 4].

4.2 Examples from Control Theory

We now consider the group of Fliess operators [11]. We fix an integer $n \geq 1$. We denote by X^* the set of words in letters $x_0, \dots, x_n$. As a set, the group of Fliess operators is isomorphic to $G_{\text{Fliess}} = \mathbb{K}\langle x_0, \dots, x_n \rangle^n \approx \mathbb{K}^n\langle x_0, \dots, x_n \rangle$. Let us now describe the composition. Let $c, d = (d_1, \dots, d_n) \in G_{\text{Fliess}}$. We put

$$c = \sum_{w \in X^*} c_w w,$$

with $c_w \in \mathbb{K}^n$ for all $w \in X^*$.

- For all $0 \leq i \leq n$, we consider

$$\tilde{D}_{x_i} : \begin{cases} G^{\text{Flieiss}} \rightarrow G^{\text{Flieiss}} \\ e \rightarrow x_i e + x_0(d_i \sqcup e) \end{cases}$$

with $d_0 = 0$. If $w = x_{i_1} \dots x_{i_k} \in X^*$, we put $\tilde{D}_w = \tilde{D}_{x_{i_1}} \circ \dots \circ \tilde{D}_{x_{i_k}}$. By convention, $\tilde{D}_\emptyset = \text{Id}_{G^{\text{Flieiss}}}$.

- The composition $c \tilde{\diamond} d$ is defined by

$$c \tilde{\diamond} d = \sum_{w \in X^*} c_w \tilde{D}_w(\emptyset).$$

In particular, $\emptyset \tilde{\diamond} d = \tilde{D}_\emptyset(\emptyset) = \emptyset$.

- The product of G^{Flieiss} is then given by $c \diamond d = c \tilde{\diamond} d + d$.

Let $0 \leq i \leq n$, $c, d \in G^{\text{Flieiss}}$. We keep the same notation as before.

$$\begin{aligned} x_i c \tilde{\diamond} d &= \sum_{w \in X^*} c_w \tilde{D}_{x_i w}(\emptyset) \\ &= \sum_{w \in X^*} c_w \tilde{D}_{x_i} \tilde{D}_w(\emptyset) \\ &= \sum_{w \in X^*} c_w (x_i \tilde{D}_w(\emptyset) + x_0(d_i \sqcup \tilde{D}_w(\emptyset))) \\ &= x_i \left(\sum_{w \in X^*} c_w \tilde{D}_w(\emptyset) \right) + x_0 \left(d_i \sqcup \left(\sum_{w \in X^*} c_w \tilde{D}_w(\emptyset) \right) \right) \\ &= x_i(c \tilde{\diamond} d) + x_0(d_i \sqcup (c \tilde{\diamond} d)). \end{aligned}$$

In other words,

- $\tilde{\diamond}$ is $\mathbb{K}^n$ -linear on the left.
- For all $d \in \mathbb{K}\langle\langle x_0, \dots, x_n \rangle\rangle^n$, $\emptyset \tilde{\diamond} d = \emptyset$.
- For all $c, d \in \mathbb{K}\langle\langle x_0, \dots, x_n \rangle\rangle^n$, $x_i c \tilde{\diamond} d = x_i(c \tilde{\diamond} d) + x_0(d_i \sqcup (c \tilde{\diamond} d))$, with the convention $d_0 = 0$.

Notation Let ϵ_i be the i -th element of the canonical basis. We put $G_i = \epsilon_i \mathbb{K}\langle\langle x_0, \dots, x_n \rangle\rangle^n$.

From the left $\mathbb{K}^n$ -linearity and $\epsilon_i \epsilon_j = \delta_{ij} \epsilon_i$ we easily get

Lemma 4.6 *If $c, d \in \mathbb{K}\langle\langle x_0, \dots, x_n \rangle\rangle$, then*

$$(\epsilon_i \emptyset) \tilde{\diamond} v = \epsilon_i \emptyset, \quad (\epsilon_i x_j d) \tilde{\diamond} d = \epsilon_i x_j (c \tilde{\diamond} d) + \epsilon_i \delta_{ij} x_0((c \tilde{\diamond} d) \sqcup d).$$

Theorem 4.7 1. For all $1 \leq i \leq n$, G_i is a subgroup of G . Moreover, it is isomorphic to the group of characters associated to (V, f_i) , with $V = \text{Vect}(x_0, \dots, x_n)$ and

$$f_i: \begin{cases} V \rightarrow V \\ x_j \rightarrow \delta_{ij}x_0. \end{cases}$$

2. The group G is the direct product of the subgroups G_i , $1 \leq i \leq n$.

Proof 1. As a set, we identify G_i with $\mathbb{K}\langle x_0, \dots, x_n \rangle$, via the multiplication by ϵ_i . In this group, if $u, v \in \mathbb{K}\langle x_0, \dots, x_n \rangle$, by Lemma 4.6,

$$\emptyset \widetilde{\diamond} v = \emptyset, \quad x_j u \widetilde{\diamond} v = x_j(u \widetilde{\diamond} v) + \delta_{ij}x_0((u \widetilde{\diamond} v) \sqcup v).$$

We consider the character group associated to f_i . As $f_i^2 = 0$, the second statement of Theorem 2.17 becomes

$$\epsilon_i x_j u \widetilde{\diamond} v = x_j(u \widetilde{\diamond} v) + \delta_{ij}x_0((u \widetilde{\diamond} v) \sqcup v).$$

So G_i is isomorphic to this group of characters, via the multiplication by ϵ_i .

2. Let $1 \leq i, j \leq n$, with $i \neq j$. Then, by Lemma 4.6, if $u, v \in \mathbb{K}\langle x_0, \dots, x_n \rangle$, then

$$\begin{aligned} (\epsilon_i \emptyset) \widetilde{\diamond} (\epsilon_j v) &= \epsilon_i \emptyset, \\ (\epsilon_i x_k u) \widetilde{\diamond} (\epsilon_j v) &= \epsilon_i x_k(u \widetilde{\diamond} (\epsilon_j v)) + \epsilon_i \delta_{ik}x_0((u \widetilde{\diamond} (\epsilon_j v)) \sqcup (\epsilon_j v)) \\ &= \epsilon_i x_k(u \widetilde{\diamond} (\epsilon_j v)) + \epsilon_i \delta_{ik}x_0((u \widetilde{\diamond} (\epsilon_j v)) \sqcup v) \\ &= x_k(\epsilon_i u \widetilde{\diamond} (\epsilon_j v)) + 0. \end{aligned}$$

An easy induction on the length then proves that for all words u and all $v \in \mathbb{K}\langle x_0, \dots, x_n \rangle$, $(\epsilon_i u) \widetilde{\diamond} (\epsilon_j v) = \epsilon_i u$. By continuity of $\widetilde{\diamond}$, for all $u, v \in \mathbb{K}\langle x_0, \dots, x_n \rangle$, $(\epsilon_i u) \widetilde{\diamond} (\epsilon_j v) = \epsilon_i u$. Hence, for all $u, v \in \mathbb{K}\langle x_0, \dots, x_n \rangle$,

$$(\epsilon_i u) \diamond (\epsilon_j v) = \epsilon_i u + \epsilon_j v.$$

As a conclusion, G is the direct sum of the subgroups G_i .

The Lie algebra of G_i is the pre-Lie algebra $T(V, f_i) = \mathbb{K}\langle x_0, \dots, x_n \rangle$ with the composition associated to f_i . Let us give a few results on this pre-Lie algebra.

Theorem 4.8 $T(V, f_i)$ inherits a gradation such that x_i is homogeneous of degree 1 for all $1 \leq i \leq n$ and x_0 is homogeneous of degree 0. The formal series of this gradation is

$$\begin{aligned} F_{T(V, f_i)}(X) &= \frac{X}{1 - nX - X^2} \\ &= \sum_{k \geq 0} \frac{1}{\sqrt{n^2 + 4}} \left(\left(\frac{n + \sqrt{n^2 + 4}}{2} \right)^k - \left(\frac{n - \sqrt{n^2 + 4}}{2} \right)^k \right) X^k. \end{aligned}$$

Proof We put $V_1 = \text{Vect}(x_1, \dots, x_n)$ and $V_2 = \text{Vect}(x_0)$. It defines a gradation of V such that f_i is homogeneous of degree 1. We apply Corollary 2.12. $\square$

Examples

$$\begin{aligned}\dim T(V, f_i)_0 &= 0, \\ \dim T(V, f_i)_1 &= 1, \\ \dim T(V, f_i)_2 &= n, \\ \dim T(V, f_i)_3 &= n^2 + 1, \\ \dim T(V, f_i)_4 &= n(n^2 + 2), \\ \dim T(V, f_i)_5 &= n^4 + 3n^2 + 1, \\ \dim T(V, f_i)_7 &= n^6 + 5n^4 + 6n^2 + 1, \\ \dim T(V, f_i)_8 &= n(n^2 + 2)(n^4 + 4n^2 + 2), \\ \dim T(V, f_i)_9 &= (n^2 + 1)(n^6 + 6n^4 + 9n^2 + 1), \\ \dim T(V, f_i)_{10} &= n(n^4 + 3n^2 + 1)(n^4 + 5n^2 + 5).\end{aligned}$$

These are the Fibonacci polynomials, see sequence A011973 in [22]. Here are several specializations of these sequences, which can be found in [22]:

n	Ref.	n	Ref.	n	Ref.
1	A000045	2	A000129	3	A006190
4	A001076	5	A052918	6	A005668
7	A054413	8	A041025	9	A099371
10	A041041	11	A049666	12	A041061
13	A140455	14	A041085	16	A041113
18	A041145	20	A041181	22	A041221

These sequences are generically called generalized Fibonacci sequences. If $n = 1$, this is the Fibonacci sequence; if $n = 2$, this is the Pell sequence.

Remark 4.9 By Corollary 3.7, $T(V, f_i)$ is generated, as a pre-Lie algebra, by $\mathbb{K}\langle x_1, \dots, x_n \rangle$. Moreover, by Corollary 3.22, $T_+(V, f_i)$ is generated, as a Com-Prelie algebra, by $\text{Vect}(x_0, \dots, x_n)$ if and only if $n = 1$. We recover in this way a result of [10].

Appendix

Admissible words and Dyck paths

Recall that a Dyck path of length $2n$ is a path from $(0, 0)$ to (n, n) , made of steps $\rightarrow$ and $\uparrow$, always staying under the diagonal, but possibly touching it in other points than $(0, 0)$ and (n, n) . The set of Dyck paths of length $2n$ is denoted by $D(n)$.

Examples

$$D(0) = \{.\},$$

$$D(1) = \{\rightarrow \uparrow\},$$

$$D(2) = \{\rightarrow \uparrow \rightarrow \uparrow, \rightarrow \rightarrow \uparrow \uparrow\},$$

$$D(3) = \{\rightarrow \uparrow \rightarrow \uparrow \rightarrow \uparrow, \rightarrow \rightarrow \uparrow \uparrow \rightarrow \uparrow, \rightarrow \uparrow \rightarrow \rightarrow \uparrow \uparrow, \rightarrow \rightarrow \uparrow \rightarrow \uparrow \uparrow, \rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow\}.$$

Note that a nonempty Dyck path always starts by $\rightarrow$ and ends by $\uparrow$.

Lemma 4.10 *Let $\rightarrow \uparrow^{a_1} \rightarrow \uparrow^{a_2} \rightarrow \dots \rightarrow \uparrow^{a_n}$ be a path starting from $(0, 0)$, with $a_1, \dots, a_n \geq 0$.*

- *It is a path from $(0, 0)$ to (n, n) if and only if $a_1 + \dots + a_n = n$.*
- *This path is under the diagonal if and only if for all $1 \leq i \leq n$, $a_1 + \dots + a_i \leq i$.*

Proof The path contains $n \rightarrow$ and $a_1 + \dots + a_n \uparrow$, so is a path from $(0, 0)$ to $(n, a_1 + \dots + a_n)$. The second property is immediate. $\square$

Notation For all $n \geq 1$, we denote by $\mathcal{W}(n)$ the set of first lines of admissible biwords of length n , that is to say words $a_1 \dots a_n$ with letters in $\mathbb{N}$, such that:

- For all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n - i$.
- $a_1 + \dots + a_n = n - 1$.

We denote by $\mathcal{W}_\sigma(n)$ the set of first lines of σ -admissible biwords of length n , that is to say words $w_1 \dots w_k$ of length k such that $w_1, \dots, w_k$ are admissible words.

Proposition 4.11 *For all $n \geq 0$, the following map is a bijection:*

$$\theta_n: \begin{cases} \mathcal{W}(n+1) \rightarrow D(n) \\ a_1 \dots a_{n+1} \rightarrow \rightarrow \uparrow^{a_n} \dots \rightarrow \uparrow^{a_1}. \end{cases}$$

Proof First, θ_n is well defined: if $a_1 \dots a_{n+1} \in \mathcal{W}(n+1)$, by definition $a_{n+1} = 0$. Moreover, $a_1 + \dots + a_n = a_1 + \dots + a_{n+1} = n+1-1 = n$. By Lemma 4.10, it is a path from $(0, 0)$ to (n, n) . For all $1 \leq i \leq n$, $a_i + \dots + a_n = a_i + \dots + a_{n+1} \leq n+1-i$. By Lemma 4.10, it is under the diagonal, so it is indeed in $D(n)$.

θ_n is injective: if $\theta_n(a_1 \dots a_{n+1}) = \theta_n(b_1 \dots b_{n+1})$, then immediately $a_1 = b_1, \dots, a_n = b_n$; moreover, $a_{n+1} = b_{n+1} = 0$.

θ_n is surjective: if P is a Dyck path of length n , as it starts by $\rightarrow$ and contains exactly $n \rightarrow$ and $\uparrow$, it can be written under the form $\rightarrow \uparrow^{a_n} \dots \rightarrow \uparrow^{a_1}$, with $a_1 + \dots + a_n = n$. As it is under the diagonal, by Lemma 4.10, for all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n+1-i$. So $a_1 \dots a_n 0 \in \mathcal{W}(n+1)$ and $\theta(a_1 \dots a_n 0) = P$. $\square$

Corollary 4.12 *For all $n \geq 1$, $|\mathcal{W}(n)|$ is the $(n-1)$ -th Catalan number $\frac{1}{n} \binom{2n-2}{n-1}$.*

Corollary 4.13 1. *For all $n \geq 0$, $|\mathcal{W}_\sigma(n)|$ is the n -th Catalan number $\frac{1}{n+1} \binom{2n}{n}$.*
 2. *Let $a_1, \dots, a_n$ be a word with letters in $\mathbb{N}$. It is σ -admissible if and only if for all $1 \leq i \leq n$, $a_i + \dots + a_n \leq n - i$.*

Proof 1. Let $C(X)$ be the formal series of Catalan numbers, then

$$C(X) = \sum_{k=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} X^n = \frac{1 - \sqrt{1-4X}}{2X}.$$

Then the formal series of admissible words is

$$f(X) = \sum_{n \geq 1} |\mathcal{W}(n)| X^n = XC(X) = \frac{1 - \sqrt{1-4X}}{2},$$

so $f^2(X) - f(X) + X = 0$ or, equivalently,

$$\frac{1}{1-f(X)} = \frac{f(X)}{X}.$$

By the unicity of decomposition of σ -admissible words in Lemma 3.17, the formal series of σ -admissible words is

$$g(X) = \sum_{n \geq 0} |\mathcal{W}_{\sigma}(n)| X^n = \frac{1}{1-f(X)} = \frac{f(X)}{X} = C(X),$$

so $|\mathcal{W}_{\sigma}(n)|$ is the n -th Catalan number.

2. We denote by $\mathcal{W}'_{\sigma}(n)$ the set of words $a_1 \dots a_n$ such that for all i , $a_i + \dots + a_n \leq n-i$. By Lemma 3.17, $\mathcal{W}_{\sigma}(n) \subseteq \mathcal{W}'_{\sigma}(n)$. By Lemma 3.17, the following map is well defined:

$$\alpha_n: \begin{cases} \mathcal{W}'_{\sigma}(n) \rightarrow \mathcal{W}(n+1) \\ a_1 \dots a_n \rightarrow (n-a_1-\dots-a_n)a_1 \dots a_n. \end{cases}$$

It is clearly injective, so $|\mathcal{W}'_{\sigma}(n)| \leq |\mathcal{W}(n+1)| = |\mathcal{W}_{\sigma}(n)|$. So $\mathcal{W}_{\sigma}(n) = \mathcal{W}'_{\sigma}(n)$. $\square$

Image of the morphism Φ_{CPL}

Proposition 4.14 $\mathfrak{g}_{\mathcal{D}} = \Phi_{\text{PL}}(\text{Prelie}(\mathcal{D}))$ is included in the space generated by admissible words and $\Phi_{\text{CPL}}(\text{ComPrelie}(\mathcal{D}))$ is included in the space generated by σ -admissible words.

Proof Let t be a partitioned tree with n vertices and k blocks, and let f be a linear extension of t . We put $\text{fert}(f(i)) = a_i$ for all i . It is not difficult to prove that $a_1 + \dots + a_n = k-1$. As $k \leq n$, $a_1 + \dots + a_n \leq n-1$; if t is a rooted tree, $k = n$ and $a_1 + \dots + a_n = n-1$. Let us prove that $a_i + \dots + a_n \leq n-i$ for all $1 \leq i \leq n$ by induction on n . It is obvious if $n = 0$. If $n \geq 1$, we already proved the result if $i = 1$. Let us assume that $i \geq 2$. We put $t' = f^{-1}(\{2, \dots, n\})$. As f is a linear extension of

t, t' is an ideal of t , and $b_1 \dots b_{n-1} = a_2 \dots a_n$ appears in $\Phi_{\text{CPL}}(t')$. By the induction hypothesis,

$$a_i + \dots + a_n = b_{i-1} + \dots + b_{n-1} \leq (n-1) - (i-1) = n-i.$$

Finally, if t is a partitioned tree, any biword appearing in $\Phi_{\text{CPL}}(t)$ is σ -admissible; if t is a rooted tree, any biword appearing in $\Phi_{\text{CPL}}(t)$ is admissible. $\square$

Remark 4.15 It is not difficult to see that $\mathfrak{g}_{\mathcal{D}}$ is strictly included in the space of admissible words and that $\Phi_{\text{CPL}}(\text{ComPrelie}(\mathcal{D}))$ is strictly included in the space of σ -admissible words.

Corollary 4.16 *The space of σ -admissible words, with the shuffle product $\sqcup$ and the pre-Lie product $\bullet$ of $T(V_{\mathcal{D}}, f_{\mathcal{D}})$, is a Com-Prelie algebra.*

Proof Let w be a σ -admissible biwords. Let us first prove that there exists a partitioned tree t such that w appears in $\Phi_{\text{CPL}}(t)$. We proceed by induction on the length n of w . If $n = 0$, then we take $t = 1$. Let us assume the result at all ranks $< n$. If $w = a_1 \dots a_n$ is admissible, then the second point implies that $a_2 \dots a_n$ is σ -admissible, and $a = 1 \binom{n-a_1-\dots-a_n}{d}$ for a certain $d \in \mathcal{D}$. By the induction hypothesis, there exists a forest t' such that $a_2 \dots a_n$ appears in $\Phi_{\text{CPL}}(t')$. Then $a_1 \dots a_n$ appears in $\Phi_{\text{CPL}}(\bullet_d \bullet t')$. If $w = w_1 \dots w_k$, with $w_1, \dots, w_k$ admissible and $k \geq 2$, then for all i there exists t_i such that w_i appears in $\Phi_{\text{CPL}}(t_i)$. Hence, $w_1 \dots w_k$, being a particular shuffling of $w_1, \dots, w_k$, appears in $\Phi_{\text{CPL}}(t_1) \sqcup \dots \sqcup \Phi_{\text{CPL}}(t_k) = \Phi_{\text{CPL}}(t_1 \dots t_k)$.

Let us consider now two σ -admissible biwords u and u' . There exists admissible forests t and t' such that u appears in $\Phi_{\text{CPL}}(t)$ and u' appears in $\Phi_{\text{CPL}}(t')$. Then any word appearing in $u \sqcup u'$ or in $u \bullet u'$ appears in $\Phi_{\text{CPL}}(t) \sqcup \Phi_{\text{CPL}}(t') = \Phi_{\text{CPL}}(t \sqcup t')$ or in $\Phi_{\text{CPL}}(t) \bullet \Phi_{\text{CPL}}(t') = \Phi_{\text{CPL}}(t \bullet t')$, so is σ -admissible by Proposition 4.14. $\square$

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