Dirichlet series solutions of generalized Riccati equations

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Abstract Dirichlet series solutions of generalized Riccati differential equations of the form

$$f' + Bf^k + Cf^\ell + D = 0, \quad k, \ell \in \mathbb{N}, \quad k > \ell,$$

where $B(\neq 0)$, C , D are polynomials with complex coefficients, are determined. For finite Dirichlet series the only possible solution is a constant function, while for infinite Dirichlet series solutions significantly depend on the parameters $k > \ell$. It is established that (a) for $\ell = 1$, $k > 1$, there is a unique solution; (b) for $\ell = 2$, $k = 3$, if $D = 0$, there is a unique solution; if $D \neq 0$, generally there are more than one solutions all of which can be explicitly determined.

Keywords Dirichlet series · Generalized Riccati differential equations · A unique solution

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1 Introduction

Throughout, by a Dirichlet series, we mean a convergent series of the form

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s}, \quad (1)$$

where $a_i \in \mathbb{C}$, $a_0 \neq 0$, $\lambda_i \in \mathbb{R}$, $\lambda_0 < \lambda_1 < \lambda_2 < \dots$, $\lim_{i \rightarrow \infty} \lambda_i = +\infty$. If all $a_i \neq 0$, we call $\varphi(s)$ an *infinite* Dirichlet series. In 1920, Ostrowski [2] proved the following results revealing astonishing properties of exponents and coefficients of a Dirichlet series satisfying an algebraic differential equation (ADE).

- (I) If a Dirichlet series $\varphi(s)$ satisfies formally [2, Satz 1] or analytically [2, Satz 6] an ADE, then there exists $i \in \mathbb{N}$ such that for each $j > i$, the exponent λ_j can be written as an integral linear combination of $\lambda_0, \lambda_1, \dots, \lambda_{i-1}$.
- (II) If a Dirichlet series $\varphi(s)$ represents a transcendental function which satisfies an ADE, then there exists a finite number of coefficients $a_0, a_1, \dots, a_k$ such that all other coefficients a_h , with $h > k$, can be expressed as rational functions with rational coefficients in $a_0, a_1, \dots, a_k$ [2, Satz 8].

Further results in this direction can be found in [3, 4].

Should one apply Ostrowski's proof of the above results to linear differential equations with constant coefficients, one is immediately led to solutions which are exponential polynomials as acquainted in introductory Calculus courses [1]. It is thus natural to consider next the case of nonlinear first order differential equations. Our objective here is to carry out a detailed analysis of Ostrowski's approach and to derive precise information about Dirichlet series solutions of generalized Riccati differential equations

$$F(f, f') = f' + Bf^k + Cf^\ell + D = 0, \quad (2)$$

where $B (\neq 0)$, C , D are polynomials with complex coefficients, and $k > \ell$ are positive integers. In the case of *finite* Dirichlet series, we now show that such a solution can only be a constant function.

Proposition 1.1 *If a finite Dirichlet series $\varphi(s) = \sum_{i=0}^n a_i e^{-\lambda_i s}$ satisfies (2), then $\varphi(s) = a_0$ is the only solution of (2) with $Ba_0^k + Ca_0^\ell + D = 0$.*

Proof If $\lambda_0 = 0$, then $\varphi(s) = a_0 + \sum_{i=1}^n a_i e^{-\lambda_i s}$ and (2) becomes

$$0 = - \sum_{i=1}^n a_i \lambda_i e^{-\lambda_i s} + B \left(a_0 + \sum_{i=1}^n a_i e^{-\lambda_i s} \right)^k + C \left(a_0 + \sum_{i=1}^n a_i e^{-\lambda_i s} \right)^\ell.$$

For $n \geq 1$, since $k > \ell$, equating the terms with highest exponent we get $Ba_n^k e^{-k\lambda_n s} = 0$ yielding $B = 0$, which is a contradiction. Hence, the only solution is $\varphi(s) = a_0$ with $Ba_0^k + Ca_0^\ell + D = 0$.

If $\lambda_0 \neq 0$, then for $n \geq 0$, equating the terms with highest exponent in (2), we get $Ba_n^k e^{-k\lambda_n s} = 0$ yielding $B = 0$, which is a contradiction. $\square$

Having dealt with finite Dirichlet series, throughout the rest of the paper, we shall consider only infinite Dirichlet series. We refine the proofs of the above two results of Ostrowski by finding explicit relations among the exponents and among the coefficients of Dirichlet series (1) satisfying (2).

2 Preliminaries

For $n \in \mathbb{N}$, rewrite the infinite Dirichlet series (1) as

$$\varphi(s) = \sum_{i=0}^{n-1} a_i e^{-\lambda_i s} + \sum_{i=n}^{\infty} a_i e^{-\lambda_i s} \stackrel{\text{def}}{=} \varphi_{1n}(s) + \varphi_{2n}(s).$$

Using the Taylor series expansion and (2), we get

$$\begin{aligned} 0 &= F(\varphi, \varphi') = F(\varphi_{1n} + \varphi_{2n}, \varphi'_{1n} + \varphi'_{2n}) \\ &= F(\varphi_{1n}, \varphi'_{1n}) + \left(\varphi_{2n} \frac{\partial F(\varphi_{1n}, \varphi'_{1n})}{\partial \varphi_{1n}} + \varphi'_{2n} \frac{\partial F(\varphi_{1n}, \varphi'_{1n})}{\partial \varphi'_{1n}} \right) \\ &\quad + \left(\frac{\varphi_{2n}^2}{2!} \frac{\partial^2 F(\varphi_{1n}, \varphi'_{1n})}{\partial^2 \varphi_{1n}} \right) + \cdots + \left(\frac{\varphi_{2n}^k}{k!} \frac{\partial^k F(\varphi_{1n}, \varphi'_{1n})}{\partial^k \varphi_{1n}} \right) \\ &= F(\varphi_{1n}, \varphi'_{1n}) + \varphi_{2n} (kB\varphi_{1n}^{k-1} + \ell C\varphi_{1n}^{\ell-1}) + \varphi'_{2n} \\ &\quad + \frac{\varphi_{2n}^2}{2!} \{k(k-1)B\varphi_{1,n}^{k-2} + \ell(\ell-1)C\varphi_{1n}^{\ell-2}\} + \cdots + \varphi_{2n}^k B. \end{aligned}$$

For brevity, let

$$\begin{aligned} A_n &= \varphi_{2n} (kB\varphi_{1n}^{k-1} + \ell C\varphi_{1n}^{\ell-1}) + \varphi'_{2n} \\ &= a_n e^{-\lambda_n s} \{kB(a_0 e^{-\lambda_0 s})^{k-1} + \ell C(a_0 e^{-\lambda_0 s})^{\ell-1}\} - \lambda_n a_n e^{-\lambda_n s} \\ &\quad + \text{terms with higher exponents} \end{aligned} \quad (3)$$

and

$$\begin{aligned} E_n &= \frac{\varphi_{2n}^2}{2!} \{k(k-1)B\varphi_{1,n}^{k-2} + \ell(\ell-1)C\varphi_{1,n}^{\ell-2}\} + \cdots + \varphi_{2n}^k B \\ &= \frac{a_n^2 e^{-2\lambda_n s}}{2!} \{k(k-1)B(a_0 e^{-\lambda_0 s})^{k-2} + \ell(\ell-1)C(a_0 e^{-\lambda_0 s})^{\ell-2}\} \\ &\quad + \text{terms with higher exponents.} \end{aligned}$$

Thus,

$$0 = F(\varphi, \varphi') = F(\varphi_{1n}, \varphi'_{1n}) + A_n + E_n. \quad (4)$$

Since our approach involves equating exponents, the term E_n can then be ignored because the exponent in its first term (i.e., the *term of least exponent*) is

$$\min\{2\lambda_n + \lambda_0(k-2), 2\lambda_n + \lambda_0(\ell-2)\} > \min\{\lambda_n + \lambda_0(k-1), \lambda_n + \lambda_0(\ell-1)\}.$$

Furthermore, if the first term of A_n is nonzero, then $F(\varphi_{1n}, \varphi'_{1n})$ must contain a term whose exponent is the same as that of A_n . If $\lambda_0 = 0$, then the first term of A_n is

$$(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_n)a_n e^{-\lambda_n s}. \quad (5)$$

If $\lambda_0 > 0$, then the first term of A_n is

$$(C - \lambda_n)a_n e^{\lambda_n s} \quad \text{if } \ell = 1, \quad \text{and} \quad -\lambda_n a_n e^{-\lambda_n s} \quad \text{if } \ell > 1. \quad (6)$$

The exponent ℓ is closely related to λ_0 , as seen in the next lemma.

Lemma 2.1 *Assume that an infinite Dirichlet series $\varphi(s)$, as in (1), satisfies (2). If $\ell \geq 2$, then $\lambda_0 = 0$, while $\lambda_0 \geq 0$ if $\ell < 2$ (i.e., $\ell = 1$).*

Proof We take $n = 1$ in (4), i.e.,

$$\begin{aligned} \varphi(s) &= a_0 e^{-\lambda_0 s} + \sum_{i=1}^{\infty} a_i e^{-\lambda_i s}, \\ F(\varphi_{11}, \varphi'_{11}) &= -\lambda_0 a_0 e^{-\lambda_0 s} + Ba_0^k e^{-k\lambda_0 s} + Ca_0^\ell e^{-\ell\lambda_0 s} + D. \end{aligned}$$

Consider first the case $\ell \geq 2$. Comparing the exponents in (4), we see that either the coefficient of the first term of A_1 is zero, or there is a term in $F(\varphi_{11}, \varphi'_{11})$ having the same exponent as the first term of A_1 . If $\lambda_0 > 0$, the expression $F(\varphi_{11}, \varphi'_{11})$ contains a nonzero term with exponent λ_0 , i.e., $-\lambda_0 a_0 e^{-\lambda_0 s}$, but A_1 starts with a term with exponent λ_1 . Thus, $\lambda_0 = 0$, which is a contradiction. If $\lambda_0 < 0$, then $k\lambda_0 < \lambda_1 + (k-1)\lambda_0 < \lambda_1 + (\ell-1)\lambda_0 < \lambda_1$, and so the first nonzero term of A_1 is $a_0^{k-1} a_1 k B e^{-(\lambda_1 + (k-1)\lambda_0)s}$, while the first nonzero term of $F(\varphi_{11}, \varphi'_{11})$ is $Ba_0^k e^{-k\lambda_0 s}$, which is again a contradiction.

We next treat the case $\ell < 2$, i.e., $\ell = 1$. If $\lambda_0 < 0$, then the first term of A_1 is $(C - \lambda_1)a_1 e^{-\lambda_1 s}$ and $F(\varphi_{11}, \varphi'_{11}) = (C - \lambda_0)a_0 e^{-\lambda_0 s} + Ba_0^k e^{-k\lambda_0 s} + D$. Comparing the first term, we get $B = 0$ which is a contradiction. $\square$

Lemma 2.1 tells us that the least exponent, λ_0 , in an infinite Dirichlet series satisfying (2) is a nonnegative real number. We proceed now to obtain information about the coefficients B, C, D in (2).

Lemma 2.2 *If an infinite Dirichlet series $\varphi(s)$, as in (1), satisfies (2), then the three polynomials B, C, D are merely complex constants.*

Proof We need only to consider the two possibilities: $\lambda_0 = 0$ and $\lambda_0 > 0$.

Case $\lambda_0 = 0$. With $n = 1$ in (4), i.e., writing $\varphi(s) = a_0 + \sum_{i \geq 1} a_i e^{-\lambda_i s}$, we see that the first term of A_1 is $(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_1)a_1 e^{-\lambda_1 s}$, and $F(\varphi_{11}, \varphi'_{11}) = Ba_0^k + Ca_0^\ell + D$. Comparing the coefficients in (4), we have

$$Ba_0^k + Ca_0^\ell + D = 0 \quad \text{and} \quad kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_1 = 0. \quad (7)$$

Next, with $n = 2$ in (4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + \sum_{i \geq 2} a_i e^{-\lambda_i s}$, the first term in A_2 is $(kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_2)a_2 e^{-\lambda_2 s} \neq 0$ (by (7)), and

$$F(\varphi_{12}, \varphi'_{12}) = -\lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 + a_1 e^{-\lambda_1 s})^k + C(a_0 + a_1 e^{-\lambda_1 s})^\ell + D.$$

The coefficient of the term $e^{-2\lambda_1}$ in $F(\varphi_{12}, \varphi'_{12})$ is

$$\left(B \binom{k}{2} a_0^{k-2} a_1^2 + C \binom{\ell}{2} a_0^{\ell-2} a_1^2 \right) e^{-2\lambda_1 s}.$$

If $\lambda_2 \neq 2\lambda_1$, then the coefficient of the term $e^{-2\lambda_1 s}$ in $F(\varphi_{1,2}, \varphi'_{12})$ is zero, i.e.,

$$B \binom{k}{2} a_0^{k-2} a_1^2 + C \binom{\ell}{2} a_0^{\ell-2} a_1^2 = 0. \quad (8)$$

Using (7) and (8), we deduce that B and C are complex constants, and using (7) again we have that D is also a complex constant.

If $\lambda_2 = 2\lambda_1$, then the coefficient of the term with exponent $2\lambda_1 s$ in $F(\varphi_{12}, \varphi'_{12})$ and the term with exponent $\lambda_2 s$ in A_2 add up to zero, i.e.,

$$B \binom{k}{2} a_0^{k-2} a_1^2 + C \binom{\ell}{2} a_0^{\ell-2} a_1^2 + (kBa_0^{k-1} + \ell Ca_0^{\ell-1} - \lambda_2)a_2 = 0. \quad (9)$$

Again using (7) and (9), we deduce that $B, C, D \in \mathbb{C}$.

Case $\lambda_0 > 0$. With $n = 1$ in (4), i.e., $\varphi(s) = a_0 e^{-\lambda_0 s} + \sum_{i \geq 1} a_i e^{-\lambda_i s}$, Lemma 2.1 yields $\ell = 1$. In this case the first term of A_1 is $(C - \lambda_1)a_1 e^{-\lambda_1 s}$, and

$$F(\varphi_{11}, \varphi'_{11}) = (C - \lambda_0)a_0 e^{-\lambda_0 s} + Ba_0^k e^{-k\lambda_0 s} + D.$$

Comparing the coefficients in (4), we have $D = 0$ and $C = \lambda_0 \in \mathbb{C}$. Thus, $(C - \lambda_1)a_1 e^{-\lambda_1 s} \neq 0$, and so $\lambda_1 = k\lambda_0$ and $B = (k - 1)\lambda_0 a_1 a_0^{-k} \in \mathbb{C}$. $\square$

From Lemma 2.2, throughout the rest of the paper, we assume that $B(\neq 0)$, C , D are complex numbers.

3 The case $\ell = 1$

In this section we treat the case $\ell = 1$, $k \geq 2$, in (2), i.e., the differential equation takes the form

$$F(f, f') = f' + Bf^k + Cf + D = 0, \quad B(\neq 0), C, D \in \mathbb{C}. \quad (10)$$

Theorem 3.1 Assume that an infinite Dirichlet series $\varphi(s)$, as in (1), satisfies (10).

(i) For $\lambda_0 = 0$, let $c_1 = 1$, $c_2 = \binom{k}{2}/\lambda_1$, and for $n \geq 3$, let

$$c_n = \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_0+q_1+\dots+q_{n-1}=k \\ q_1+2q_2+\dots+(n-1)q_{n-1}=n}} \binom{k}{q_0, \dots, q_{n-1}} \left(\frac{c_2(\lambda_1 - C)}{k} \right)^{q_2} \dots \left(\frac{c_{n-1}(\lambda_1 - C)}{k} \right)^{q_{n-1}}.$$

If $c_n \neq 0$, $n \geq 3$, then $Ba_0^k + Ca_0 + D = 0$, $\lambda_1 = kBa_0^{k-1} + C$, and for $n \geq 2$, we have $\lambda_n = n\lambda_1$ and $a_n = Bc_n a_0^{k-n} a_1^n$, where $\binom{k}{q_0, \dots, q_{n-1}}$ is a multinomial coefficient.

(ii) For $\lambda_0 > 0$, let $d_1 = 1/(k-1)C$, and for $n \geq 2$, let

$$d_n = \frac{d_1}{n} \sum_{\substack{t_0+t_1+\dots+t_{n-1}=k \\ t_0+kt_1+\dots+(n-1)k-(n-2)t_{n-1}=nk-(n-1)}} \binom{k}{t_0, \dots, t_{n-1}} (Bd_1)^{t_1} \dots (Bd_{n-1})^{t_{n-1}}.$$

Then $D = 0$, $\lambda_0 = C$, and for $n \geq 1$, we have $\lambda_n = (nk - (n-1))C$ and $a_n = Bd_n a_0^{nk-(n-1)}$.

Proof (i) From (3), we have $A_n = \varphi_{2n}(kBa_0^{k-1} + C) + \varphi'_{2n}$. Since $\lambda_0 = 0$, using (5), the first term of A_n is $(kBa_0^{k-1} + C - \lambda_n)a_n e^{-\lambda_n s}$. Taking $n = 1$ in (4), i.e., $\varphi(s) = a_0 + \sum_{i \geq 1} a_i e^{-\lambda_i s}$, the first term of A_1 is $(kBa_0^{k-1} + C - \lambda_1)a_1 e^{-\lambda_1 s}$, and $F(\varphi_{11}, \varphi'_{11}) = Ba_0^k + Ca_0 + D$. Comparing the coefficients in (4), we get $Ba_0^k + Ca_0 + D = 0$ and $\lambda_1 = kBa_0^{k-1} + C$, i.e.,

$$0 \neq B = \frac{\lambda_1 - C}{ka_0^{k-1}}.$$

Taking $n = 2$ in (4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + \sum_{i \geq 2} a_i e^{-\lambda_i s} = \varphi_{12} + \varphi_{22}$, the first term of A_2 is $(\lambda_1 - \lambda_2)a_2 e^{-\lambda_2 s} \neq 0$. The term with exponent $2\lambda_1$ in

$$F(\varphi_{12}, \varphi'_{12}) = -\lambda_1 a_1 e^{-\lambda_1 s} + B(a_0 + a_1 e^{-\lambda_1 s})^k + C(a_0 + a_1 e^{-\lambda_1 s}) + D,$$

is $\binom{k}{2}Ba_0^{k-2}a_1^2 e^{-2\lambda_1 s} \neq 0$. Comparing the coefficients in (4), we get $\lambda_2 = 2\lambda_1$ and $(\lambda_1 - \lambda_2)a_2 + \binom{k}{2}Ba_0^{k-2}a_1^2 = 0$, and so $a_2 = Bc_2 a_0^{k-2} a_1^2$, where $c_2 = \binom{k}{2}/\lambda_1$. Taking

$n = 3$ in (4), i.e., $\varphi(s) = a_0 + a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s} + \sum_{i \geq 3} a_i e^{-\lambda_i s} = \varphi_{13} + \varphi_{23}$, the first term of A_3 is $(\lambda_1 - \lambda_3)a_3 e^{-\lambda_3 s} \neq 0$. The term with exponent $3\lambda_1$ in

$$F(\varphi_{13}, \varphi'_{13}) = -\lambda_1 a_1 e^{-\lambda_1 s} - \lambda_2 a_2 e^{-\lambda_2 s} + B(a_0 + a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s})^k \\ + C(a_0 + a_1 e^{-\lambda_1 s} + a_2 e^{-\lambda_2 s}) + D,$$

has a coefficient equal to

$$B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} a_0^{t_0} a_1^{t_1} a_2^{t_2} \\ = B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} a_0^{t_0} a_1^{t_1} \left(\frac{\lambda_1 - C}{k a_0^{k-1}} c_2 a_0^{k-2} a_1^2 \right)^{t_2} \\ = B \sum_{\substack{t_0+t_1+t_2=k \\ t_1+2t_2=3}} \binom{k}{t_0, t_1, t_2} \left(\frac{c_2(\lambda_1 - C)}{k} \right)^{t_2} a_0^{k-3} a_1^3 = 2B\lambda_1 c_3 a_0^{k-3} a_1^3 \neq 0.$$

Equating these terms, we get $\lambda_3 = 3\lambda_1$ and $(\lambda_1 - \lambda_3)a_3 e^{-\lambda_3 s} \neq 0$, i.e., $a_3 = Bc_3 a_0^{k-3} a_1^3$. Continuing inductively, assume $\lambda_n = n\lambda_1$, $n > 1$, and $a_n = Bc_n a_0^{k-n} a_1^n$. Writing

$$\varphi(s) = \sum_{i=0}^n a_i e^{-\lambda_i s} + \sum_{i \geq n+1} a_i e^{-\lambda_i s} = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s),$$

we see that the first term of A_{n+1} is $(\lambda_1 - \lambda_{n+1})a_{n+1} e^{-\lambda_{n+1} s} \neq 0$. The coefficient of the term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi'_{1,n+1})$ is

$$B \sum_{\substack{t_0+\dots+t_n=k \\ t_1+\dots+nt_n=n+1}} \binom{k}{t_0, \dots, t_n} a_0^{t_0} a_1^{t_1} \dots a_n^{t_n} \\ = B \sum_{\substack{t_0+\dots+t_n=k \\ t_1+\dots+nt_n=n+1}} \binom{k}{t_0, \dots, t_n} \left(\frac{c_2(\lambda_1 - C)}{k a_0^{k-1}} a_0^{k-2} a_1^2 \right)^{t_2} \\ \dots \left(\frac{c_n(\lambda_1 - C)}{k a_0^{k-1}} a_0^{k-n} a_1^n \right)^{t_n} \\ = B \sum_{\substack{t_0+\dots+t_n=k \\ t_1+\dots+nt_n=n+1}} \binom{k}{t_0, \dots, t_n} \left(\frac{c_2(\lambda_1 - C)}{k} \right)^{t_2} \\ \dots \left(\frac{c_n(\lambda_1 - C)}{k} \right)^{t_n} a_1^{n+1} a_0^{k-(n+1)} \\ = nB\lambda_1 c_{n+1} a_1^{n+1} a_0^{k-(n+1)} \neq 0.$$

Comparing the coefficients in (4), we get $\lambda_{n+1} = (n+1)\lambda_1$ and

$$(\lambda_1 - \lambda_{n+1})a_{n+1} + nB\lambda_1c_{n+1}a_1^{n+1}a_0^{k-(n+1)} = 0,$$

i.e., $a_{n+1} = Bc_{n+1}a_0^{k-(n+1)}a_1^{n+1}$.

(ii) We assume for the time being that $d_n \neq 0$ for all $n \in \mathbb{N}$. Since $\lambda_0 > 0$, from (6), the first term of A_n is $(C - \lambda_n)a_ne^{-\lambda_ns}$. Taking $n = 1$ in (4), i.e., $\varphi(s) = a_0e^{-\lambda_0s} + \sum_{i \geq 1} a_ie^{-\lambda_is}$, the first term of A_1 is $(C - \lambda_1)a_1e^{-\lambda_1s}$, and $F(\varphi_{11}, \varphi'_{11}) = (C - \lambda_0)a_0e^{-\lambda_0s} + Ba_0^k e^{-k\lambda_0s} + D$. Comparing the coefficients in (4), we get $D = 0$, $C = \lambda_0 > 0$. Since $(C - \lambda_1)a_1 \neq 0$, we have $\lambda_1 = k\lambda_0 = kC$, and $Ba_0^k + (C - \lambda_1)a_1 = 0$, i.e., $a_1 = Bd_1a_0^k$, where $d_1 = 1/(k-1)C$. Taking $n = 2$ in (4), i.e., $\varphi(s) = a_0e^{-\lambda_0s} + a_1e^{-\lambda_1s} + \sum_{i \geq 2} a_ie^{-\lambda_is} = \varphi_{12} + \varphi_{22}$, the first term of A_2 is $(C - \lambda_2)a_2e^{-\lambda_2s} \neq 0$. The term with exponent $(2k-1)\lambda_0$ in

$$\begin{aligned} F(\varphi_{12}, \varphi'_{12}) &= -\lambda_0a_0e^{-\lambda_0s} - \lambda_1a_1e^{-\lambda_1s} \\ &\quad + B(a_0e^{-\lambda_0s} + a_1e^{-\lambda_1s})^k + C(a_0e^{-\lambda_0s} + a_1e^{-\lambda_1s}) + D, \end{aligned}$$

which is the term right after the one with exponent $k\lambda_0$, has a coefficient equal to

$$\begin{aligned} B \sum_{\substack{t_0+t_1=k \\ t_0+kt_1=2k-1}} \binom{k}{t_0, t_1} a_0^{t_0} a_1^{t_1} \\ = B \sum_{\substack{t_0+t_1=k \\ t_0+kt_1=2k-1}} \binom{k}{t_0, t_1} (Bd_1)^{t_1} a_0^{2k-1} = 2(k-1)d_2CBa_0^{2k-1} \neq 0. \end{aligned}$$

Comparing these terms, we get $\lambda_2 = (2k-1)\lambda_0$ and $(C - \lambda_2)a_2 + 2(k-1)d_2CBa_0^{2k-1} = 0$, i.e., $a_2 = Bd_2a_0^{2k-1}$. To carry out the induction step, assume that $\lambda_n = (nk - (n-1))C$ and $a_n = Bd_na_0^{nk-(n-1)}$. Writing $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, we see that the first term of A_{n+1} is $(C - \lambda_{n+1})a_{n+1}e^{-\lambda_{n+1}s} \neq 0$. The term with exponent $((n+1)k - n)C$ in $F(\varphi_{1,n+1}, \varphi'_{1,n+1})$, which is this term following the term with exponent $(nk - (n-1))C$, has a coefficient equal to

$$\begin{aligned} B \sum_{\substack{t_0+t_1+\dots+t_n=k \\ t_0+kt_1+\dots+(nk-(n-1))t_n=(n+1)k-n}} \binom{k}{t_0, \dots, t_n} a_0^{t_0} \dots a_n^{t_n} \\ = B \sum_{\substack{t_0+\dots+t_n=k \\ t_0+\dots+(nk-(n-1))t_n=(n+1)k-n}} \binom{k}{t_0, \dots, t_n} (Bd_1)^{t_1} \dots (Bd_n)^{t_n} a_0^{(n+1)k-n} \\ = (n+1)(k-1)Cd_{n+1}Ba_0^{(n+1)k-n} \neq 0. \end{aligned}$$

Comparing these terms, we get $\lambda_{n+1} = ((n+1)k - n)C$ and

$$(C - \lambda_{n+1})a_{n+1} + (n+1)(k-1)Cd_{n+1}Ba_0^{(n+1)k-n} = 0,$$

i.e., $a_{n+1} = Bd_{n+1}a_0^{(n+1)k-n}$. There remains to verify that $d_n \neq 0$ for all $n \in \mathbb{N}$. This follows immediately from $d_1 = 1/(k-1)C$ and the following identity:

$$d_n = \frac{1}{n} B^{n-1} d_1^n D_n, \quad n \geq 2, \quad (11)$$

where all D_n are positive real numbers.

Let us proof (11). Note first that

$$d_2 = \frac{d_1}{2} \binom{k}{k-1, 1} B d_1 = \frac{1}{2} B d_1^2 D_2$$

with $D_2 = k$, i.e., the identity (11) holds for $n = 2$. Proceeding by induction and replacing the second condition in the summation of (ii) by subtracting the first condition, we have

$$\begin{aligned} d_{n+1} &= \frac{d_1}{n+1} \sum_{\substack{t_0+t_1+\dots+t_n=k \\ t_1+2t_2+\dots+nt_n=n}} \binom{k}{t_0, \dots, t_n} (Bd_1)^{t_1} \dots (Bd_n)^{t_n} \\ &= \frac{B^n d_1^{n+1}}{n+1} \sum_{\substack{t_0+t_1+\dots+t_n=k \\ t_1+2t_2+\dots+nt_n=n}} \binom{k}{t_0, \dots, t_n} \left(\frac{D_2}{2}\right)^{t_2} \dots \left(\frac{D_n}{n}\right)^{t_n} \\ &= \frac{B^n d_1^{n+1} D_{n+1}}{n+1} \neq 0. \end{aligned} \quad \square$$

Applying Lemma 2.2 and the case $k = 2$ in (10) of Theorem 3.1, which corresponds to the classical Riccati equation, we have

Corollary 3.2 Assume that an infinite Dirichlet series $\varphi(s)$, as in (1), satisfies the Riccati equation

$$f' + Bf^2 + Cf + D = 0.$$

(i) If $\lambda_0 = 0$, then

$$\varphi(s) = a_0 + \frac{a_1 \lambda_1}{\lambda_1 e^{\lambda_1 s} - Ba_1}$$

with $a_0 = (\lambda_1 - C)/2B$, $C^2 - \lambda_1^2 = 4BD$ and $|Ba_1 e^{-\lambda_1 s}/\lambda_1| < 1$.

(ii) If $\lambda_0 > 0$, then $D = 0$, $\lambda_0 = C$, and

$$\varphi(s) = \frac{a_0 C}{C e^{Cs} - a_0 B}$$

with $|Ba_0 e^{-Cs}/C| < 1$.

Proof (i) When $\lambda_0 = 0$, from the proof of Theorem 3.1 (i), we get $B = (\lambda_1 - C)/2a_0$ and $Ba_0^2 + Ca_0 + D = 0$, i.e., $a_0 = (\lambda_1 - C)/2B$ and $C^2 - \lambda_1^2 = 4BD$. Since $c_2 = 1/\lambda_1$, $c_3 = Ba_0/\lambda_1^2$, assuming $c_{n-1} = (Ba_0)^{n-3}/\lambda_1^{n-2}$, we see that

$$\begin{aligned} c_n &= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_0+\dots+q_{n-1}=2 \\ q_1+\dots+(n-1)q_{n-1}=n}} \binom{2}{q_0, \dots, q_{n-1}} \left(\frac{c_2(\lambda_1 - C)}{2} \right)^{q_2} \\ &\quad \dots \left(\frac{c_{n-1}(\lambda_1 - C)}{2} \right)^{q_{n-1}} \\ &= \frac{1}{(n-1)\lambda_1} \sum_{\substack{q_1+\dots+q_{n-1}=2 \\ q_1+\dots+(n-1)q_{n-1}=n}} \binom{2}{q_1, \dots, q_{n-1}} \frac{(Ba_0)^{n-2}}{\lambda_1^{n-2}} = \frac{(Ba_0)^{n-2}}{\lambda_1^{n-1}}. \end{aligned}$$

Substituting $a_n = (Ba_1/\lambda_1)^{n-1}a_1$ from Theorem 3.1 (i) into $\varphi(s)$, we get

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s} = a_0 + \sum_{i=1}^{\infty} \left(\frac{Ba_1}{\lambda_1} \right)^{i-1} a_1 e^{-i\lambda_1 s} = a_0 + \frac{a_1 \lambda_1}{\lambda_1 e^{\lambda_1 s} - Ba_1}.$$

(ii) When $\lambda_0 > 0$, since $d_1 = 1/C$, $d_2 = B/C^2$, assuming that $d_{n-1} = B^{n-2}/C^{n-1}$, by Theorem 3.1 (ii), we get

$$d_n = \frac{1}{nC} \sum_{\substack{t_0+\dots+t_{n-1}=2 \\ t_0+\dots+nt_{n-1}=n+1}} \binom{2}{t_0, \dots, t_{n-1}} (Bd_1)^{t_1} \dots (Bd_{n-1})^{t_{n-1}} = \frac{B^{n-1}}{C^n}.$$

Using Theorem 3.1 again, we have $a_n = (B/C)^n a_0^{n+1}$ and $\lambda_n = (n+1)C$, and so

$$\varphi(s) = \sum_{i=0}^{\infty} a_i e^{-\lambda_i s} = \sum_{i=0}^{\infty} \left(\frac{B}{C} \right)^i a_0^{i+1} e^{-(n+1)Cs} = \frac{a_0 C}{C e^{Cs} - a_0 B}.$$

□

4 The case $\ell = 2, k = 3$

In this section we treat the case $k = 3, \ell = 2$ of (2). The differential equation takes the form

$$F(f, f') = f' + Bf^3 + Cf^2 + D = 0, \quad B(\neq 0), C, D \in \mathbb{C}. \quad (12)$$

We will show that for $D = 0$, equation (12) has a unique infinite Dirichlet series solution, while for $D \neq 0$, there generally are more than one infinite Dirichlet series solutions.

Theorem 4.1 *If an infinite Dirichlet series $\varphi(s)$, as in (1), satisfies*

$$f' + Bf^3 + Cf^2 = 0, \quad B(\neq 0), C \in \mathbb{C},$$

then $Ba_0 + C = 0$, $\lambda_0 = 0$, $\lambda_1 = Ba_0^2$ and for $n \geq 2$, we have $\lambda_n = n\lambda_1$, $a_n = c_n a_1^n / a_0^{n-1}$, where $c_1 = 1$ and $(n-1)c_n$, $n \geq 2$, is the coefficient of x^n in the expansion of

$$(1 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})^2(c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}). \quad (13)$$

Proof From (3), we have $A_n = \varphi_{2n}(3B\varphi_{1n}^2 + 2C\varphi_{1n}) + \varphi'_{2n}$. By Lemma 2.1, we know that $\lambda_0 = 0$. From (5), the first term of A_n is $(3Ba_0^2 + 2Ca_0 - \lambda_n)a_ne^{-\lambda_ns}$. Taking $n = 1$ in (4), i.e., $\varphi(s) = a_0 + \sum_{i \geq 1} a_ie^{-\lambda_is}$, the first term of A_1 is $(3Ba_0^2 + 2Ca_0 - \lambda_1)a_1e^{-\lambda_1s} \neq 0$, and $F(\varphi_{11}, \varphi'_{11}) = Ba_0^3 + Ca_0^2$. Comparing terms, we get $\lambda_1 = 3Ba_0^2 + 2Ca_0$ and $Ba_0^3 + Ca_0^2 = 0$. Thus, $Ba_0 + C = 0$ and $\lambda_1 = Ba_0^2$. Next, taking $n = 2$ in (4), i.e., $\varphi(s) = a_0 + a_1e^{-\lambda_1s} + \sum_{i \geq 2} a_ie^{-\lambda_is}$. In this case the first term of A_2 is $(\lambda_1 - \lambda_2)a_2e^{-\lambda_2s} \neq 0$. The term with exponent $2\lambda_1$ in $F(\varphi_{12}, \varphi'_{12})$ is $2Ba_0a_1^2e^{-2\lambda_1s} \neq 0$. Comparing these terms, we get $\lambda_2 = 2\lambda_1$ and $(\lambda_1 - \lambda_2)a_2 + 2Ba_0a_1^2 = 0$. Thus,

$$a_2 = \frac{2Ba_0a_1^2}{\lambda_1} = \frac{2a_1^2}{a_0} = \frac{c_2a_1^2}{a_0},$$

where c_2 is the coefficient of x^2 in the expansion of $(1 + c_1x)^2(c_1x)$.

For $n \in \mathbb{N}$, assume that $\lambda_n = n\lambda_1$ and $a_n = c_n a_1^n / a_0^{n-1}$, where $(n-1)c_n$ is the coefficient of x^n of $(1 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})^2(c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})$. Writing $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, the first term of A_{n+1} is $(\lambda_1 - \lambda_{n+1})a_{n+1}e^{-\lambda_{n+1}s} \neq 0$. The term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi'_{1,n+1})$ is

$$\begin{aligned} & B \sum_{\substack{t_0 + \cdots + t_n = 3 \\ t_1 + \cdots + nt_n = n+1}} \binom{3}{t_0, \dots, t_n} a_0^{t_0} a_1^{t_1} \cdots a_n^{t_n} \\ & \quad + C \sum_{\substack{q_0 + \cdots + q_n = 2 \\ q_1 + \cdots + nq_n = n+1}} \binom{2}{q_0, \dots, q_n} a_0^{q_0} a_1^{q_1} \cdots a_n^{q_n} \\ & = \left(\sum_{\substack{t_0 + \cdots + t_n = 3 \\ t_1 + \cdots + nt_n = n+1}} \binom{3}{t_0, \dots, t_n} c_2^{t_2} \cdots c_n^{t_n} - \sum_{\substack{q_0 + \cdots + q_n = 2 \\ q_1 + \cdots + nq_n = n+1}} \binom{2}{q_0, \dots, q_n} c_2^{q_2} \cdots c_n^{q_n} \right) \frac{Ba_1^{n+1}}{a_0^{n-2}} \\ & \stackrel{\text{def}}{=} Q \frac{Ba_1^{n+1}}{a_0^{n-2}}. \end{aligned}$$

In the second sum, we must have $q_0 = 0$, which shows that the second sum is less than the case $t_0 = 1$ in the first sum, and this implies that $Q > 0$. Consequently, $\lambda_{n+1} = (n+1)\lambda_1$ and $(\lambda_1 - \lambda_{n+1})a_{n+1} + QBa_1^{n+1}/a_0^{n-2} = 0$, i.e., $a_{n+1} = QBa_1^{n+1}/n\lambda_1 a_0^{n-2} =$

Qa_1^{n+1}/na_0^n . Since Q is the coefficient of x^{n+1} in the expansion $(1 + c_1x + \cdots + c_nx^n)^3 - (1 + c_1x + \cdots + c_nx^n)^2 = (1 + c_1x + \cdots + c_nx^n)^2(c_1x + \cdots + c_nx^n)$, putting $nc_{n+1} = Q$, we get $a_{n+1} = c_{n+1}a_1^{n+1}/a_0^n$, as desired. $\square$

Remark 4.2 The following supplement to Theorem 4.1 is generously provided by one of the referees, to whom we express our sincere gratitude. It is indeed possible to derive an explicit formula for the coefficient c_n alluded to in Theorem 4.1. If $c_1 = 1$ and $(n-1)c_n, n \geq 2$, is the coefficient of x^n in the expression of (13), then setting $c_0 = 1$, we have

$$c_n = \frac{1}{n-1} \left(\sum_{v=1}^{n-1} c_v c_{n-v} + \sum_{v_1} c_{v_2} c_{n-v_1-v_2} \right).$$

To see this, we write

$$\begin{aligned} & (1 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1})^2 (c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1}) \\ &= \sum_{v_1=0}^{n-1} c_{v_1} x^{v_1} \sum_{v_2=0}^{n-1} c_{v_2} x^{v_2} \sum_{v_3=0}^{n-1} c_{v_3} x^{v_3} = \sum_{v_1=0}^{n-1} \sum_{v_2=0}^{n-1} \sum_{v_3=0}^{n-1} c_{v_1} c_{v_2} c_{v_3} x^{v_1+v_2+v_3}. \end{aligned}$$

Thus,

$$\begin{aligned} (n-1)c_n &= \sum_{v_1=0}^{n-1} \sum_{v_2=0}^{n-1} \sum_{v_3=0}^{n-1} c_{v_1} c_{v_2} c_{v_3} = \sum_{\substack{v_1=0, v_2=0 \\ 1 \leq v_1+v_2 \leq n-1}}^{n-1} c_{v_1} c_{v_2} c_{n-v_1-v_2} \\ &= \sum_{v_2=0}^{n-1} c_0 c_{v_2} c_{n-v_2} + \sum_{\substack{v_1=1, v_2=0 \\ v_1+v_2 \leq n-1}}^{n-1} c_{v_1} c_{v_2} c_{n-v_1-v_2} \\ &= \sum_{v=1}^{n-1} c_v c_{n-v} + \sum_{v_1=1}^{n-1} \sum_{v_2=0}^{n-1-v_1} c_{v_1} c_{v_2} c_{n-v_1-v_2}. \end{aligned}$$

We next solve (12) with $D \neq 0$. We show that it generally has more than one infinite Dirichlet series solutions; one such solution is given explicitly in the next theorem.

Theorem 4.3 The infinite Dirichlet series $\varphi(s) = a_0 + \sum_{i=1}^{\infty} a_i e^{-\lambda_i s}$ with

$$\lambda_1 = 3Ba_0^2 + 2C, \quad a_0 = -C/3B, \quad Ba_0^3 + Ca_0^2 + D = 0$$

and

$$\lambda_n = (2n-1)\lambda_1, \quad a_n = \left(\frac{Ba_1^2}{4\lambda_1} \right)^{n-1} \binom{2(n-1)}{n-1} a_1, \quad n \in \mathbb{N},$$

satisfies the differential equation

$$f' + Bf^3 + Cf^2 + D = 0, \quad B(\neq 0), C, D(\neq 0) \in \mathbb{C}, \quad (14)$$

provided that $|Ba_1^2 e^{-2\lambda_1 s} / 4\lambda_1| < 1/4$.

Proof Setting $Y = Ba_1^2 e^{-2\lambda_1 s} / 4\lambda_1$ and assuming $|Y| < 1/4$, we have

$$\varphi(s) = -\frac{C}{3B} + \sum_{i=1}^{\infty} \left(\frac{Ba_1^2}{4\lambda_1} \right)^{i-1} \binom{2(i-1)}{i-1} a_1 e^{-(2i-1)\lambda_1 s} = -\frac{C}{3B} + \frac{a_1 e^{-\lambda_1 s}}{\sqrt{1-4Y}}.$$

It is easily checked that this infinite Dirichlet series satisfies (12). $\square$

To find another possible infinite Dirichlet series solution of (12), we begin with

Theorem 4.4 *The infinite Dirichlet series $\varphi(s) = a_0 + \sum_{i \geq 1} a_i e^{-\lambda_i s}$, as in (1), with*

$$\lambda_0 = 0 < \lambda_1 = 3Ba_0^2 + 2Ca_0, \quad \lambda_n = n\lambda_1, \quad n \geq 2, \quad (15)$$

$$Ba_0^3 + Ca_0^2 + D = 0, \quad a_2 = \frac{(3Ba_0 + C)a_1^2}{\lambda_1} \neq 0 \quad (16)$$

and the remaining coefficients recursively determined by

$$\begin{aligned} a_{n+1} = & \frac{1}{n\lambda_1} \sum_{\substack{q_1 + \dots + q_n = 2 \\ q_1 + \dots + nq_n = n+1}} (3Ba_0 + C) \binom{2}{q_1, \dots, q_n} a_1^{q_1} a_2^{q_2} \dots a_n^{q_n} \\ & + \frac{1}{n\lambda_1} \sum_{\substack{t_1 + \dots + t_n = 3 \\ t_1 + \dots + nt_n = n+1}} B \binom{3}{t_1, \dots, t_n} a_1^{t_1} a_2^{t_2} \dots a_n^{t_n}, \quad n \geq 3, \end{aligned} \quad (17)$$

satisfies the differential equation (14).

Proof By Lemma 2.1, we have $\lambda_0 = 0$. From (5), the first term of A_n is $(3Ba_0^2 + 2Ca_0 - \lambda_n)a_n e^{-n\lambda_1 s}$. For $n = 1$, using similar arguments as in the proof of Theorem 4.1, we get

$$0 < \lambda_1 = 3Ba_0^2 + 2Ca_0, \quad Ba_0^3 + Ca_0^2 + D = 0.$$

For $n = 2$, using (15), the first term of A_2 is $-\lambda_1 a_2 e^{-\lambda_2 s}$, while the term with exponent $2\lambda_1$ in $F(\varphi_{12}, \varphi'_{12})$ is $3Ba_0 a_1^2 + Ca_1^2$. Comparing these terms, we get a_2 as given in (16). Proceeding by induction from $\varphi(s) = \varphi_{1,n+1}(s) + \varphi_{2,n+1}(s)$, using (15), we see that the first term of A_{n+1} is $-n\lambda_1 a_{n+1} e^{-(n+1)\lambda_1 s}$, while the coefficient of the term with exponent $(n+1)\lambda_1$ in $F(\varphi_{1,n+1}, \varphi'_{1,n+1})$ is

$$\begin{aligned}
& B \sum_{\substack{t_0+\dots+t_n=3 \\ t_1+\dots+nt_n=n+1}} \binom{3}{t_0, \dots, t_n} a_0^{t_0} a_1^{t_1} \dots a_n^{t_n} \\
& \quad + C \sum_{\substack{q_0+\dots+q_n=2 \\ q_1+\dots+nq_n=n+1}} \binom{2}{q_0, \dots, q_n} a_0^{q_0} a_1^{q_1} \dots a_n^{q_n} \\
& = \sum_{\substack{q_1+\dots+q_n=2 \\ q_1+\dots+nq_n=n+1}} \left\{ B \binom{3}{1, q_1, \dots, q_n} a_0 + C \binom{2}{q_1, \dots, q_n} \right\} a_1^{q_1} \dots a_n^{q_n} \\
& \quad + \sum_{\substack{t_1+\dots+t_n=3 \\ t_1+\dots+nt_n=n+1}} B \binom{3}{t_1, \dots, t_n} a_1^{t_1} \dots a_n^{t_n} \quad (q_0 = 0 \text{ and } t_0 = 0 \text{ or } 1) \\
& = \sum_{\substack{q_1+\dots+q_n=2 \\ q_1+\dots+nq_n=n+1}} (3Ba_0 + C) \binom{2}{q_1, \dots, q_n} a_1^{q_1} \dots a_n^{q_n} \\
& \quad + \sum_{\substack{t_1+\dots+t_n=3 \\ t_1+\dots+nt_n=n+1}} B \binom{3}{t_1, \dots, t_n} a_1^{t_1} \dots a_n^{t_n}.
\end{aligned}$$

Comparing the terms, (17) follows. $\square$

Using Theorem 4.4, another solution of (12) is now easily constructed.

Corollary 4.5 *Adopting the notation of Theorem 4.4, if $|3Ba_0 + C| \leq 1$, and $|a_1| = 4/|B| < \lambda_1$, then for $|e^{-\lambda_1 s}| < 1/4$ the corresponding Dirichlet series (1) converges and satisfies (14).*

Proof We first show that $|a_n| \leq \binom{2(n-1)}{n-1} 4/|B|$. This is trivial for $n = 1, n = 2$. Using Theorem 4.4, and induction hypothesis, we get

$$\begin{aligned}
|a_{n+1}| & \leq \frac{1}{n\lambda_1} \sum_{\substack{q_1+\dots+q_n=2 \\ q_1+\dots+nq_n=n+1}} \binom{2}{q_1, \dots, q_n} |a_1|^{q_1} \dots |a_n|^{q_n} \\
& \quad + \frac{1}{n\lambda_1} \sum_{\substack{t_1+\dots+t_n=3 \\ t_1+\dots+nt_n=n+1}} |B| \binom{3}{t_1, \dots, t_n} |a_1|^{t_1} \dots |a_n|^{t_n} \\
& \leq \frac{1}{n\lambda_1} \sum_{\substack{q_1+\dots+q_n=2 \\ q_1+\dots+nq_n=n+1}} \binom{2}{q_1, \dots, q_n} \binom{2}{1}^{q_2} \\
& \quad \dots \binom{2(n-1)}{n-1}^{q_n} \left(\frac{4}{|B|} \right)^{q_1+\dots+q_n} \\
& \quad + \frac{1}{n\lambda_1} \sum_{\substack{t_1+\dots+t_n=3 \\ t_1+\dots+nt_n=n+1}} |B| \binom{3}{t_1, \dots, t_n} \binom{2}{1}^{t_2}
\end{aligned}$$

$$\begin{aligned} & \dots \binom{2(n-1)}{n-1}^{t_n} \left(\frac{4}{|B|} \right)^{t_1 + \dots + t_n} \\ & \leq \frac{4}{n|B|} \left(\sum_{\substack{q_1 + \dots + q_n = 2 \\ q_1 + \dots + nq_n = n+1}} \binom{2}{q_1, \dots, q_n} 2^{q_2} \dots \binom{2(n-1)}{n-1}^{q_n} \right. \\ & \quad \left. + \sum_{\substack{t_1 + \dots + t_n = 3 \\ t_1 + \dots + nt_n = n+1}} 4 \binom{3}{t_1, \dots, t_n} 2^{t_2} \dots \binom{2(n-1)}{n-1}^{t_n} \right). \end{aligned}$$

Observe that the term

$$\sum_{\substack{q_1 + \dots + q_n = 2 \\ q_1 + \dots + nq_n = n+1}} \binom{2}{q_1, \dots, q_n} \binom{2}{1}^{q_2} \dots \binom{2(n-1)}{n-1}^{q_n}$$

is the coefficient of x^{n-1} in the expansion

$$\left(1 + 2x + \dots + \binom{2(n-1)}{n-1} x^{n-1} \right)^2$$

which is also the coefficient of x^{n-1} in the expansion

$$\left(\sum_{j=0}^{\infty} \binom{2(j-1)}{j-1} x^{j-1} \right)^2 = \frac{1}{1-4x}, \quad \text{convergent when } |x| < \frac{1}{4}.$$

Similarly, the term

$$\sum_{\substack{t_1 + \dots + t_n = 3 \\ t_1 + \dots + nt_n = n+1}} \binom{3}{t_1, \dots, t_n} \binom{2}{1}^{t_2} \dots \binom{2(n-1)}{n-1}^{t_n}$$

is the coefficient of x^{n-2} in the expansion $(1-4x)^{-2/3}$. It follows that

$$|a_{n+1}| \leq \frac{4}{|B|} \left\{ \frac{4^{n-1}}{n} + \frac{3 \cdot 5 \dots (2n-3)}{n(n-2)!} \right\} \leq \frac{4}{|B|} \binom{2n}{n}.$$

The corresponding Dirichlet series solution thus converges as the following majorization shows:

$$\begin{aligned}
 |\varphi(s)| &= \left| \sum_{j=0}^{\infty} a_j e^{-\lambda_j s} \right| \leq \sum_{j=0}^{\infty} \binom{2(j-1)}{j-1} \frac{4}{|B|} |e^{-(j-1)\lambda_1 s}| |e^{-\lambda_1 s}| \\
 &= \frac{4|e^{-\lambda_1 s}|}{|B|\sqrt{1-4|e^{-\lambda_1 s}|}}, \quad \text{provided } |e^{-\lambda_1 s}| < \frac{1}{4}.
 \end{aligned}$$

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