Existence of slices on a tame context

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Abstract We study the ramification theory for actions involving group schemes, focusing on the tame ramification. We consider the notions of tame quotient stack introduced in Abramovich et al. (Ann Inst Fourier (Grenoble) 58(4):1057–1091, 2008) and of tame action introduced in Chinburg et al. (Duke Math J 82(2):269–308, 1996). We establish a local slice theorem for unramified actions, prove interesting lifting properties for linearly reductive group schemes, and establish a slice theorem for actions by commutative group schemes inducing tame quotient stacks. We show that these actions are induced from an action of an extension of the inertia group on a finitely presented flat neighborhood. Finally, we consider the notion of tame action and its relation to tame quotient stacks.

Keywords Group schemes · Ramification · Freeness · Tameness · Slice theorem · Linearly reductive group schemes · Lifting · Quotient stack

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1 Introduction

We consider an action, denoted by (X, G) , of a flat group scheme G , of finite type over S , on a scheme X , locally of finite type over S , where the base scheme S is the spectrum of a commutative, unitary noetherian ring R . Throughout, we suppose that any orbit of the action is contained in an affine scheme.

A notion of *slices* for the action (X, G) was introduced in [8], generalizing the one defined previously in [15].

Definition 1.1 We say that the action (X, G) over S admits *étale* (resp. *finitely presented flat*) *slices* if:

- The quotient $Q = X/G$ exists (the quotient X/G stands for the categorical quotient for the action (X, G) in the category of algebraic spaces).
- For any topological point $q \in Q$, there exist:
 - an S -scheme Q' and an étale (resp. finitely presented flat) S -morphism $Q' \rightarrow Q$ such that there is $q' \in Q'$ which maps to q via this morphism;
 - a closed subgroup G_q of the group scheme $G_{Q'} = G \times_Q Q'$ over Q' which stabilizes some point p' of $X \times_Q Q'$ above q' . That is, there is a group scheme isomorphism over the residue field $k(p')$ at p' ,

$$G_q \times_{Q'} \text{Spec}(k(p')) \simeq I_{G_{Q'}}(p'),$$

where $I_{G_{Q'}}(p')$ denotes the inertia group at p' of the base changed action $(X \times_Q Q', G_{Q'})$;

- a Q' -scheme Z_q with a G_q -action such that the quotient Z_q/G_q exists and equals Q' . The group G_q acts on $Z_q \times_{Q'} G_{Q'}$ on the right by $(z, g)h = (zh, h^{-1}g)$ (on points), we require also that the quotient $(Z_q \times_{Q'} G_{Q'})/G_q$ exists and that the base changed action $(X \times_Q Q', G_{Q'})$ is induced by (Z_q, G_q) . That is, there exists a Q' -scheme $G_{Q'}$ -equivariant isomorphism, called *balanced product*:

$$X \times_Q Q' \simeq (Z_q \times_{Q'} G_{Q'})/G_q,$$

where $G_{Q'}$ acts on $(Z_q \times_{Q'} G_{Q'})/G_q$ via the factor $G_{Q'}$ (so that, the quotient for the action $(X \times_Q Q', G_{Q'})$ exists and equals Q').

We call the subgroup G_q the *slice group* and Z_q the *slice scheme*.

The action $(X \times_Q Q', G_{Q'})$ should be thought of as a $G_{Q'}$ -stable neighborhood of an orbit. Such a neighborhood induced from an action (Z, H) , where H is a lifting of an inertia group of a point, is called a *tubular neighborhood*.

Roughly speaking, for the étale (resp. fppf) topology, an action which admits slices can be described by the action of a lifting of an inertia group at some given point. It is well known that locally, for the étale topology, actions of constant group schemes (and slightly more generally étale group schemes) are induced from actions of their inertia groups: one can reconstruct the original action from the action of an inertia group at a point, Lemma 4.1 (this is a direct extension of the classical result on decomposition of finite extensions of valuation field after moving to the completion [20, II, Section 3,

Théorème 1, Corollaire 4 and Proposition 4]). This statement is an instance of a *slice theorem*, Theorem 4.2.

Here, we study the existence of slices, Theorem 5.5, under a tameness hypothesis on the action (X, G) . One motivation for studying slices in this generality is the theory of tame covers, in the sense of Grothendieck and Murre: these admit étale slices, see Abhyankar's lemma in [7, 12]. Another motivation is the fundamental theorem of Luna [15] which states that actions of linearly reductive algebraic groups on affine varieties are tame and admit slices.

In the simple case, when the action by a finite flat group scheme over S admits a trivial inertia group at some topological point x , we establish a local slice theorem proving that in this case the action is locally free on the fppf topology, Theorem 5.5.

Tame stacks were introduced by Abramovich, Olsson and Vistoli in [1, Section 3] in their study of certain ramification issues arising in moduli theory. [1, Theorem 3.2], Theorem 7.4, characterized the tameness applied to the quotient stack $[X/G]$ for the action (X, G) by the property that all the inertia groups at topological points of X are linearly reductive (it suffices to require this only at geometric points). As a consequence, this notion of tameness on $[X/G]$ generalizes the classical notion of tameness in algebraic number theory, Remark 7.5. Moreover, it allows one to define a notion of “local tameness” on topological points of X , Definition 7.7. As a *slice theorem*, we manage to characterize tameness for quotient stacks for actions of finite, flat commutative group schemes over S via the existence of finitely presented flat slices with linearly reductive slice groups.

Theorem 1.2 *Let S be the spectrum of a noetherian ring R and X a scheme of finite type over S . Assume that the group scheme G is flat, commutative and finite over S and that any orbit of the action of G on X is included in an affine scheme. The following assertions are equivalent:*

- (a) *The quotient stack $[X/G]$ is tame.*
- (b) *The action (X, G) admits finitely presented flat slices at any topological point $x \in X$ such that the slice group at $x \in X$ is linearly reductive and the slice scheme is a closed subscheme of some flat, finitely presented base change of X .*

Furthermore, this lifting of the inertia group can be constructed also in the non-commutative case, as a subgroup of the initial group and as a flat and linearly reductive group scheme.

Theorem 1.3 (key technical result) *Let S be the spectrum of a noetherian ring R , $s \in S$ a point, G a finite, flat group scheme over S , and H_0 a finite, linearly reductive closed subgroup scheme of $G_{k(s)}$ over $\text{Spec}(k(s))$. Then there exist a flat, finitely presented morphism $S' \rightarrow S$, a point $s' \in S'$ mapping to s , and a flat linearly reductive closed subgroup scheme H of $G_{S'}$ over S' whose pullback $H_{k(s')}$ is isomorphic to the pullback of $H_{0k(s')}$.*

Independently, when X and G are affine schemes, Chinburg, Pappas, Erez and Taylor [8, Section 2] introduced the notion of *tame actions* via the existence of a total integral, Definition 8.2. Thus it is natural to anticipate a relationship between these two notions of tameness. We prove that under quite general hypotheses, *tame actions*

define *tame quotient stacks* and under additional hypotheses such as finiteness and flatness of the quotient, the two notions are equivalent.

Theorem 1.4 *Let S be the spectrum of a noetherian ring, X be affine, of finite type over S , and G be affine, flat of finite type over S . Suppose that the inertia stack is finite.*

- *If the action (X, G) is tame then the quotient stack $[X/G]$ is tame.*
- *Suppose, moreover, G is finite, flat over S , and X is flat over the quotient X/G . Then the action (X, G) is tame if and only if the quotient stack $[X/G]$ is tame.*

Thus, the previous results apply to this notion of tameness and we obtain a more precise answer to [8, Section 4, Questions 2 and 3].

2 Hypotheses and notation

Throughout, we fix the following notation. We write fppf for “faithfully flat and finitely presented” (fidèlement plate de présentation finie). As mentioned in the introduction,

- S refers to the spectrum of a noetherian ring R ,
- X to a scheme locally of finite type over S ,
- G to a flat group scheme of finite type over S .

All schemes are S -schemes, and $x \in X$ denotes a topological point of X .

For V a scheme over S and S' an S -scheme, let $V_{S'}$ (resp. ${}_S V$) be the base change $V \times_S S'$ (resp. $S' \times_S V$). In particular, for V a scheme over S and R' an R -algebra, we write $V_{R'}$ (resp. ${}_{R'} V$) instead of $V_{\mathrm{Spec}(R')}$ (resp. ${}_{\mathrm{Spec}(R')} V$) for the base change $V \times_S \mathrm{Spec}(R')$ (resp. $\mathrm{Spec}(R') \times_S V$). Given S -morphisms of schemes $V \rightarrow W$ and $Z \rightarrow W$, we have the fiber product

$$\begin{array}{ccc} V \times_W Z & \xrightarrow{\mathrm{pr}_2} & Z \\ \mathrm{pr}_1 \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

together with the natural projections.

When X or G are affine schemes, we write

- B for the commutative algebra such that $X = \mathrm{Spec}(B)$,
- $A = (A, \delta, \epsilon, s)$ for the commutative R -Hopf algebra such that $G = \mathrm{Spec}(A)$, where δ denotes the comultiplication, ϵ the unit map and s the antipode.

We consider an action of G on X over S which we denote by (X, G) ; we write $\mu_X: X \times_S G \rightarrow X$ for its structure map, (we write $x \cdot g = \mu_X(x, g)$, for any point $x \in X$ and $g \in G$) and $\rho_B: B \rightarrow B \otimes_R A$ for the structure map giving the A -comodule structure on B , when X and G are affine schemes, see [8, Section 1.b].

We will use a *sigma notation*, well-established in the Hopf algebra literature. Specifically, for $a \in A$, we write $\delta(a) = \sum_{(a)} a_1 \otimes a_2$. This presentation itself is purely

symbolic; the terms a_1 and a_2 do not stand for particular elements of A . The comultiplication δ takes values in $A \otimes_R A$, and we know that

$$\delta(a) = (a_{11} \otimes a_{12}) + (a_{21} \otimes a_{22}) + (a_{31} \otimes a_{32}) + \cdots + (a_{n1} \otimes a_{n2}),$$

for some elements a_{ij} of A and some integer n . The sigma notation is a way to separate a_{i1} from a_{j2} . In other words, the notation a_1 stands for the generic a_{i1} and the notation a_2 stands for the generic a_{j2} . Similarly, for any right comodule (resp. left comodule) M , any $m \in M$, we write $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ (resp. ${}_M \rho(m) = \sum m_{(-1)} \otimes m_{(0)}$).

For T an S -scheme and $f: T \rightarrow X$, the *inertia group at f* (denoted as $I_G(f)$) is the fiber product

$$\begin{array}{ccc} I_G(f) & \xrightarrow{\text{pr}_2} & T \\ \text{pr}_1 \downarrow & & \downarrow \Delta \circ f \\ X \times_S G & \xrightarrow{(\mu_X, \text{pr}_1)} & X \times_S X, \end{array}$$

where

$$(\mu_X, \text{pr}_1): X \times_S G \rightarrow X \times_S X$$

is the *Galois map* sending (w, g) to $(w \cdot g, w)$.

For $x \in X$, we denote by $k(x)$ the *residue field at x* and by $\xi_x: \text{Spec}(k(x)) \rightarrow X$ the canonical morphism defined by x . We write $I_G(x)$ instead of $I_G(\xi_x)$ for the inertia group at x .

We denote by $\mathcal{M}^A$ (resp. ${}^A\mathcal{M}$) the category of the right (resp. left) A -comodules and by ${}_R\mathcal{M}$ the category of R -modules. Let (M, ρ_M) and (N, ρ_N) be two right A -comodules. Denote by $\text{Com}_A(M, N)$ the R -module of A -comodule morphisms from M to N . Recall that an R -linear map $g: M \rightarrow N$ is called *A -comodule morphism* if $\rho_N \circ g = (g \otimes \text{Id}_A) \circ \rho_M$.

We denote by ${}_B\mathcal{M}^A$ the category of (B, A) -modules. Its objects are R -modules N which are both right A -comodules and B -modules such that the structural map of comodules is B -linear, i.e.,

$$\rho_N(bn) = \rho_B(b) \rho_N(n) = \sum b_{(0)} n_{(0)} \otimes b_{(1)} n_{(1)} \quad \text{for all } n \in N, \quad b \in B.$$

The morphisms of ${}_B\mathcal{M}^A$ are morphisms that are simultaneously B -linear and A -comodule morphisms. The set of morphisms from M to N , $M, N \in {}_B\mathcal{M}^A$, we denote by ${}_B\text{Bim}^A(M, N)$. We denote by $\text{Qcoh}^G(X)$ the G -equivariant quasi-coherent sheaves over X , see [1, 2.1]. We have $\text{Qcoh}^G(X) \simeq {}_B\mathcal{M}^A$. For $\mathcal{X}$ an Artin stack, we denote by $\text{Qcoh}(\mathcal{X})$ the quasi-coherent sheaves over $\mathcal{X}$.

3 Generalities about quotients

3.1 Categorical and universal quotients

Definition 3.1 We say that the action (X, G) admits a *categorical quotient* of (X, G) in the category of algebraic spaces, denoted by X/G , if

- X/G is an algebraic space,
- there is a morphism $\phi: X \rightarrow X/G$ such that $\phi \circ \mu_X = \phi \circ \text{pr}_1$, and for any algebraic space Z and any morphism $\psi: X \rightarrow Z$ such that $\psi \circ \mu_X = \psi \circ \text{pr}_1$, ψ factorizes via ϕ , e.g. there is a morphism $X/G \rightarrow Z$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Z \\ \phi \downarrow & \nearrow & \\ X/G & & \end{array}$$

When X/G exists, we refer to it simply as the *quotient* of (X, G) .

Remark 3.2 For G finite, flat over S the quotient X/G exists, see [9, Theorem 3.1]. Hence we can consider X as a scheme over $Q = X/G$. Then the data of the S -scheme morphism $\mu_X: X \times_S G \rightarrow X$, defining an action (X, G) over S , is equivalent to the data of the Q -scheme morphism $\mu_X: X \times_Q G_Q \rightarrow X$, defining an action (X, G_Q) over Q .

Definition 3.3 Let $X = \text{Spec}(B)$ and $G = \text{Spec}(A)$ be affine schemes, we call the subring of B defined by

$$B^A = \{b \in B : \rho_B(b) = b \otimes 1\}$$

the *ring of invariants of the action* (X, G) . We denote by $Y = \text{Spec}(B^A)$ the associated affine scheme and by $\pi: X \rightarrow Y$ the morphism induced by the inclusion $B^A \subseteq B$.

Remark 3.4 If S is the spectrum of a noetherian ring R , $X = \text{Spec}(B)$ is affine, of finite type over S , and $G = \text{Spec}(A)$ is affine, finite, flat over S , then the quotient X/G is precisely Y , see [9, Theorem 3.1].

Definition 3.5 A quotient $X \rightarrow Q = X/G$ is called a *universal quotient* if, for any flat morphism of algebraic spaces $Q' \rightarrow Q$, the map $X \times_Q Q' \rightarrow Q'$ given by the projection is a quotient for the action $(X \times_Q Q', G_{Q'})$.

3.2 Torsors

Definition 3.6 We say that a U -scheme Z together with a G -action over S , where U is an S -scheme, is a *G -torsor over U* if

- $Z \rightarrow U$ is fppf, and

- the Galois map $(\mu_Z, \text{pr}_1): Z \times_S G \simeq Z \times_U Z$ is an isomorphism.

Remark 3.7 We consider torsor on the fppf topology since it is a more reasonable topology to work with under our assumptions.

3.3 Quotient stack

Definition 3.8 Let T be an S -scheme. The *quotient stack associated to the action* (X, G) , denoted by $[X/G]$, is the following groupoid:

$$[X/G](T) = \{(P, \varphi) : P \text{ is a } G\text{-torsor over } T, \\ \varphi \text{ is a } G\text{-equivariant morphism } P \rightarrow X\}.$$

The quotient stack associated to the trivial action of G on S is called the *classifying stack*, denoted by $B_S G$.

Remark 3.9 We get a canonical S -morphism $p: X \rightarrow [X/G]$ by sending a T -point $T \rightarrow X$ to the trivial G -torsor $T \times_S G$ over T together with the G -equivariant morphism $T \times_S G \rightarrow X$. The morphism p is fppf hence, by the Artin criterium [14, Théorème 10.1], $[X/G]$ is an Artin stack. Moreover, we have an isomorphism $(\mu_X, \text{pr}_1): X \times_S G \simeq X \times_{[X/G]} X$.

It is important to notice that for G a flat group scheme the quotient stack $[X/G]$ can always be defined, while X/G does not necessarily exist. Even if X/G does exist, the quotient stack $[X/G]$ gives more information about the action than the categorical quotient X/G . More precisely, unlike the categorical quotient, the topological points of the quotient stack $[X/G]$ have automorphisms non-necessarily trivial, they correspond to the inertia groups of the action.

Definition 3.10 Given a morphism $f: T \rightarrow [X/G]$, where T is a scheme, we define the group of automorphisms of f , denoted $\underline{\text{Aut}}_T(f)$, as the fiber product

$$\begin{array}{ccc} \underline{\text{Aut}}_T(f) & \longrightarrow & T \\ \downarrow & & \downarrow f \\ \mathcal{I}_{[X/G]} & \xrightarrow{p_2} & [X/G], \end{array}$$

where $\mathcal{I}_{[X/G]}$, called the *inertia stack*, is defined as the fiber product

$$\begin{array}{ccc} \mathcal{I}_{[X/G]} & \longrightarrow & [X/G] \\ \downarrow & & \downarrow \Delta_{[X/G]/S} \\ [X/G] & \xrightarrow{\Delta_{[X/G]/S}} & [X/G] \times_S [X/G]. \end{array}$$

It turns out that these automorphisms correspond to the inertia groups for the action (X, G) .

Lemma 3.11 *Let $f: T \rightarrow X$ be a T -point of X . Set $\bar{f} = p \circ f: T \rightarrow X \rightarrow [X/G]$. Then, the automorphism group $\underline{\text{Aut}}_T(\bar{f})$ of the T -point $\bar{f}$ and the inertia group $I_G(f)$ at the T -point f are isomorphic as group schemes over T .*

Proof

$$\begin{aligned} \underline{\text{Aut}}_T(\bar{f}) &\simeq ([X/G] \times_{[X/G] \times_S [X/G], \Delta, \Delta} [X/G]) \times_{[X/G]} T \\ &\simeq ([X/G] \times_{[X/G] \times_S [X/G], \Delta, (p, p)} (X \times_S X)) \times_{X \times_S X, \text{pr}_2, \Delta \circ f} T \\ &\simeq (X \times_{[X/G]} X) \times_{X \times_S X} T \\ &\simeq (G \times_S X) \times_{X \times_S X, (\mu_X, \text{pr}_1), \Delta \circ f} T \simeq I_G(f). \end{aligned} \quad \square$$

Remark 3.12 A slice theorem reducing, fppf locally, the action (X, G) to an action by a lifting of an inertia group on a scheme built only by means of $X \rightarrow X/G$ would imply that the data of the quotient stack associated to X and G is enough to rebuild locally the action (X, G) . This is the case for actions of finite and étale group schemes, Lemma 4.1.

3.4 Coarse moduli spaces

Definition 3.13 A coarse moduli space for the quotient stack $[X/G]$ is a pair (M, ϕ) , where M is an algebraic space and $\phi: [X/G] \rightarrow M$ a morphism of stacks, such that

- any morphism from $[X/G]$ to an algebraic space factors through ϕ , and
- for any algebraically closed field Ω ,

$$|[X/G](\Omega)| \simeq M(\Omega),$$

where $[X/G](\Omega)$ is the set of isomorphism classes of the geometric points of $[X/G]$ taking value on Ω .

We say that the homomorphism ϕ is *universal* if, for any flat morphism $M' \rightarrow M$, the pair

$$(M', \phi': [X/G] \times_M M' \rightarrow M')$$

is a coarse moduli space for $[X/G] \times_M M'$.

Remark 3.14 The notion of coarse moduli spaces can be generalized to general Artin stacks. Coarse moduli spaces for quotient stacks are, in particular, categorical quotients in the category of algebraic spaces for the action (X, G) . More precisely,

- Let M be an algebraic space. The datum of a morphism $\phi: [X/G] \rightarrow M$ is equivalent to the datum of a morphism of S -algebraic spaces $f: X \rightarrow M$ such that $f \circ \text{pr}_1 = f \circ \mu_X$. In particular, when X and G are affine schemes, the canonical map $\pi: X \rightarrow Y$ induces a map $\rho: [X/G] \rightarrow Y$.
- If $f: X \rightarrow M$ is a geometric quotient which is categorical in the category of algebraic spaces, then $[X/G] \rightarrow M$ is a coarse moduli space.

If S is the spectrum of a noetherian ring, X is affine, of finite type over S , and G is affine, finite, flat over S , then $\rho: [X/G] \rightarrow Y$ is a coarse moduli space for $[X/G]$, see [9, Section 3].

Theorem 3.15 *Let S be the spectrum of a noetherian ring R , X locally of finite type over S , and G flat of finite type over S . Suppose that the inertia stack is finite. Then the quotient stack $[X/G]$ admits a coarse moduli space $\phi: [X/G] \rightarrow M$ such that ϕ is proper. Moreover, ϕ induces a functor $\phi_*: \mathrm{Qcoh}([X/G]) \rightarrow \mathrm{Qcoh}(M)$.*

Proof Since G is finitely presented over S , so is $X \rightarrow [X/G]$ (X is a G -torsor over $[X/G]$). Since $X \rightarrow [X/G]$ is surjective, flat of finite presentation and $X \rightarrow S$ is also of finite presentation, $[X/G] \rightarrow S$ is of finite presentation. By [9, Theorem 1.1], the hypotheses ensures that the quotient stack admits a coarse moduli space that we denote by $\phi: [X/G] \rightarrow M$ such that ϕ is proper. In particular, ϕ is quasi-compact and quasi-separated, thus by [18, Lemma 6.5(i)], the induced morphism on the quasi-coherent sheaves $\phi_*: \mathrm{Qcoh}([X/G]) \rightarrow \mathrm{Qcoh}(M)$ is well defined. $\square$

3.5 On the exactness of the functor of invariants

Definition 3.16 Let S be the spectrum of a noetherian ring R , $X = \mathrm{Spec}(B)$ affine, of finite type over S , and $G = \mathrm{Spec}(A)$ affine, finite, flat over S . We write $Y = \mathrm{Spec}(B^A)$. For a (B, A) -module M , we call the submodule of M defined by the exact sequence

$$0 \rightarrow (M)^A \xrightarrow[M \otimes 1]{\rho_M} M \otimes A$$

the *submodule of invariants* $(M)^A$. This defines a functor called the *functor of invariants*

$$(-)^A: {}_B\mathcal{M}^A \rightarrow {}_C\mathcal{M}, \quad M \mapsto (M)^A,$$

which is left exact, by definition.

By [1, 2.1] or [16, Chapter IV, Section 9], we have $\mathrm{Qcoh}([X/G]) \simeq \mathrm{Qcoh}^G(X)$. As a consequence we easily obtain the following lemma.

Lemma 3.17 *Let S be the spectrum of a noetherian ring R , $X = \mathrm{Spec}(B)$ affine, of finite type over S , and $G = \mathrm{Spec}(A)$ affine, finite, flat over S . $Y = \mathrm{Spec}(B^A)$. The map $\rho: [X/G] \rightarrow Y$ (defined in Remark 3.14) induces a functor $\rho_*: \mathrm{Qcoh}([X/G]) \rightarrow \mathrm{Qcoh}(Y)$. Moreover, the functor ρ_* is exact if and only if the functor of invariants $(-)^A: {}_B\mathcal{M}^A \rightarrow {}_C\mathcal{M}$ is exact.*

Theorem 3.18 *Let S be the spectrum of a noetherian ring R , $X = \mathrm{Spec}(B)$ affine, of finite type over S , and $G = \mathrm{Spec}(A)$ affine, finite, flat over S . $Y = \mathrm{Spec}(B^A)$. Suppose that the inertia stack is finite. If the functor of invariants*

$$(-)^A: {}_B\mathcal{M}^A \rightarrow {}_{B^A}\mathcal{M}$$

is exact, then the map $\rho: [X/G] \rightarrow Y$ is a coarse moduli space of $[X/G]$ and ρ is proper.

Proof By [2, Theorem 6.6], the quotient $\pi: X \rightarrow Y$ is categorical in the category of algebraic spaces. [9, Theorem 1.1] insures that the quotient stack admits a coarse moduli space that we denote $\rho: [X/G] \rightarrow M$ such that ρ is proper. But, since $\rho \circ \rho: X \rightarrow [X/G] \rightarrow M$ and $\pi: X \rightarrow Y$ are categorical quotients in the category of algebraic spaces, we obtain by unicity that $M \simeq Y$ and $\rho: [X/G] \rightarrow Y$ is a coarse moduli space. $\square$

4 Slice theorem for actions by finite étale (smooth) group scheme

Let Γ be an abstract finite group. Denote by $\underline{\Gamma}$ the constant group scheme associated to Γ over $S = \text{Spec}(R)$, where R is a commutative, noetherian, unitary ring. The associated Hopf algebra is then of the form $\text{Map}(\Gamma, R)$. We write C for the ring of invariants B^A .

We recall first some important facts.

- The data of an action of $\underline{\Gamma}$ on $X = \text{Spec}(B)$ is equivalent to the data of an action of Γ on B , $C = B^A \simeq B^\Gamma = \{b \in B : \gamma b = b\}$, see [8, Section 1.d], and $X/\underline{\Gamma} = Y = \text{Spec}(C)$, see [9, Theorem 3.1]. This particular case also permits to make the transition from algebra and classic number theory to the algebraic geometry context and to understand the later definitions.
- For $\mathfrak{q} \in Y$, Γ operates transitively on the prime ideals above $\mathfrak{q}$, see [4, Chapter V, Section 2, Theorem 2].
- The inertia group scheme at a point $\mathfrak{p} \in X$ is the constant group scheme associated to the algebraic inertia group

$$\Gamma_0(\mathfrak{p}) = \{\gamma \in \Gamma : \gamma \mathfrak{p} = \mathfrak{p} \text{ and } \gamma.x = x, x \in k(\mathfrak{p})\}$$

for the action Γ on B induced by (X, G) at the ideal $\mathfrak{p}$, see [10, III, Section 2, no. 2, Example 2.4].

In order to alleviate writing, we use the fact that the morphism $B \rightarrow C$ is finite [4, Chapter V, Section 1, no. 9, Proposition 22]. Let $\mathfrak{q}$ be a prime ideal on C , if we write $C_{\mathfrak{q}}^{\text{sh}}$ for the strict henselization of C at $\mathfrak{q}$, then the base change $B \otimes_C C_{\mathfrak{q}}^{\text{sh}} \rightarrow C_{\mathfrak{q}}^{\text{sh}}$ is also finite and since we want to obtain a local result for the étale topology, without loss of generality, we may suppose in what follows that the base is the ring of the invariant $C = B^A$ and a local strictly henselian ring with maximal ideal $\mathfrak{q}$ and that B is semi-local over C that is, the product of its local components $(B_{\mathfrak{m}})$, where $\mathfrak{m}$ runs through the set of prime (maximal) ideals of B over the maximal ideal $\mathfrak{q}$ of C . Let $\mathfrak{p}$ be a prime ideal over $\mathfrak{q}$. We define $\text{Map}^{\Gamma_0(\mathfrak{p})}(\Gamma, B_{\mathfrak{p}})$ to be the set of maps

$$u: \Gamma \rightarrow B_{\mathfrak{p}}$$

such that $u(\gamma i) = i^{-1}u(\gamma)$, for any $\gamma \in \Gamma$ and $i \in \Gamma_0(\mathfrak{P})$. It carries a Γ -action via

$$\lambda.u(\gamma) = u(\lambda^{-1}\gamma), \quad \lambda, \gamma \in \Gamma.$$

With these notations, we easily obtain the following Γ -equivariant isomorphism:

$$(\mathrm{Map}(\Gamma, C) \otimes_C B_{\mathfrak{p}})^{\Gamma_0(\mathfrak{p})} \simeq \mathrm{Map}^{\Gamma_0(\mathfrak{p})}(\Gamma, B_{\mathfrak{p}}).$$

Using the above, one concludes that the action of Γ on B can be rebuilt locally at $\mathfrak{q}$ thanks to the action of the inertia group at $\mathfrak{p}$ on $B_{\mathfrak{p}}$.

Lemma 4.1 [19, Chapitre X] *We have a canonical isomorphism ϕ compatible with the actions of Γ defined by*

$$\phi: B \rightarrow \mathrm{Map}^{\Gamma_0(\mathfrak{p})}(\Gamma, B_{\mathfrak{p}}), \quad b \mapsto u: \gamma \mapsto (\gamma^{-1}b)_{\mathfrak{p}}.$$

Moreover, $B^{\Gamma} \simeq B_{\mathfrak{p}}^{\Gamma_0(\mathfrak{p})}$.

Proof Since $B_{\mathfrak{p}} \simeq iB_{\mathfrak{p}}$, for any $i \in \Gamma_0(\mathfrak{p})$, ϕ is well defined. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the prime ideals over $\mathfrak{q}$, we put $\mathfrak{p}_1 = \mathfrak{p}$. Since the action of Γ is transitive on $\mathfrak{p}_i$, for any $i \neq 1$, we have $\gamma_i \in \Gamma$ such that $\mathfrak{p}_i = \gamma_i \mathfrak{p}$. Moreover, $B_{\mathfrak{p}_j} \simeq \gamma_j B_{\mathfrak{p}}$. Since we suppose C strictly henselian and B finite over C , then we know that $B = \bigoplus_{i=1}^s B_{\mathfrak{p}_i} \simeq \bigoplus_{i=1}^s \gamma_i B_{\mathfrak{p}}$. As a consequence, any $b \in B$ can be written uniquely as $(\gamma_1 b_1, \gamma_2 b_2, \dots, \gamma_s b_s)$ in $\bigoplus_{i=1}^s \gamma_i B_{\mathfrak{p}}$ with $b_i \in B_{\mathfrak{p}}$. Each γ_i acts on this representation by left translation and permuting the term on the direct sum. In particular, we obtain $(\gamma_j^{-1}b)_{\mathfrak{p}} = \gamma_j^{-1} \gamma_j b_j = b_j$ and this proves that ϕ is an isomorphism. Finally, the composite map

$$B \xrightarrow{\phi} \mathrm{Map}^{\Gamma_0(\mathfrak{p})}(\Gamma, B_{\mathfrak{p}}) \rightarrow B_{\mathfrak{p}}, \quad u \mapsto u(1_{\Gamma})$$

induces an isomorphism between C and $B_{\mathfrak{p}}^{\Gamma_0(\mathfrak{p})}$. $\square$

Rewriting the previous theorem in algebro-geometric terms gives us exactly étale slices for any action by a finite étale group scheme.

Theorem 4.2 *An action of a finite étale group scheme G on X a scheme of finite type over S admits étale slices.*

Proof By [9, Theorem 3.1], there exists a universal categorical quotient $Q = X/G$ in the category of algebraic space such that $X \rightarrow X/G$ is finite. As a consequence, for any $q \in Q$, up to taking an étale neighborhood of q , without lost of generality, we can suppose that G is constant associated to an abstract finite group Γ , Q is the spectrum of a local strictly henselian ring with maximal ideal $\mathfrak{q}$ and X is the spectrum of a semi local ring. This permits us to obtain the result directly by applying the previous theorem and the universality of the quotient. $\square$

5 Unramified case

5.1 Definitions

Definition 5.1 We say that the action (X, G) is *unramified at $\mathfrak{p} \in X$* if the inertia group $I_G(\mathfrak{p})$ is trivial. We say that the action (X, G) is *unramified* if (X, G) is unramified at

any $\mathfrak{p}$. The action is called *free* if the Galois morphism $(\mu_X, \text{pr}_1): X \times_S G \rightarrow X \times_S X$ is a closed immersion.

Lemma 5.2 [10, III, Section 2, no. 2, 2.2] *Let S be locally noetherian, X locally of finite type over S , and G flat of finite type over S . Then the following assertions are equivalent:*

- (X, G) is free;
- (X, G) is unramified.

5.2 Global slice theorem

Theorem 5.3 [10, III, Section 2, no. 2, 3] *Let S be locally noetherian, X locally of finite type over S , and G finite, flat over S . Then the following assertions are equivalent:*

- (X, G) is unramified;
- (X, G) is free;
- X is a G -torsor over Y .

5.3 Local slice theorem

Lemma 5.4 *Let S be the spectrum of a noetherian ring R , $X = \text{Spec}(B)$ an affine scheme locally of finite type over S , and $G = \text{Spec}(A)$ affine, finite, flat over S . Suppose, moreover, that S is the spectrum of a noetherian local ring $R = B^A$, with maximal ideal $\mathfrak{q}$. Then the following assertions are equivalent:*

- (a) (X, G) is unramified;
- (b) (X, G) is free;
- (c) $I_G(\mathfrak{p})$ is trivial for some $\mathfrak{p}$ over $\mathfrak{q}$.

Proof The first implication follows from the definition. Let us prove that (b) $\Rightarrow$ (a). Let $k(\mathfrak{q}) = R/\mathfrak{q}$ be the residue field of R at $\mathfrak{q}$ and $f: X \rightarrow S$ be the quotient morphism: since G is finite, f is finite. Set $S_0 = \text{Spec}(k(\mathfrak{q}))$ and

$$X_0 = f^{-1}(\mathfrak{q}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\} = X \times_S S_0,$$

where for any $i \in 1, \dots, r$, $\mathfrak{p}_i$ are the primes above $\mathfrak{q}$. Since the inertia is trivial at some prime over $\mathfrak{q}$, it is trivial at any prime over $\mathfrak{q}$ as all inertia groups at the prime ideals over $\mathfrak{q}$ are conjugate. Then, the action (X_{S_0}, G_{S_0}) is free, in other words, $X_0 \times_{S_0} G_{S_0} \rightarrow X_0 \times_{S_0} X_0$ is a closed immersion, see [10, III, Section 2, no. 2, Proposition 2.2]. That is, we have the surjection

$$(B \otimes_R B) \otimes_R R/\mathfrak{q} \rightarrow (B \otimes_R A) \otimes_R R/\mathfrak{q}.$$

Since $B \otimes_R B$ and $B \otimes_R A$ are finite R -algebras, by Nakayama's lemma, $B \otimes_R B \rightarrow B \otimes_R A$ is a surjection, hence the action is free. $\square$

From this lemma, we can deduce easily the following theorem which is a local slice theorem for free action.

Theorem 5.5 *Let S be the spectrum of a noetherian ring, X locally of finite type over S , and G finite, flat over S . Let for $x \in X$, $q \in Q = X/G$ be the image of x via the morphism $\phi: X \rightarrow Q$. Suppose that the inertia group at x is trivial. Then, there is an étale morphism $Q' \rightarrow Q$ containing y in its image such that the action $(X \times_Q Q', G_{Q'})$ over Q' is free. In particular, the action (X, G) admits a slice at x with trivial slice group and $X \times_Q Q'$ as a slice scheme.*

Proof Up to taking an étale neighborhood of q , $Q' \rightarrow Q$, we can assume that S is the spectrum of a strictly henselian local ring of maximal ideal q and, since X is finite over Q , that X is the spectrum of a semi local ring. So that we can apply the previous lemma and prove that $(X \times_Q Q', G_{Q'})$ is free, and since $G_{Q'}$ is finite over Q' , $X \times_Q Q'$ is a $G_{Q'}$ -torsor over Q' , see Theorem 5.3. Hence $X \times_Q Q'$ is an fppf morphism over Q' and there is a $G_{Q'}$ -equivariant isomorphism

$$(\mu_{X \times_Q Q'}, \text{pr}_1): ((X \times_Q Q') \times_{Q'} G_{Q'})/e \simeq (X \times_Q Q') \times_{Q'} (X \times_Q Q'),$$

where e denotes the trivial group scheme over $X \times_Q Q'$. Moreover, on the left hand side $G_{Q'}$ acts only on the factor $G_{Q'}$ by translation and on the right hand side $G_{Q'}$ acts only on the first factor. $\square$

6 Linearly reductive group schemes

6.1 Definition

From the notion of tameness, we can define a class of group schemes with nice properties, which will be very important for the following.

Definition 6.1 We say that G is a *linearly reductive group scheme*, if

$$(-)^A: {}_R\mathcal{M}^A \rightarrow {}_R\mathcal{M}$$

is exact.

6.2 Cohomological properties

The cohomology of linearly reductive group schemes has some very interesting cohomological vanishing properties that are useful for the study of deformations involving such group schemes. Among others, linear reductivity can be characterized by the vanishing of the cohomology.

Lemma 6.2 [13, Proposition 1] *Suppose G is flat, finite over S , X is finitely presented over S , and $S = \text{Spec}(k)$, where k is a field. Then the following assertions are equivalent:*

- G is linearly reductive over S ;
- $H^1(G, M) = 0$ for any $M \in \mathcal{M}^A$.

The following lemma is a part of the proof of [1, Lemma 2.13].

Lemma 6.3 *If G is linearly reductive over S , then*

$$\mathrm{Ext}^i(L_{B_k G/k}, \mathcal{F}) = 0, \quad i \neq -1, 0,$$

for any coherent sheaf $\mathcal{F}$ on $B_k G$. If we suppose that G is also smooth, then this holds for all $i \neq -1$.

6.3 Liftings of linearly reductive group schemes

Proof of Theorem 1.3 Let $s \in S$. By [1, Proposition 2.18], there exists an étale morphism $U \rightarrow S$, with $U \ni u \mapsto s$, and a linearly reductive group scheme H over U such that $H_{k(u)} \simeq H_{0k(u)}$. By a standard limit argument, we can assume that U is the spectrum of a local henselian ring D with maximal ideal u . Set $U_n = \mathrm{Spec}(D/u^{n+1})$. One has that H_0 is a subgroup scheme of $G_{k(s)}$, and this defines a representable morphism of algebraic stacks $i_0: B_{k(u)} H_0 \rightarrow B_{k(u)} G_{k(u)}$ over $k(u)$. We want to prove the existence of a representable morphism of algebraic stacks $i: B_U H \rightarrow B_U G_U$ completing the following 2-commutative diagram:

$$\begin{array}{ccc}
 B_{k(u)} H_0 & \xrightarrow{\quad} & B_U H \\
 \searrow i_0 & & \searrow i \\
 & B_{k(u)} G_{k(u)} & \xrightarrow{\quad} B_U G_U \\
 \downarrow g & & \downarrow \\
 U_0 & \xrightarrow{\quad} & U.
 \end{array}$$

In order to be able to apply Artin's approximation theorem [3, Théorème 1.12] and to obtain an algebraization of some formal data, we can view R as a limit of rings R_i which are henselizations of $\mathbb{Z}$ -algebras of finite type. Since the functor X is locally of finite type over S locally noetherian, one can use Grothendieck's Existence theorem for algebraic stacks [18, Theorem 11.1] to reduce to the case where R is the henselization of a $\mathbb{Z}$ -algebra of finite type. Using again Grothendieck's Existence theorem for algebraic stacks [18, Theorem 11.1], to go to the completion, one can then obtain the morphism i via Artin's Approximation theorem. As a consequence, the existence of such i depends only on the existence of a formal deformation, that is morphisms $i_n: B_{U_n} H_{U_n} \rightarrow B_{U_n} G_{U_n}$ filling in the following 2-commutative diagram:

$$\begin{array}{ccccc}
B_{k(u)}H_0 & \cdots \rightarrow & B_{U_{n-1}}H_{U_{n-1}} & \xrightarrow{\quad} & B_{U_n}H_{U_n} \\
& \searrow i_1 & \searrow i_{n-1} & & \searrow i_n \\
& B_{k(u)}G_{k(u)} & \cdots \rightarrow & B_{U_{n-1}}G_{U_{n-1}} & \xrightarrow{\quad} & B_{U_n}G_{U_n} \\
& \downarrow & \downarrow & & \downarrow \\
& \text{Spec}(k(u)) & \cdots \rightarrow & U_{n-1} & \xrightarrow{\quad} & U_n
\end{array}$$

By [17, Theorem 1.5], the obstruction to extend the morphism i_{n-1} to i_n lies in

$$\text{Ext}^1(Lx^*L_{B_{k(u)}G_{k(u)}/k(u)}, u^n/u^{n+1} \otimes_{k(u)} \mathcal{O}_{B_{k(u)}G_{k(u)}}),$$

which is trivial for any $n \in \mathbb{N}$ by Lemma 6.3. It follows that there exists an arrow i_n filling the previous diagram. This leads to the existence of a representable morphism of stacks $i: B_U H_U \rightarrow B_U G_U$. Let $S' \rightarrow U$ be the G_U -torsor which is the image via i of the trivial H_U -torsor $H_U \rightarrow U$. Since $S' \rightarrow U$ is surjective as a G_U -torsor, up to an fppf base change, this morphism contains u in its image, and we denote by $s' \in S$ the point mapping to u via $S' \rightarrow U$. Furthermore, the functor i induces a homomorphism from $\text{Aut}_{S'}(H_{S'} \rightarrow S') = H_{S'}$ to the automorphism group scheme of the image of the $H_{S'}$ -torsor $H_{S'} \rightarrow S'$ in $B_{S'}G_{S'}$. Since this image is the pullback of S' to S' over $B_{S'}G_{S'}$, which is canonically a trivial torsor, its automorphism group is $G_{S'}$. This defines a group morphism $H_{S'} \rightarrow G_{S'}$, with $S' \rightarrow U$ an fppf morphism as a G_U -torsor. Since i is representable, this morphism is injective. Finally, since H is proper and G is separated, $H_{S'}$ is closed in $G_{S'}$. $\square$

Remark 6.4 It would be desirable to obtain an étale neighborhood instead of the finitely presented and flat one that we obtain, but in the construction $S' \rightarrow U$ is only fppf as a G_U -torsor.

7 Tame quotient stack

7.1 Definition

In order to define a tame quotient stack, we need the existence of a coarse moduli space. Let S be locally noetherian, X locally of finite type over S , and G flat of finite type over S . Suppose that the inertia stack is finite. Under these assumptions we know that a coarse moduli space for $[X/G]$ exists and we denote it by M . Then the morphism $\phi: [X/G] \rightarrow M$ is a proper map by Theorem 3.15.

Definition 7.1 We say that the quotient stack $[X/G]$ is *tame* if the functor

$$\phi_*: \text{Qcoh}[X/G] \rightarrow \text{Qcoh}(M)$$

is exact.

Remark 7.2 The notion of tameness can be defined similarly for Artin stacks of finite type over S locally noetherian with finite inertia stack, see [1, Definition 3.1].

Let S be affine, noetherian, X affine, of finite type over S , and G affine, flat, finite over S . The quotient stack $[X/G]$ is tame if and only if the functor of invariants $(-)^A$ is exact. It is a consequence of Lemma 3.17 since Y is a coarse moduli space for G finite flat over S , by [9, Section 3].

Let S be locally noetherian, X locally of finite type over S , and G flat, finite over S . Then G is linearly reductive if and only if the classifying stack $B_S G$ is tame.

Tameness is local. In fact,

Lemma 7.3 [2, Proposition 3.10] *Let S be locally noetherian, X locally of finite type over S , and G flat of finite type over S . Suppose that the inertia stack is finite. For any morphism of S -schemes $g: M' \rightarrow M$, we consider the following 2-cartesian diagram:*

$$\begin{array}{ccc} [X/G]_{M'} & \xrightarrow{g'} & [X/G] \\ \phi' \downarrow & & \downarrow \phi \\ M' & \xrightarrow{g} & M. \end{array}$$

Suppose that ϕ (resp. ϕ') is the coarse moduli space for $[X/G]$ (resp. $[X/G]_{M'}$).

- If g is faithfully flat and the quotient stack $[X/G]_{M'}$ is tame then the quotient stack $[X/G]$ is tame.
- If the quotient stack $[X/G]$ is tame then the quotient stack $[X/G]_{M'}$ is also tame.

7.2 Local definition of tameness

Theorem 7.4 [1, Theorem 3.2] *Let S be locally noetherian, X locally of finite type over S , and G flat of finite type over S . Suppose that the inertia stack is finite. Then the following assertions are equivalent:*

- The quotient stack $[X/G]$ is tame.
- The inertia groups $I_G(\xi) \rightarrow \mathrm{Spec}(k)$ are linearly reductive groups, for any $\xi: \mathrm{Spec}(k) \rightarrow X$, where k is a field.
- The inertia groups $I_G(\xi) \rightarrow \mathrm{Spec}(k)$ are linearly reductive groups, for any geometric point $\xi: \mathrm{Spec}(k) \rightarrow X$, where k is an algebraically closed field.
- The inertia groups $I_G(x) \rightarrow \mathrm{Spec}(k(x))$ are linearly reductive groups, for any $x \in X$.
- For any point $x \in X$, denote by m its image via the morphism $X \rightarrow M$. There exist an fppf (also can be chosen étale surjective) morphism $M' \rightarrow M$ with some m' mapping to m , a linearly reductive group scheme $H \rightarrow M'$, such that $H_{k(m')} \simeq I_G(x)$, acting on a finite and finitely presented scheme $P \rightarrow M'$, and an isomorphism of algebraic stacks over M'

$$[X/G] \times_M M' \simeq [P/H].$$

Remark 7.5 This theorem shows that the tameness of the quotient stack $[X/G]$ associated to the action (X, G) generalizes the classical notion of tameness in algebraic number theory. In fact, let us consider a Γ -extension of Dedekind rings B/B^Γ , where Γ is an abstract finite group and B^Γ is the invariant ring for the action of Γ on B . In this classical algebro-number theory context, tameness on B/B^Γ is defined by requiring the surjectivity of the trace map $\text{tr}: B \rightarrow B^\Gamma$ sending $b \in B$ to $\sum_{\gamma \in \Gamma} \gamma \cdot b$. For any prime ideal $\mathfrak{p}$ of B we write $k(\mathfrak{p})$; then, it is well known, that B/B^Γ is tame if and only if for all prime $\mathfrak{p}$ of B , the inertia group $\Gamma_0(\mathfrak{p})$ has order prime to the characteristic of $k(\mathfrak{p})$. This last condition is equivalent to requiring that the group algebra $k(\mathfrak{p})[\Gamma_0(\mathfrak{p})]$ is semisimple, i.e., that the constant group scheme $\underline{\Gamma_0(\mathfrak{p})}$ attached to $\Gamma_0(\mathfrak{p})$ is linearly reductive. But as a $k(\mathfrak{p})$ -group scheme, $\underline{\Gamma_0(\mathfrak{p})}$ is exactly the inertia group of the action of the constant group scheme attached to Γ on $X = \text{Spec}(B)$, cf. [10, III, Section 2, no. 2, Example 2.4].

We obtain the following corollary.

Corollary 7.6 [1, Corollary 3.5] *Let S be locally noetherian, X locally of finite type over S , and G flat of finite type over S . Suppose that the inertia stack is finite. The stack $[X/G] \rightarrow S$ is tame if and only if for any morphism $\text{Spec}(k) \rightarrow S$, where k is an algebraically closed field, the geometric fiber $[X/G] \times_S \text{Spec}(k) \rightarrow \text{Spec}(k)$ is tame.*

The previous theorem justifies a local definition of tameness on topological points.

Definition 7.7 We say that $[X/G]$ is *tame at* $x \in X$ if the inertia group $I_G(x)$ is linearly reductive.

Remark 7.8 By the previous theorem, $[X/G]$ is tame if and only if $[X/G]$ is tame at x , for any $x \in X$.

7.3 Existence of a torsor

We state now an interesting consequence of the previous theorem which permits to define a torsor for a tame quotient stack for actions of finite commutative group schemes.

Proposition 7.9 *Let S be locally noetherian, X locally of finite type over S , and G flat, finite and commutative over S . For any $x \in X$ denote by $q \in Q = X/G$ its image by the morphism $X \rightarrow Q$. If the quotient stack $[X/G]$ is tame, then for any point $x \in X$ there exist a finitely presented flat morphism $Q' \rightarrow Q$ containing q in its image and a subgroup H of $G_{Q'}$ over Q' lifting the inertia group at x such that $X_{Q'}^Q/H$ is a $G_{Q'}/H$ -torsor over Q' .*

Proof Notice first that under the assumption of the theorem, $G_{Q'}/H$ is a finite affine group scheme. By the previous theorem, we know that the inertia at x is linearly reductive. So, by Theorem 1.3, there are a finitely presented flat morphism $Q' \rightarrow Q$ containing q in its image and a linearly reductive group $H \rightarrow Q'$ lifting this inertia group as a subgroup of $G_{Q'}$. Moreover, the inertia group at $\bar{x}$, image of x by the quotient morphism $X \times_Q Q' \rightarrow (X \times_Q Q')/H$ for the action $((X \times_Q Q')/H, G_{Q'}/H)$, is equal to

$$I_{G_{Q'}/H}(\bar{x}) = I_G(x)/(I_G(x) \cap H_{k(x)}) = \{e\}.$$

By Theorem 5.3, up to passing to a finitely presented flat neighborhood of q , since the quotients induced by an action of H are universal [9, Theorem 3.1], the action $((X \times_Q Q')/H, G_{Q'}/H)$ is free. Thus $(X \times_Q Q')/H$ is a $G_{Q'}/H$ -torsor over Q' since $G_{Q'}/H$ is finite over Q' , see Theorem 5.3. $\square$

Remark 7.10 It would be useful to establish the previous proposition in general if G is not commutative. But, in this case G/H is not necessarily a group scheme and we cannot even define the notion of action or of torsor.

7.4 Proof of Theorem 1.2

In order to prove Theorem 1.2 we need the following lemma.

Lemma 7.11 [8, Proposition 6.5] *Let H be a flat subgroup of a flat group scheme G over S such that the quotient for the natural translation action exists and is universal. Let $\psi: X \rightarrow G/H$ be a morphism of schemes preserving the G -actions. Let*

$$Z = X \times_{G/H} e$$

be defined as the fibered product of two maps ψ and the inclusion $e = H/H \rightarrow G/H$. Assume that the balanced product $(Z \times_S G)/H$, see Definition 1.1, exists and is a universal quotient. Then, we have an isomorphism of G -actions

$$(X, G) \simeq ((Z \times_S G)/H, G).$$

Remark 7.12 The scheme Z in the lemma is a closed subscheme of X as a fiber product of X and e (which is trivial) over G/H .

Finally, we prove Theorem 1.2 which extends [8, Theorem 6.4].

Proof of Theorem 1.2 Denote $Q = X/G$.

(a) $\Rightarrow$ (b) By Proposition 7.9, as G is commutative, up to passing to a finitely presented flat neighborhood of q , there is a subgroup G_q of G such that $(X/G_q, G/G_q)$ defines a torsor over Q , hence $X/G_q \times_Q X/G_q \simeq X/G_q \times_Q G/G_q$. This gives a G -equivariant morphism $\psi: X \rightarrow G/G_q$ after the fppf base change $X/G_q \rightarrow Q$ and permits to construct Z_q thanks to Lemma 7.11. Now, since H is finite, flat over R , $(Z_q \times_S G)/H$ is a universal categorical quotient. So, (X, G) is induced by the action (Z_q, G_q) by Lemma 7.11.

(b) $\Leftarrow$ (a) This is due to [1, Propositions 2.4 and 2.5], since all slice groups are linearly reductive. $\square$

8 Tame action by affine group scheme vs tame quotient stack

In this section, the base $S = \operatorname{Spec}(R)$ is affine and noetherian, $G = \operatorname{Spec}(A)$ is an affine group scheme flat, of finite type over S acting on an affine scheme locally of finite type $X = \operatorname{Spec}(B)$.

8.1 Algebraic interlude

The following lemma is quite important in the sense that it relates the functor of invariants to the functor $\text{Com}_A(-, B)$. As a consequence, it relates in particular the exactness of this functors which will be important in order to compare the two notions of tameness, see Proposition 8.7.

Lemma 8.1 *Let A be a commutative, flat Hopf algebra over R , $B \in \mathcal{M}^A$ and $M \in {}_R\mathcal{M}$ be finitely presented as R -modules. Then there is a natural (functorial on M) isomorphism*

$$(B \otimes_R M^*)^A \simeq \text{Com}_A(M, B).$$

Proof The commutativity of the right square of the diagram below insures the existence of λ which is an isomorphism and hence existence of α and γ as well.

$$\begin{array}{ccccccc} 0 & \longrightarrow & (B \otimes_R M^*)^A & \longrightarrow & B \otimes_R M^* & \xrightarrow{\phi} & (B \otimes_R M^*) \otimes_R A \\ & & \downarrow \lambda & & \downarrow \alpha & & \downarrow \gamma \\ 0 & \longrightarrow & \text{Com}_A(M, B) & \longrightarrow & \text{Hom}_R(M, B) & \xrightarrow[\psi]{} & \text{Hom}_R(M, B \otimes_R A), \end{array}$$

where $\phi = \rho_{B \otimes_R M^*} - \text{Id}_B \otimes \text{Id}_{M^*} \otimes 1$, $\psi = (\rho_B \circ -) - \rho_{M^*}$ and

$$\gamma(b \otimes f \otimes a)(m) = \sum b \cdot f(m_{(0)}) \otimes a \cdot m_{(-1)}$$

for any $b \otimes f \otimes a \in B \otimes_R M^* \otimes_R A$ and $m \in M$. Indeed, the map γ is an isomorphism as a composition of the isomorphism

$$\gamma_1: B \otimes_R M^* \otimes_R A \rightarrow B \otimes_R A \otimes_R M^*, \quad b \otimes f \otimes a \mapsto b \otimes a \cdot m_{(-1)} \otimes f(m_{(0)})$$

and the canonical isomorphism

$$\gamma_2: B \otimes_R A \otimes_R M^* \rightarrow \text{Hom}_R(M, B \otimes_R A),$$

where the inverse map of γ_1 is defined for any $b \otimes a \otimes f \in B \otimes_R A \otimes_R M^*$ by

$$\gamma_1^{-1}(b \otimes a \otimes f) = \sum b \otimes f(m_{(0)}) \otimes a \cdot S(m_{(-1)}).$$

For any $b \otimes f \in B \otimes_R M^*$, we have

$$\psi \circ \alpha(b \otimes f) = \psi([m \mapsto bf(m)]) = [m \mapsto \rho_B(b) f(m) - \sum bf(m_{(0)}) \otimes m_{(-1)}]$$

and

$$\begin{aligned}
 \gamma \circ \phi(b \otimes f) &= \gamma \left(\left[m \mapsto \sum b_{(0)} \otimes f(m_{(0)}) \otimes b_{(1)} S(m_{(-1)}) - b \otimes f(m) \otimes 1 \right] \right) \\
 &= \left[m \mapsto \left(b_{(0)} \otimes b_{(1)} S(m_{(1)}) (m_{(0)})_{(-1)} \right) (f(m_{(0)}))_{(0)} \right. \\
 &\quad \left. - \sum b f(m_{(0)}) \otimes m_{(-1)} \right] \\
 &= \left[m \mapsto \sum \left(b_{(0)} \otimes b_{(1)} \epsilon(m_{(-1)}) \right) f(m_{(0)}) - \sum b f(m_{(0)}) \otimes m_{(-1)} \right] \\
 &= \left[m \mapsto \rho_B(b) f(m) - \sum b f(m_{(0)}) \otimes m_{(-1)} \right]. \quad \square
 \end{aligned}$$

8.2 Tame actions

Chinburg, Erez, Pappas and Taylor defined in the article [8] the notion of tame action. We recall this definition as well as some useful properties.

Definition 8.2 Let S be the spectrum of a noetherian ring, $X = \operatorname{Spec}(B)$ affine, of finite type over S , and $G = \operatorname{Spec}(A)$ affine, flat of finite type over S . We say that an action (X, G) is *tame* if there is a unitary (that is $\alpha(1_A) = 1_B$) morphism of A -comodules $\alpha: A \rightarrow B$, which means that α is an R -linear map such that $\rho_B \circ \alpha = (\alpha \otimes \operatorname{Id}_A) \circ \delta$, such a morphism is called a *total integral*.

Tame actions are stable under the base change.

Lemma 8.3 Let S be the spectrum of a noetherian ring, $X = \operatorname{Spec}(B)$ affine, of finite type over S , and $G = \operatorname{Spec}(A)$ affine, flat of finite type over S . If the action (X, G) is tame, then after an affine base change $R' \rightarrow R$ the action $(X_{R'}, G_{R'})$ is also tame.

Proof Since the action (X, G) is tame, there is an A_C -comodule map $A \rightarrow B$, which induces naturally a comodule map $A_{R'} \rightarrow B_{R'}$. $\square$

The next lemma will allow us to assume that the base is equal to the quotient, if the structural map $X \rightarrow S$ has the same properties as the quotient morphism $X \rightarrow Y$ where $Y = \operatorname{Spec}(B^A)$.

Lemma 8.4 Let S be the spectrum of a noetherian ring, X affine, of finite type over S , and G affine, flat of finite type over S . Suppose $C = B^A$. Then the following assertions are equivalent:

- (a) The action $(X, {}_C G)$ over C is tame.
- (b) The action (X, G) over S is tame.
- (c) The action $({}_C X, {}_C G)$ over C is tame.

Proof (a) $\Rightarrow$ (b) Let $\alpha: {}_C A \rightarrow B$ be a total integral for the tame action $(X, {}_C G)$. Since B is a ${}_C A$ -comodule via $\rho_B: B \rightarrow B \otimes_R A \simeq B \otimes_C {}_C A$, the composite $\alpha' =$

$\alpha \circ i_A: A \rightarrow B$, where $i_A: A \rightarrow C \otimes_R A$ maps a to $1 \otimes a$, is a unitary A -comodule map so a total integral for the action (X, G) .

(b) $\Rightarrow$ (c) Follows from the base change, by Lemma 8.3.

(c) $\Rightarrow$ (a) Denote by $\beta: {}_C A \rightarrow {}_C B$ the total integral for the action $({}_C X, {}_C G)$. Recall that $\text{Id}_C \otimes_R A$ is a ${}_C A$ -comodule via $\text{Id}_C \otimes \delta$ and ${}_C B$ is a ${}_C A$ -comodule map via $\text{Id}_C \otimes \rho_B$. Consider the composition $\beta' = \mu_C \circ \beta: C \otimes_R A \rightarrow B$ where $\mu_C: C \otimes_R B \rightarrow B$ comes from the algebra multiplication of B . For any $b \in B$ and $c \in C$,

$$\begin{aligned} (\mu_C \otimes \text{Id}_A)((\text{Id}_C \otimes \rho_B)(c \otimes b)) &= (\mu_C \otimes \text{Id}_A)\left(\sum c \otimes b_{(0)} \otimes b_{(1)}\right) \\ &= \sum cb_{(0)} \otimes b_{(1)} = \rho_B(c) \rho_B(b) \quad (\text{since } C = B^A) \\ &= \rho_B(\mu_C(b \otimes c)) \quad (\text{since } B \text{ is an } A\text{-comodule algebra}). \end{aligned}$$

Thus, μ_C is an A_C -comodule map and β' also being compositions of A_C -comodule maps. $\square$

8.3 Exactness of the functor of invariants for tame actions

In the case of a constant group scheme, by [8, Lemma 2.2], the tameness of an action is equivalent to the surjectivity of the trace map, which generalizes a characterization of tameness in number theory, see [6, Chapter I, Section 5, Theorem 2], and justifies the choice of the terminology. In the general case, we can define a projector which plays a role of the trace map in the case of an action by a constant group scheme, see [11, Section 1].

Lemma 8.5 [11, Section 1] *For a total integral map $\alpha: A \rightarrow B$ and $M \in {}_B \mathcal{M}^A$, we can define an R -linear projector called the Reynold operator*

$$\text{pr}_M: M \rightarrow M^A, \quad m \mapsto \text{pr}_M(m) = \text{pr}_{M,\alpha}(m) = \sum_{(m)} \alpha(S(m_{(1)}))m_{(0)}.$$

The existence of this projector insures that the functor of invariants is exact for tame actions. The following lemma allows to relate the above two notions of tameness.

Lemma 8.6 [8, Lemma 2.3] *If the action (X, G) is tame then the functor of invariants $(-)^A: {}_B \mathcal{M}^A \rightarrow {}_C \mathcal{M}$ is exact.*

Adding the finite hypothesis, we are able to prove that the exactness of the functor of invariants is equivalent to tameness of the action.

Proposition 8.7 *Let S be affine, noetherian, X affine, of finite type over S , and G affine, flat and finite over S . Suppose X is flat over Y . Then the following assertions are equivalent:*

- (a) *The action (X, G) is tame.*
- (b) *The functor $(-)^A: {}_B \mathcal{M}^A \rightarrow {}_C \mathcal{M}$, $N \mapsto (N)^A$ is exact.*

(a') The action $(X, {}_C G)$ is tame.

(b') The functor $(-)^{cA}: {}_B \mathcal{M}^{cA} \rightarrow {}_C \mathcal{M}, N \mapsto (N)^{cA}$ is exact.

Proof (a) $\Leftrightarrow$ (a') follows from Lemma 8.4. (b) $\Leftrightarrow$ (b') follows from ${}_B \mathcal{M}^A = {}_B \mathcal{M}^{Ac}$. (a) $\Rightarrow$ (b) follows from Lemma 8.5.

(b') $\Rightarrow$ (a') Suppose that $(-)^A$ is exact. Since ${}_C A$ is finite over C (base change of A which is finite over R) and B is finite over C , $B \otimes_C {}_C A$ is finite over C . Moreover, as C is locally noetherian, B and $B \otimes_R A$ are also of finite presentation as algebras over C hence, in particular, as C -modules. By Lemma 8.1, since we suppose B is flat over C , we have the following isomorphism:

$$({}_B \otimes_R B^*)^{cA} \simeq \text{Com}_{{}_C A}(B, B), \quad (B \otimes_R (B \otimes_R A)^*)^{cA} \simeq \text{Com}_{{}_C A}(B \otimes_C {}_C A, B).$$

Since $(B \otimes \epsilon)$ is a C -linear section of $\rho_B: B \rightarrow B \otimes_R A$, $(B \otimes_C {}_C A)^* \rightarrow B^*$ is surjective. Thus, from the exactness of $(-)^{cA}$, we obtain the surjectivity of

$$(B \otimes_R (B \otimes_R A)^*)^{cA} \rightarrow (B \otimes_R B^*)^{cA}.$$

Finally, the isomorphisms above imply the surjectivity of the natural map

$$\text{Com}_{{}_C A}(B \otimes_C {}_C A, B) \rightarrow \text{Com}_{{}_C A}(B, B).$$

This insures the existence of a ${}_C A$ -comodule map $\lambda_B: B \otimes_C {}_C A \rightarrow B$ such that $\lambda_B \circ \rho_B = B$. The previous lemma permits to conclude the proof. $\square$

8.4 Relationship between the two notions of tameness

Now Theorem 1.4 follows easily from the previous results. The first statement follows directly from Lemma 8.6 and Theorem 3.18, whereas the second follows from Proposition 8.7.

Remark 8.8 The flatness is not a necessary condition. In fact, let k be a field of characteristic distinct from 2, we consider the action of the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ on $X = \text{Spec}(k[x, y]/(xy))$, defined by the action of the abstract group $\mathbb{Z}/2\mathbb{Z}$ on $k[x, y]/(xy)$ sending $-1 \cdot x$ to x and $-1 \cdot y$ to $-y$. So that $B^{\mathbb{Z}/2\mathbb{Z}} \simeq k[x]$. The constant group scheme $\mathbb{Z}/2\mathbb{Z}$ is linearly reductive since $\text{char}(k) \neq 2$, so the action defined is tame as well as the quotient stack $[X/G]$ but $k[x, y]/(xy)$ is not flat as a $k[x]$ -module (in fact, the inclusion $(t) \rightarrow k[x, t]$ is no longer injective when we tensor by $\otimes_{k[x]} k[x, y]/(xy)$, since by the map $(t) \otimes_{k[x]} k[x, y]/(xy) \rightarrow k[x, t] \otimes_{k[x]} k[x, y]/(xy)$ any $0 \neq t \otimes x$ is sent to 0).

We easily obtain the following corollary.

Corollary 8.9 *Let S be the spectrum of a noetherian ring, X affine, of finite type over S , and G affine, flat of finite type over S . Then, G is linearly reductive over S if and only if the trivial action (S, G) is tame over S .*

Remark 8.10 We say that a Hopf algebra A is *relatively cosemisimple* if for all $M \in \mathcal{M}^A$ the submodules which are direct summands in $\mathcal{M}_R$ are also direct summands in $\mathcal{M}^A$. The Hopf algebra A is relatively cosemisimple if and only if the trivial action $(\text{Spec}(R), G)$ is tame, see [5, 16.10]. In particular, when G is finite, flat over S , this is also equivalent to G being linearly reductive.

The following result can be seen as an analogue of the trace surjectivity that we have proved in the constant case.

Corollary 8.11 *Let S be affine, noetherian, X affine, of finite type over S , and G affine, flat and finite over S . Then the following assertions are equivalent:*

- (a) *The action (X, G) is tame.*
- (b) *The quotient stack $[X/G]$ is tame.*
- (c) *The functor $(-)^A: {}_B\mathcal{M}^A \rightarrow {}_C\mathcal{M}$, $N \mapsto (N)^A$ is exact.*
- (d) *There is a Reynold operator $\text{pr}_M: M \rightarrow M^A$ for any $M \in {}_B\mathcal{M}^A$.*

Proof (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows from the previous theorem. (a) $\Rightarrow$ (d) follows from Lemma 8.5.

(d) $\Rightarrow$ (b) As the functor $(-)^A$ is left exact, it is enough to prove exactness on the right. So, let $\xi: M \rightarrow N \in {}_B\text{Bim}^A(M, N)$ be an epimorphism. It induces a morphism $\xi^A: M^A \rightarrow N^A$. For $n \in N^A$, by the surjectivity of ξ , there is $m \in M$ such that $\xi(m) = n$. Moreover, $\text{pr}_M(m) \in M^A$, so $\xi(\text{pr}_M(m)) = \text{pr}_N(\xi(m)) = n$. Therefore, ξ is surjective. $\square$

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