DG-modules over de Rham DG-algebra

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Abstract For a morphism of smooth schemes over a regular affine base we define functors of derived direct image and extraordinary inverse image on coderived categories of DG-modules over de Rham DG-algebras. Positselski proved that for a smooth algebraic variety X over a field k of characteristic zero the coderived category of DG-modules over $\Omega_{X/k}^\bullet$ is equivalent to the unbounded derived category of quasi-coherent right $\mathcal{D}_X$ -modules. We prove that our functors correspond to the functors of the same name for $\mathcal{D}_X$ -modules under Positselski equivalence.

Keywords de Rham complex · D-module · Coderived category

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1 Introduction

Let X be a smooth algebraic variety over a field k of characteristic zero. Its de Rham complex $\Omega_X = \Omega_{X/k}^\bullet$ is a sheaf of DG-algebras, and quasi-coherent sheaves of DG-modules over Ω_X (we call them Ω_X -modules) form a DG-category, but the derived category of this category is usually not interesting. Recently Positselski defined the coderived category $D^{co}(\Omega_X\text{-qcoh})$ of quasi-coherent Ω_X -modules and proved that the unbounded derived category of quasi-coherent left $\mathcal{D}_X$ -modules $D(\mathcal{D}_X\text{-qcoh})$ and the coderived category $D^{co}(\Omega_X\text{-qcoh})$ are equivalent [11]. This result is a generalization of Koszul duality for symmetric and exterior algebras. In this paper we construct functors of derived tensor product, direct image and extraordinary inverse image on Ω_X -modules and prove that these functors correspond to the functors of the same name for $\mathcal{D}_X$ -modules under Positselski equivalence (see Sect. 9). For coherent Ω_X -modules we define the duality functor which correspond to duality on $\mathcal{D}_X$ -modules. Before this, we prove some preliminary results which have obvious $\mathcal{D}_X$ -module counterparts.

Some of these constructions are well-known for DG-categories of Ω -modules (for example, [7] and [2, 7.2]), but as far as we know there was no attempt to write down these definitions for coderived categories of Ω -modules. We feel it is important to do it.

2 Quasi-coherent sheaves over Ω_X

2.1 DG-rings and DG-modules A *DG-ring* is a graded ring $R = \bigoplus_{i \in \mathbb{Z}} R^i$ endowed with an odd derivation $d: R \rightarrow R$ of degree 1 such that $d^2 = 0$. We write R^{gr} when we view R as a graded ring (or a DG-ring with zero differential). We denote by $|x|$ the degree of a homogeneous element $x \in R^{|x|}$. A DG-ring is called *commutative* if $xy = (-1)^{|x||y|}yx$ for any pair of homogeneous elements x and y of R . In what follows we assume that R is a commutative DG-ring.

A (left) *DG-module* M over R is a graded R -module $M = \bigoplus_{i \in \mathbb{Z}} M^i$ endowed with differential $d_M: M \rightarrow M$ of degree 1 such that $d_M^2 = 0$, and for any pair of homogeneous elements $x \in R$ and $m \in M$ the following Leibniz rule holds:

$$d_M(xm) = dx \cdot m + (-1)^{|x|} x d_M(m).$$

The module $M[j] = \bigoplus_{i \in \mathbb{Z}} M^{i+j}$ is called the *shift* of M by an integer j . The differential on $M[j]$ is defined as $(-1)^j d_M$.

Let M and N be DG-modules over R . The *DG-module of homomorphisms* $\text{Hom}_R(M, N)$ from M to N over R is defined as follows. The component $\text{Hom}_R^i(M, N)$ consists of all homogeneous maps $f: M \rightarrow N$ of degree i such that $f(xm) = (-1)^i |x| x f(m)$ for homogeneous $x \in R$ and $m \in M$. Since R is commutative, the formula $(xf)(m) = xf(m)$ defines an action of R on $\text{Hom}_R(M, N)$. The differential in the complex $\text{Hom}_R(M, N)$ is given by the formula $df(m) = d_N(f(m)) - (-1)^{|f|} f(d_M(m))$. Clearly, $d^2(f) = 0$, and for any composable morphisms f and g one has $d(fg) = df \cdot g + (-1)^{|f|} f dg$.

We define *tensor product DG-module* $M \otimes_R N$ of DG-modules M and N over R as the tensor product $M^{\text{gr}} \otimes_{R^{\text{gr}}} N^{\text{gr}}$ with the differential given by the formula $d(m \otimes n) = dm \otimes n + (-1)^{|m|} m \otimes dn$.

2.2 DG-modules over the de Rham DG-algebra In this paper we always assume that all schemes are smooth, quasi-projective and connected over a regular connected affine base S . We denote by $\dim X$ the Krull dimension of X , and by Ω_X the sheaf of de Rham DG-algebras $\Omega_{X/S}^\bullet$. Note that Ω_X is a sheaf of commutative DG-algebras over $\mathcal{O}_X$. In what follows sheaves of DG-modules over Ω_X are assumed to be quasi-coherent over $\mathcal{O}_X$. We call them Ω_X -modules for short. Ω_X -modules form a DG-category $\Omega_X\text{-qcoh}$ with shifts, cones and arbitrary direct sums. The *complex of homomorphisms* $\text{Hom}_{\Omega_X}(M, N)$ from M to N over Ω_X is defined in a similar way as in the previous section. The component $\text{Hom}_{\Omega_X}^i(M, N)$ consists of all homogeneous maps $f: M \rightarrow N$ of degree i such that over any affine open $U \subset X$ and for any $\omega \in \Omega_U$ and $m \in M(U)$ the following relation holds: $f(\omega m) = (-1)^{i|\omega|} \omega f(m)$. The formula for the differential is the same as before. Note that $\text{Hom}_{\Omega_X}^\bullet(M, N)$ is a complex of $\mathcal{O}_S(S)$ -modules.

An Ω_X -module M is called *coherent* if M is coherent as an $\mathcal{O}_X$ -module. In particular, there is a finite set of indexes i such that the graded component M^i is not zero. Coherent Ω_X -modules form a DG-category $\Omega_X\text{-coh}$ with shifts, cones and finite direct sums.

Let M be a coherent Ω_X -module. In this case we can define an Ω_X -module $\mathcal{H}om_{\Omega_X}(M, N)$ of homomorphisms from M to N over Ω_X . For any open $U \subset X$ the DG-module $\mathcal{H}om_{\Omega_X}(M, N)(U)$ is the DG-module of homomorphisms $\text{Hom}_{\Omega_U}(M(U), N(U))$. Since M is coherent, $\mathcal{H}om_{\Omega_X}(M, N)$ is quasi-coherent.

Let M and N be Ω_X -modules. One defines the *tensor product* Ω_X -module $M \otimes_{\Omega_X} N$ as the associated sheaf to the presheaf $U \mapsto M(U) \otimes_{\Omega_U} N(U)$.

We define a *closed morphism* f as an element of $\text{Hom}_{\Omega_X}^0(M, N)$ such that $d(f) = 0$. The Ω_X -module M is called *contractible* if the identity morphism is homotopic to zero, i.e. $\text{Id}_M = d(h)$ for some $h \in \text{Hom}_{\Omega_X}^{-1}(M, N)$. Let $f: M \rightarrow N$ be a closed morphism of Ω_X -modules. The *cone* of f is the following Ω_X -module. As a graded module $\text{cone}(f) = M[1] \oplus N$, and the differential is given by the formula

$$d_{\text{cone}(f)}(m, n) = (-d_M(m), f(m) + d_N(n)).$$

There is an exact sequence of Ω_X -modules with closed differentials

$$0 \rightarrow N \rightarrow \text{cone}(f) \rightarrow M[1] \rightarrow 0.$$

We need the following generalization of the cone. Let $\cdots \rightarrow M_i \rightarrow M_{i-1} \rightarrow \cdots$ be a complex of Ω_X -modules with closed differentials $d_i: M_i \rightarrow M_{i-1}$. Take the graded module $\text{Tot}^\oplus(M_\bullet) = \bigoplus_i M_i[i]$ with differential

$$d(m) = (-1)^i d_{M_i}(m) + d_i(m), \quad m \in M_i.$$

The Ω_X -module $\mathrm{Tot}^\oplus(M_\bullet)$ is called the *total object* of the complex $M_\bullet$ formed by (infinite) direct sums.

Let D be a DG-category with shifts and cones. The *homotopy category* $\mathrm{Hot}(D)$ is the additive category with the same class of objects as D and groups of morphisms given by $\mathrm{Hom}_{\mathrm{Hot}(D)}(X, Y) = H^0(\mathrm{Hom}_D(X, Y))$. The homotopy category $\mathrm{Hot}(D)$ is a triangulated category [3].

3 Positselski results on coderived categories

This section is an overview of some results on exotic derived categories due to Positselski. We present here a down to earth treatment of the subject [12]. The reader can find the results in full generality in the papers [9–11, 13].

3.1 Fix an exact subcategory E in the abelian category of quasi-coherent graded Ω_X^{gr} -modules. We say that Ω_X -module is an *E-module* if its underlying graded module belongs to E . An Ω_X -module is called *absolutely acyclic* with respect to E if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(\Omega_X\text{-qcoh})$ containing total objects of exact triples of E -modules. We denote this subcategory by $\mathrm{Acycl}_E^{\mathrm{abs}}$. The quotient category of the homotopy category of E -modules by $\mathrm{Acycl}_E^{\mathrm{abs}}$ is called the *absolute derived category of E-modules*.

For example, let E be the subcategory of coherent Ω_X^{gr} -modules. Then $D^{\mathrm{abs}}(\Omega_X\text{-coh}) = \mathrm{Hot}(\Omega_X\text{-coh})/\mathrm{Acycl}_{\mathrm{coh}}^{\mathrm{abs}}$ is the absolute derived category of coherent Ω_X -modules.

Suppose that E is closed under arbitrary direct sums. An Ω_X -module is called *coacyclic* with respect to E if it belongs to the minimal thick subcategory of the homotopy category $\mathrm{Hot}(\Omega_X\text{-qcoh})$ containing total objects of exact triples of E -modules and closed under arbitrary direct sums. We denote this subcategory by $\mathrm{Acycl}_E^{\mathrm{co}}$. The quotient category of the homotopy category of E -modules by $\mathrm{Acycl}_E^{\mathrm{co}}$ is called the *coderived category of E-modules*.

If E is the category of quasi-coherent Ω_X^{gr} -modules, we call coacyclic Ω_X -modules with respect to E *coacyclic*, and denote the subcategory of coacyclic Ω_X -modules by $\mathrm{Acycl}^{\mathrm{co}}$. Similarly, the category of *absolutely acyclic* Ω_X -modules $\mathrm{Acycl}^{\mathrm{abs}}$ is the category of absolutely acyclic Ω_X -modules with respect to E . The category $D^{\mathrm{co}}(\Omega_X\text{-qcoh}) = \mathrm{Hot}(\Omega_X\text{-qcoh})/\mathrm{Acycl}^{\mathrm{co}}$ is called the *coderived category of Ω_X -modules*. We call two Ω_X -modules *co-isomorphic* if they are isomorphic in $D^{\mathrm{co}}(\Omega_X\text{-qcoh})$. A curious example of co-isomorphism is given in Lemma 9.2.

A coherent absolutely acyclic Ω_X -module is absolutely acyclic with respect to the category of coherent Ω_X^{gr} -modules, and the natural functor from $D^{\mathrm{abs}}(\Omega_X\text{-coh})$ to $D^{\mathrm{co}}(\Omega_X\text{-qcoh})$ is full and faithful [11, 3.11].

- Lemma 3.2** (a) Let $M = \varinjlim_{n \in \mathbb{N}} M_n$ be the limit of an inductive system of coacyclic Ω_X -modules with closed morphisms. Then M is coacyclic.
- (b) Let $M_0 \xrightarrow{d_0} M_1 \rightarrow \cdots$ be an exact sequence of Ω_X -modules with closed differentials. Then $M = \ker d_0$ and $\mathrm{Tot}^\oplus M_\bullet$ are co-isomorphic.
- (c) Let $M = \bigcup_{n \in \mathbb{N}} M_n$ be an Ω_X -module with an increasing filtration such that M_{n+1}/M_n is coacyclic for any $n \in \mathbb{N}$. Then M/M_0 is coacyclic.

Proof The first assertion follows from definitions since $M = \varinjlim_{n \in \mathbb{N}} M_n$ is the cone of the injective map $\text{id} - d: \bigoplus_{n \in \mathbb{N}} M_n \rightarrow \bigoplus_{n \in \mathbb{N}} M_n$. To prove the second assertion note that the total object of the sequence $0 \rightarrow M \rightarrow M_0 \rightarrow \cdots \rightarrow M_n \rightarrow \ker d_{n+1} \rightarrow 0$ is absolutely acyclic. Hence the direct limit N of such total objects is coacyclic, and we have an exact sequence of Ω_X -modules

$$0 \rightarrow \text{Tot}^{\oplus} M_{\bullet} \rightarrow N \rightarrow M[-1] \rightarrow 0.$$

The third assertion follows from the fact that $M/M_0 = \varinjlim_{n \in \mathbb{N}} M_n/M_0$. $\square$

3.3 An Ω_X -module I is called *injective* if I is injective as a graded Ω_X^{gr} -module. By [13, Theorem A3], I is injective if and only if the restriction of I to any open subscheme U is an injective Ω_U -module. An Ω_X -module F is called *flat* if F is flat as a graded Ω_X^{gr} -module. Let M be an Ω_X -module. We would like to find an injective Ω_X -module I and a flat Ω_X -module F which are co-isomorphic to M . As a first approximation to I and F we construct an injective Ω_X -module I_0 with an injection $M \rightarrow I_0$ and a locally free (over Ω_X^{gr}) Ω_X -module P_0 and a surjection $P_0 \rightarrow M$.

There is an injective morphism of graded $\mathcal{O}_X$ -modules $M \rightarrow J$ to an injective quasi-coherent graded $\mathcal{O}_X$ -module J and a surjective $\mathcal{O}_X$ -morphism $Q \rightarrow M$ from a locally free graded $\mathcal{O}_X$ -module Q . We are going to construct functors DG^+ and DG^- from graded $\mathcal{O}_X$ -modules to Ω_X -modules such that $I_0 = \text{DG}^-(J)$, and $P_0 = \text{DG}^+(Q)$.

3.3.1 Let J be a quasi-coherent $\mathcal{O}_X$ -module. The module $\text{DG}^-(J)$ is cofreely generated by J over Ω_X . Namely, take a coinduced module $J' = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{\text{gr}}, J)$. It is a graded Ω_X^{gr} -module such that for any Ω_X^{gr} -module N there is a natural isomorphism

$$\Phi: \mathcal{H}om_{\mathcal{O}_X}(N, J) \rightarrow \mathcal{H}om_{\Omega_X^{\text{gr}}}(N, J').$$

The graded sheaf of abelian groups $\text{DG}^-(J)$ is defined as follows. For an open $U \subset X$ the group $\text{DG}^-(J)(U)$ consists of all formal expressions of the form $d^{-1}x + y$, where $x, y \in J'(U)$, and $\deg d^{-1}x = \deg x - 1$. The differential and the action of Ω_U on $\text{DG}^-(J)(U)$ is given by the formulas: $d(d^{-1}x + y) = x$ and $\omega(d^{-1}x + y) = \omega y + d^{-1}(d(\omega)y) + (-1)^{|\omega|}d^{-1}(\omega x)$ for $\omega \in \Omega_U$.

For any Ω_X -module M there is a bijective correspondence between morphisms of graded $\mathcal{O}_X$ -modules $f: M \rightarrow J$ and closed morphisms of Ω_X -modules $g: M \rightarrow \text{DG}^-(J)$ which is described by the formula $g(m) = d^{-1}(\Phi(f)(dm)) + \Phi(f)(m)$. The module $\text{DG}^-(J)$ is contractible. If J is an injective $\mathcal{O}_X$ -module, then $\text{DG}^-(J) \cong J' \oplus J'[-1]$ as a graded Ω_X^{gr} -module.

3.3.2 The module $\text{DG}^+(Q)$ is constructed in a similar way. As a graded sheaf of abelian groups

$$\text{DG}^+(Q)(U) = \{x + dy : x, y \in (Q \otimes_{\mathcal{O}_X} \Omega_X)(U)\},$$

where $\deg dy = \deg y + 1$. The differential and the action of Ω_U on $\text{DG}^+(Q)(U)$ is given by the formulas: $d(x + dy) = dx$ and $\omega(x + dy) = \omega x - (-1)^{|\omega|}d(\omega y) + (-1)^{|\omega|}d(\omega y)$ for $\omega \in \Omega_U$.

As before, there is a bijective correspondence between morphisms of graded $\mathcal{O}_X$ -modules $f: Q \rightarrow M$ and closed morphisms of Ω_X -modules $g: \mathrm{DG}^+(Q) \rightarrow M$, and $\mathrm{DG}^+(Q)$ is contractible. If Q is locally free over $\mathcal{O}_X$, then $\mathrm{DG}^+(Q)$ is locally free as a graded Ω_X^{gr} -module.

3.4 Injective resolutions Denote the category of injective Ω_X -modules by $\Omega_X\text{-qcoh}_{\mathrm{inj}}$.

Theorem *The coderived category $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ is equivalent to the homotopy category $\mathrm{Hot}(\Omega_X\text{-qcoh}_{\mathrm{inj}})$.*

Proof Let M be an Ω_X -module. Construct an *injective resolution* I of M as follows. By 3.3, we have an inclusion $M \rightarrow I_0$, where I_0 is injective. A common argument gives an exact complex $I_\bullet = (I_0 \rightarrow I_1 \rightarrow \cdots)$ of injective Ω_X -modules with closed differentials. By Lemma 3.2(a), $I = \mathrm{Tot}^\oplus I_\bullet$ is co-isomorphic to M . Since Ω_X^{gr} is Noetherian, the direct sum of injective modules is injective, thus I is injective.

By [11, Theorem 3.5], for any coacyclic Ω_X -module L the complex $\mathrm{Hom}_{\Omega_X}(L, I)$ is acyclic. Thus $\mathrm{Hot}(\Omega_X\text{-qcoh}_{\mathrm{inj}})$ and coacyclic Ω_X -modules form a semiorthogonal decomposition of $\mathrm{Hot}(\Omega_X\text{-qcoh})$. By [11, Lemma 1.3], this proves the theorem. $\square$

3.5 Thick Ω_X -modules We call an Ω_X -module A *(co)induced* if A is (co)induced as an Ω_X^{gr} -module, i.e., there exists an $\mathcal{O}_X$ -module L such that $A^{\mathrm{gr}} \cong L \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{gr}}$ (or $A^{\mathrm{gr}} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{\mathrm{gr}}, L)$). Note that any induced Ω_X^{gr} -module is coinduced and vice versa.

Theorem *Let A be a quasi-coherent Ω_X^{gr} -module. The following conditions are equivalent:*

- (a) $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, A) = 0$ for any $i > 0$;
- (b) $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(F, A) = 0$ for any $i > 0$, and for any $\mathcal{O}_X$ -flat Ω_X^{gr} -module F ;
- (c) $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(A, J) = 0$ for any $i > 0$, and for any $\mathcal{O}_X$ -injective Ω_X^{gr} -module J such that $\Omega_X^1 J = 0$;
- (d) $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(A, I) = 0$ for any $i > 0$, and for any $\mathcal{O}_X$ -injective Ω_X^{gr} -module I ;
- (e) $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, A) = 0$ for any $i > 0$;
- (f) $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(P, A) = 0$ for any $i > 0$, and for any Ω_X^{gr} -module P which is locally projective over $\mathcal{O}_X$.

If A is (co)induced, then (a)–(f) hold.

Proof Clearly, (b) $\Rightarrow$ (a), (d) $\Rightarrow$ (c), (f) $\Rightarrow$ (e), and if A is (co)induced, then (a)–(f) hold. (a) $\Leftrightarrow$ (c). Take a locally free over Ω_X^{gr} resolution for A

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0. \quad (*)$$

The complex

$$\cdots \rightarrow P_n \otimes_{\Omega_X^{\mathrm{gr}}} \mathcal{O}_X \rightarrow \cdots \rightarrow P_0 \otimes_{\Omega_X^{\mathrm{gr}}} \mathcal{O}_X \rightarrow 0$$

computes $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, A)$. Apply $\mathcal{H}om_{\mathcal{O}_X}(-, J)$ to this complex. We obtain a complex which computes $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^\bullet(A, J)$. Since J is an arbitrary $\mathcal{O}_X$ -injective module, the groups $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(A, J)$ are trivial if and only if $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, A) = 0$.

(d) $\Rightarrow$ (b). Let J be an injective $\mathcal{O}_X$ -module. The module $I = \mathcal{H}om_{\mathcal{O}_X}(F, J)$ is $\mathcal{O}_X$ -injective Ω_X^{gr} -module. Apply $\mathcal{H}om_{\Omega_X^{\mathrm{gr}}}(-, I)$ to the resolution (*). If (d) holds, we get a resolution for $\mathcal{H}om_{\mathcal{O}_X}(A \otimes_{\Omega_X^{\mathrm{gr}}} F, J)$. Since J is an arbitrary injective $\mathcal{O}_X$ -module, $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(F, A) = 0$ for any $i > 0$.

(a) $\Rightarrow$ (f). First, we prove a lemma.

Lemma Assume (a). Then A has a finite left resolution over Ω_X^{gr} which is locally (co)induced.

Proof Take the truncation of the resolution (*)

$$0 \rightarrow Q \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0, \quad (**)$$

where $n = \dim X$. Then Q is locally projective $\mathcal{O}_X$ -module. Tensor (**) with $\mathcal{O}_X$ over Ω_X^{gr} . By condition (a), the module $L = Q \otimes_{\Omega_X^{\mathrm{gr}}} \mathcal{O}_X$ is also locally projective over $\mathcal{O}_X$, and $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, Q) = 0$ for any $i > 0$. We have the natural surjective morphism of Ω_X^{gr} -modules

$$Q \rightarrow L \cong Q / \Omega_X^1 Q.$$

This implies that locally there is a morphism of $\mathcal{O}_X$ -modules $L \rightarrow Q$ which induces a surjective morphism of Ω_X^{gr} -modules

$$\varphi: L \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{gr}} \rightarrow Q.$$

Let $K = \ker \varphi$. Clearly, $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, K) = 0$ for any $i \geq 0$, and thus $K = 0$. We proved that locally Q is a (co)induced module. $\blacksquare$

By the lemma, the module A has a resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

such that for any P_k the property (f) holds: $\mathcal{E}xt_{\Omega_X^{\mathrm{gr}}}^i(P, P_k) = 0$ for any $i > 0$. It follows that (f) is true for A . To prove (e) $\Rightarrow$ (d) one uses almost the same argument involving right resolution of A instead of the resolution (*). $\square$

We call an Ω_X^{gr} -module A *thick* if conditions (a)–(f) hold for A . We call Ω_X -module *thick* if it is thick as a graded Ω_X^{gr} -module. Positselski call such modules *weakly relatively admissible*. The following result can be proved using the argument given in the previous lemma.

Corollary *Let A be a thick Ω_X^{gr} -module. Suppose that A is locally projective (flat or injective) over $\mathcal{O}_X$. Then A is locally projective (respectively flat or injective) over Ω_X^{gr} .*

3.6 Thick resolutions

Theorem *The coderived category $D^{\text{co}}(\Omega_X\text{-qcoh})$ is equivalent to the absolute derived category of $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{thick}})$ of thick Ω_X -modules.*

Proof We prove that subcategory of absolutely acyclic thick Ω_X -modules and the homotopy category of injective Ω_X -modules form a semiorthogonal decomposition of the homotopy category of thick Ω_X -modules. The theorem then follows from [11, Lemma 1.3] and Theorem 3.4.

Let A be a thick Ω_X -module. Find an $\mathcal{O}_X$ -injection of A into an injective $\mathcal{O}_X$ -module J_0 . By 3.3 we obtain an inclusion $A \rightarrow I_0 = \text{DG}^-(J_0)$ into a coinduced injective Ω_X -module. In this way we construct first terms of the resolution for A by coinduced injective Ω_X -modules

$$I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_n,$$

where all morphisms are closed. Let J be the cokernel of the map $I_{n-1} \rightarrow I_n$. Then J is thick. Moreover, if $n \geq \dim X$, then J is injective over $\mathcal{O}_X$. By Corollary 3.5, J is an injective Ω_X -module. It follows that

$$I = \text{Tot}^{\oplus}(I_0 \rightarrow \cdots \rightarrow I_n \rightarrow J)$$

is injective too. Moreover, the cokernel L of the natural morphism $A \rightarrow I$ is an absolutely acyclic thick Ω_X -module. By [11, Theorem 3.5], $\text{Hom}_{\Omega_X}(L, I)$ is acyclic (see also Lemma 4.1). $\square$

Remark In fact, we proved that any Ω_X -module M has a *coinduced resolution*, i.e. there exists a coinduced Ω_X -module A and a morphism $M \rightarrow A$ with absolutely acyclic cone.

3.7 Flat resolutions

Theorem *The coderived category $D^{\text{co}}(\Omega_X\text{-qcoh})$ is equivalent to the absolute derived category $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}})$ of flat Ω_X -modules.*

Proof By Theorem 3.6, $D^{\text{co}}(\Omega_X\text{-qcoh})$ is equivalent to $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{thick}})$. Let us prove that the latter is equivalent to $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}})$. Let A be a thick Ω_X -module. As before, using 3.3 find a finite exact complex

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow A \rightarrow 0$$

of Ω_X -modules such that $P_0, \dots, P_{n-1}$ are flat (and even locally free over Ω_X^{gr}) and differentials are closed. If n is greater than dimension of X , the module P_n is also flat

by Corollary 3.5. It follows that $P = \text{Tot}^\oplus P_\bullet$ is flat and $\text{cone}(P \rightarrow A)$ is absolutely acyclic with respect to $\Omega_X\text{-qcoh}_{\text{thick}}$.

We proved that the natural functor

$$D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{thick}}) \rightarrow D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}})$$

is essentially surjective. We have to show that it is full and faithful. The key point is to prove that any object of $\Omega_X\text{-qcoh}_{\text{fl}}$ that is absolutely acyclic with respect to $\Omega_X\text{-qcoh}_{\text{thick}}$ is absolutely acyclic with respect to $\Omega_X\text{-qcoh}_{\text{fl}}$. This is done in [9, Sect. 3.2]. $\square$

3.8 Perfect resolutions We need a one more type of resolutions. Denote the category of $\mathcal{O}_X$ -locally free coherent Ω_X -modules by $\Omega_X\text{-perf}$. We call such Ω_X -modules *perfect*. This is an exact category and we can define the absolute derived category $D^{\text{abs}}(\Omega_X\text{-perf})$.

Theorem *The absolute derived category $D^{\text{abs}}(\Omega_X\text{-coh})$ of coherent Ω_X -modules is equivalent to the absolute derived category $D^{\text{abs}}(\Omega_X\text{-perf})$ of perfect Ω_X -modules.*

Proof Let M be a coherent Ω_X -module. By 3.3, there is a finite exact complex $0 \rightarrow P_n \rightarrow \dots \rightarrow P_0 \rightarrow M \rightarrow 0$ of Ω_X -modules such that $P_0, \dots, P_{n-1}$ are perfect (and even locally free over Ω_X^{gr}) and differentials are closed. If $n \geq \dim X$, the module P_n is also perfect. It follows that $\text{Tot}^\oplus P_\bullet$ is perfect and co-isomorphic to M . As before, one can prove that any object of $\Omega_X\text{-perf}$ that is absolutely acyclic with respect to $\Omega_X\text{-coh}$ is absolutely acyclic with respect to $\Omega_X\text{-perf}$ (see [9, Sect. 3.2]). $\square$

Remark For any Ω_X -module M there is a flat resolution F which is locally free over Ω_X^{gr} . Indeed, the same construction as before gives a resolution P of M which is locally free over $\mathcal{O}_X$, and the coinduced resolution of P is locally free over Ω_X^{gr} .

4 Inverse and direct images

Let $f: X \rightarrow Y$ be a morphism of smooth schemes over S . Denote by $f^\bullet$ and $f_\bullet$ the inverse and direct image of sheaves of abelian groups associated to f . We have a morphism of sheaves of DG-algebras on X : $f^\bullet \Omega_Y \rightarrow \Omega_X$. This gives a morphism of DG-ringed spaces $(X, \Omega_X) \rightarrow (Y, \Omega_Y)$, and allows one to define inverse and direct image functors on coderived categories of Ω -modules.

4.1 Derived Hom and tensor product First we define functors of derived Hom and (shifted) tensor product:

$$\begin{aligned} R\text{Hom}_{\Omega_X} : D^{\text{abs}}(\Omega_X\text{-coh}) \times D^{\text{co}}(\Omega_X\text{-qcoh}) &\rightarrow D^{\text{co}}(\Omega_X\text{-qcoh}), \\ \otimes_{\Omega_X}^D : D^{\text{co}}(\Omega_X\text{-qcoh}) \times D^{\text{co}}(\Omega_X\text{-qcoh}) &\rightarrow D^{\text{co}}(\Omega_X\text{-qcoh}). \end{aligned}$$

Lemma (a) *For any absolutely acyclic coherent Ω_X -module M and injective Ω_X -module I the Ω_X -module $\text{Hom}_{\Omega_X}(M, I)$ is absolutely acyclic.*

- (b) For any coherent Ω_X -module M and coacyclic injective Ω_X -module I the Ω_X -module $\mathcal{H}om_{\Omega_X}(M, I)$ is coacyclic.
- (c) For any coacyclic Ω_X -module M and flat Ω_X -module F the Ω_X -module $M \otimes_{\Omega_X} F$ is coacyclic.
- (d) Let F be a flat absolutely acyclic (with respect to the additive category of flat Ω_X -modules) Ω_X -module. Then for any Ω_X -module M the Ω_X -module $M \otimes_{\Omega_X} F$ is coacyclic.

Proof The proof of (a) is almost the same as [11, Theorem 3.5]. Indeed, the functor $\mathcal{H}om_{\Omega_X}(-, I)$ respects shifts, cones and direct sums. Thus it is enough to check (a) when M is the total object of an exact triple of Ω_X -modules. Clearly, $\mathcal{H}om_{\Omega_X}(M, I)$ is the total object of an exact triple of Ω_X -modules. Similarly, if I is co-acyclic, then $\mathcal{H}om_{\Omega_X}(I, I)$ is acyclic, and I is contractible. It follows that $\mathcal{H}om_{\Omega_X}(M, I)$ is coacyclic. This proves (b).

The functor $-\otimes_{\Omega_X} F$ respects shifts, cones and arbitrary direct sums. We have to check (c) when M is the total object of an exact triple of Ω_X -modules and (d) when F is the total object of an exact triple of flat Ω_X -modules. Both times, $M \otimes_{\Omega_X} F$ is the total object of an exact triple of Ω_X -modules. $\square$

4.1.1 We are ready to define the functor $R\mathcal{H}om_{\Omega_X}$. Take the restriction of the functor

$$\mathcal{H}om_{\Omega_X} : \text{Hot}(\Omega_X\text{-coh}) \times \text{Hot}(\Omega_X\text{-qcoh}) \rightarrow \text{Hot}(\Omega_X\text{-qcoh})$$

to the subcategory $\text{Hot}(\Omega_X\text{-coh}) \times \text{Hot}(\Omega_X\text{-qcoh}_{\text{inj}})$. By Lemma 4.1 (a), this restriction factors through the functor

$$D^{\text{abs}}(\Omega_X\text{-coh}) \times \text{Hot}(\Omega_X\text{-qcoh}_{\text{inj}}) \rightarrow D^{\text{co}}(\Omega_X\text{-qcoh}) \quad (***)$$

By Theorem 3.4, $\text{Hot}(\Omega_X\text{-qcoh}_{\text{inj}})$ is equivalent to $D^{\text{co}}(\Omega_X\text{-qcoh})$. Combining this equivalence with the functor (***) we get the functor $R\mathcal{H}om_{\Omega_X}$.

4.1.2 Let us define the derived tensor product. Let n be the relative dimension of X over S . By Lemma 4.1 (c)&(d), the shifted tensor product

$$\otimes_{\Omega_X}[2n] : \text{Hot}(\Omega_X\text{-qcoh}) \times \text{Hot}(\Omega_X\text{-qcoh}_{\text{fl}}) \rightarrow \text{Hot}(\Omega_X\text{-qcoh})$$

factors through

$$D^{\text{co}}(\Omega_X\text{-qcoh}) \times D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}}) \rightarrow D^{\text{co}}(\Omega_X\text{-qcoh}).$$

By Theorem 3.7, $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}})$ is equivalent to $D^{\text{co}}(\Omega_X\text{-qcoh})$, and we are done.

The tensor product can be obtained in two equivalent ways

$$\begin{aligned} D^{\text{co}}(\Omega_X\text{-qcoh}) \times D^{\text{co}}(\Omega_X\text{-qcoh}) &\rightarrow D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}}) \times D^{\text{co}}(\Omega_X\text{-qcoh}) \\ &\xrightarrow{\otimes_{\Omega_X}[2n]} D^{\text{co}}(\Omega_X\text{-qcoh}), \end{aligned}$$

where the first arrow comes from the equivalence 3.7, or

$$\begin{aligned} D^{\text{co}}(\Omega_X\text{-qcoh}) \times D^{\text{co}}(\Omega_X\text{-qcoh}) &\rightarrow D^{\text{co}}(\Omega_X\text{-qcoh}) \times D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}}) \\ &\xrightarrow{\otimes_{\Omega_X}[2n]} D^{\text{co}}(\Omega_X\text{-qcoh}). \end{aligned}$$

These definitions of tensor product are equivalent since both functors factors through $D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}}) \times D^{\text{abs}}(\Omega_X\text{-qcoh}_{\text{fl}})$.

Theorem 4.2 *Let P be a perfect Ω_X -module, and A be a thick Ω_X -module. Then*

- (a) $R\mathcal{H}om_{\Omega_X}(P, A) \cong \mathcal{H}om_{\Omega_X}(P, A)$, and
- (b) $P \otimes_{\Omega_X}^D A \cong P \otimes_{\Omega_X} A[2n]$

in $D^{\text{co}}(\Omega_X\text{-qcoh})$.

Proof We prove (a) and leave (b) to the reader. Take an injective resolution of A from the proof of Theorem 3.6:

$$I = \text{Tot}^{\oplus}(I_0 \rightarrow \cdots \rightarrow I_n \rightarrow J).$$

We know that exact sequence $I_{\bullet} = (I_0 \rightarrow \cdots \rightarrow I_n \rightarrow J)$ gives an injective resolution of A over Ω_X^{gr} . Since P is perfect, the complex of Ω_X^{gr} -modules $\mathcal{H}om_{\Omega_X^{\text{gr}}}(P^{\text{gr}}, I_{\bullet}^{\text{gr}})$ has homology at zero term only. It follows that $\mathcal{H}om_{\Omega_X}(P, I)$ is co-isomorphic to $\mathcal{H}om_{\Omega_X}(P, A)$. $\square$

4.3 Direct image In this section we define the functor of direct image

$$f_*: D^{\text{co}}(\Omega_X\text{-qcoh}) \rightarrow D^{\text{co}}(\Omega_Y\text{-qcoh}).$$

For any injective Ω_X module I the direct image $f_{\bullet}(I)$ is an Ω_Y -module since f is a morphism of DG-ringed spaces $(X, \Omega_X) \rightarrow (Y, \Omega_Y)$. This gives the functor

$$f_{\circ}: \text{Hot}(\Omega_X\text{-qcoh}_{\text{inj}}) \rightarrow D^{\text{co}}(\Omega_Y\text{-qcoh}).$$

Any Ω_X -module has an injective resolution, and we define f_* as a composition of $f_{\circ}$ with the equivalence 3.4. It is well known that for an injective sheaf I the sheaf $f_{\bullet}(I)$ is flabby, and $f_{\bullet}$ is exact on flabby sheaves. Thus, for a pair of composable maps f and g we have $(f \circ g)_* \cong f_* \circ g_*$.

By definition, $\Omega_S \cong \mathcal{O}_S$, and $\text{Hom}_{\Omega_X}(M, N)$ is a complex of $\mathcal{O}_S(S)$ -modules. Since $\mathcal{O}_S$ has finite homological dimension and affine,

$$D^{\text{co}}(\Omega_S\text{-qcoh}) \cong D(\mathcal{O}_S(S)\text{-mod})$$

is an unbounded derived category of $\mathcal{O}_S(S)$ -modules [10, Remark 2.1].

Proposition Let f be the morphism from X to S , $N \in \mathbf{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ and $M \in \mathbf{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$. Then there is a canonical isomorphism in $\mathbf{D}(\mathcal{O}_S(S)\text{-mod})$

$$\Gamma(S, f_* R\mathcal{H}om_{\Omega_X}(M, N)) \rightarrow \mathrm{Hom}_{\Omega_X}(M, I),$$

where I is an injective resolution of N . In particular, the space of homomorphisms $\mathrm{Hom}_{\mathbf{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})}(M, N)$ is isomorphic to $H^0(S, f_* R\mathcal{H}om_{\Omega_X}(M, N))$.

Proof The Ω_X -module $\mathcal{H}om_{\Omega_X}(M, I)$ is flabby. Thus

$$\Gamma(S, f_* R\mathcal{H}om_{\Omega_X}(M, N)) \cong \Gamma(S, f_* \mathcal{H}om_{\Omega_X}(M, I)) \cong \mathrm{Hom}_{\Omega_X}(M, I). \quad \square$$

4.4 Extraordinary inverse image Now, we define two inverse image functors:

$$f^+, f^!: \mathbf{D}^{\mathrm{co}}(\Omega_Y\text{-qcoh}) \rightarrow \mathbf{D}^{\mathrm{co}}(\Omega_X\text{-qcoh}).$$

By Theorem 3.7, it is enough to define the functors on $\mathbf{D}^{\mathrm{co}}(\Omega_Y\text{-qcoh}_{\mathrm{fl}})$. We do it as follows:

$$f^+(F) = f^\bullet(F) \otimes_{f^\bullet \Omega_Y} \Omega_X,$$

where F is flat. The *extraordinary inverse image* is defined as $f^!(M) = f^+(M)[2(\dim X - \dim Y)]$. Clearly, for a pair of composable maps f and g we have $(f \circ g)^! \cong g^! \circ f^!$.

Lemma Suppose that $f: X \rightarrow Y$ is smooth. Then Ω_X is flat over $f^\bullet \Omega_Y$.

Proof It follows from the smoothness of f that Ω_X^{gr} is locally free over $f^\bullet(\Omega_Y^{\mathrm{gr}}) \otimes_{f^\bullet \mathcal{O}_Y} \mathcal{O}_X$ and $\mathcal{O}_X$ is flat over $f^\bullet \mathcal{O}_Y$. Thus Ω_X^{gr} is flat over $f^\bullet(\Omega_Y^{\mathrm{gr}})$. $\square$

Corollary Suppose that $f: X \rightarrow Y$ is smooth, and M is an Ω_Y -module. Then

- (a) $f^+(M) \cong f^\bullet(M) \otimes_{f^\bullet \Omega_Y} \Omega_X$.
- (b) The extraordinary inverse image of a coherent Ω_Y -module is coherent.
- (c) If f is an open embedding, then $f^!(M) \cong f^+(M) \cong f^\bullet(M) \otimes_{f^\bullet \mathcal{O}_Y} \mathcal{O}_X$.

Theorem 4.5 Suppose that f is smooth. Then functors f_* and f^+ are adjoint. Let $N \in \mathbf{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ and let $M \in \mathbf{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$. Then

$$R\mathcal{H}om_{\Omega_Y}(M, f_* N) \cong f_* R\mathcal{H}om_{\Omega_X}(f^+ M, N).$$

Proof We prove the second assertion. The same argument proves that the functors are adjoint. Take an injective resolution I for N . Since the functor $f^\bullet$ is exact, and functors f_* and $f^\bullet$ are adjoint, the module $f_\bullet(I)$ is injective. Moreover, homomorphism sheaf to an injective sheaf is flabby. Thus we have to prove that

$$\mathcal{H}om_{\Omega_Y}(M, f_\bullet I) \cong f_\bullet \mathcal{H}om_{\Omega_X}(f^\bullet M \otimes_{f^\bullet \Omega_Y} \Omega_X, I).$$

And this can be shown in the usual way. $\square$

Theorem 4.6 (projection formula) *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over S . Let M be an Ω_X -module and let N be an Ω_Y -module. Then $N \otimes_{\Omega_X}^D f_* M \cong f_*(f^! N \otimes_{\Omega_Y}^D M)$ in $D^{\text{co}}(\Omega_Y\text{-qcoh})$.*

Proof Let I be an injective resolution of M , and F be a flat resolution of N , which is locally free over Ω_Y^{gr} . We have the natural morphism

$$F \otimes_{\Omega_Y} f_* I \rightarrow f_*(f^+ F \otimes_{\Omega_X} I).$$

Take the composition Φ of this morphism with

$$f_*(f^+ F \otimes_{\Omega_X} I) \rightarrow f_*(f^+ F \otimes_{\Omega_X} I).$$

We claim that Φ is an isomorphism. It is enough to check the claim for graded modules. Clearly, Φ is an isomorphism when $F = \Omega_Y^{\text{gr}}$. Thus the claim holds for any locally free Ω_Y^{gr} -module. The projection formula now follows from the definition of the (shifted) derived tensor product and of the extraordinary inverse image. $\square$

4.7 Extraordinary inverse image for a closed embedding Let $i: Z \rightarrow X$ be a closed embedding of smooth schemes over S . In this case Ω_Z is coherent over Ω_X , and we can give the second definition of the extraordinary inverse image

$$i^b(M) = \mathcal{H}om_{i^*\Omega_X}(\Omega_Z, i^*(M)).$$

We define the derived functor $Ri^b: D^{\text{co}}(\Omega_Y\text{-qcoh}) \rightarrow D^{\text{co}}(\Omega_X\text{-qcoh})$ using the equivalence 3.4.

Let I be an injective Ω_X -module. Since functors $i_*, \mathcal{H}om_{i^*\Omega_X}(\Omega_Z^{\text{gr}}, i^*(-))$ are adjoint, and i_* is exact, the Ω_X -module $i^b(I)$ is injective. It follows that for any pair of composable closed embeddings f and g we have $(f \circ g)^b \cong g^b \circ f^b$. For the same reason, we have

Theorem 4.7 *The functors i_* and Ri^b are adjoint. Let $N \in D^{\text{co}}(\Omega_X\text{-qcoh})$ and let $M \in D^{\text{abs}}(\Omega_X\text{-coh})$. Then*

$$i_* R\mathcal{H}om_{\Omega_Z}(M, Ri^b(N)) \cong R\mathcal{H}om_{\Omega_X}(i_* M, N).$$

In Theorem 7.5 we prove that if S is defined over $\mathbb{Q}$, then Ri^b and $i^!$ are equivalent.

Corollary 4.8 *Let $i: Z \rightarrow X$ be a closed embedding, and $j: U = X \setminus Z \rightarrow X$ be the corresponding open embedding. Then for any Ω_U -module N we have $Ri^b j_* N = 0$.*

Proof By Corollary 4.4(c), for any Ω_Z -module M we have $j^+ i_* M = 0$. The result follows from Theorems 4.5 and 4.7. $\square$

5 Differential graded commutative algebra

5.1 The injective hull Let M be an Ω_X -module. We say that injective Ω_X^{gr} -module E is an *injective hull* of M^{gr} , if $M^{\text{gr}} \subset E$, and every nonzero submodule of E intersects M^{gr} non-trivially. The injective hull of a graded module exists and is unique up to an isomorphism [4]. We say that injective Ω_X -module E is an *injective hull* of M if the graded module E^{gr} is the injective hull of M^{gr} .

Proposition For any Ω_X -module M there exists an injective hull $E(M)$ of M .

Proof Take an injective Ω_X -module I which contains M as a submodule. Let E^{gr} be the injective hull of the graded module M^{gr} in I^{gr} . The module E^{gr} generates an Ω_X -submodule E_1 of I . Since the hull E^{gr} is injective, $E_1^{\text{gr}} \cong E^{\text{gr}} \oplus E_2$ for some graded Ω_X^{gr} -submodule E_2 of E_1^{gr} . The Ω_X^{gr} -module E_2 can be generated by (some set of) linear combinations of de for $e \in E^{\text{gr}}$, so E_2 is an Ω_X -submodule of E_1 . Put $E(M) = E_1/E_2$. $\square$

Remark By definition, the graded module $E(M)^{\text{gr}}$ is unique up to isomorphism. However, here is an example of an Ω_X -module M such that $E(M)^{\text{gr}}$ has two different differentials compatible with differential on M . Suppose X is affine. Let M be an Ω_X -module such that $df \cdot M = 0$ for some $f \in \mathcal{O}_X(X)$. For example, $M = df \wedge \Omega_X$. By the proposition, there exists a module $E(M)$ with some differential d . Then $d + df$ is the second differential on $E(M)$, and $dm = (d + df)m$ for any $m \in M(X)$.

5.2 The stupid filtration Let M be an Ω_X -module. Define the *stupid filtration* $\text{st}_\bullet$ on M by the formula $\text{st}_i M = \Omega_X^i M$. Clearly, $\text{st}_i M$ is an Ω_X -submodule of M . The associated graded module $\text{Gr}^{\text{st}} M$ is a DG-module over $\text{Gr}^{\text{st}} \Omega_X = \Omega_X^{\text{gr}}$. In particular, the differential on $\text{Gr}^{\text{st}} M$ is Ω_X^{gr} -linear. We define the stupid filtration for Ω_X^{gr} -modules in the same way.

Proposition 5.3 Let M be an Ω_X^{gr} -module. Suppose that $\text{Tor}_{\Omega_X^{\text{gr}}}^1(\mathcal{O}_X, M) = 0$. Then $\text{Gr}_i^{\text{st}} M \cong \Omega_X^i \otimes_{\Omega_X^{\text{gr}}} M$, and $\text{Gr}^{\text{st}} M \cong \Omega_X^{\text{gr}} \otimes_{\mathcal{O}_X} \text{Gr}_0^{\text{st}} M$.

Proof Take two short exact sequences of perfect Ω_X^{gr} -modules

$$\begin{aligned} 0 \rightarrow \text{st}_i \Omega_X \rightarrow \Omega_X \rightarrow \Omega_X / \text{st}_i \Omega_X \rightarrow 0, \\ 0 \rightarrow \text{st}_{i+1} \Omega_X \rightarrow \text{st}_i \Omega_X \rightarrow \Omega_X^i \rightarrow 0. \end{aligned}$$

Take the tensor product over Ω_X^{gr} of these sequences with M . We get that $\text{st}_i M \cong \text{st}_i \Omega_X \otimes_{\Omega_X^{\text{gr}}} M$ and that

$$\text{Gr}_i^{\text{st}} M \cong \Omega_X^i \otimes_{\Omega_X^{\text{gr}}} M \cong \Omega_X^i \otimes_{\mathcal{O}_X} \text{Gr}_0^{\text{st}} M. \quad \square$$

Corollary 5.4 The functor Gr^{st} is exact on the category of thick Ω_X^{gr} -modules.

Proof $\mathrm{Gr}_0^{\mathrm{st}} \cong \mathcal{O}_X \otimes_{\Omega_X^{\mathrm{gr}}} -$ and $\Omega_X^i \otimes_{\mathcal{O}_X} -$ are exact on $\Omega_X\text{-qcoh}_{\mathrm{thick}}$. $\square$

Theorem 5.5 *The following conditions on an Ω_X^{gr} -module A are equivalent:*

- (a) A is thick;
- (b) $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^1(\mathcal{O}_X, A) = 0$;
- (c) $\mathrm{Gr}^{\mathrm{st}} A$ is (co)induced;
- (d) $\mathrm{Gr}^{\mathrm{st}} A$ is thick.

Proof By Theorem 3.5, (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d). By Proposition 5.3, (b) $\Rightarrow$ (c). We now prove (d) $\Rightarrow$ (a). Take a finite resolution for $\mathrm{Gr}^{\mathrm{st}} A$ by modules which are locally free over Ω_X^{gr}

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathrm{Gr}^{\mathrm{st}} A \rightarrow 0.$$

Let $P_0 = Q_0 \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{gr}}$. Then there is a surjection $\varphi: Q_0 \cong \mathrm{Gr}_0^{\mathrm{st}} P_0 \rightarrow \mathrm{Gr}_0^{\mathrm{st}} A$. For any point of X there exists a neighborhood U such that the restriction $\varphi|_U$ lifts to a surjection of Ω_U^{gr} -modules $P_{0|U}^{\mathrm{gr}} \rightarrow A|_U^{\mathrm{gr}}$. This gives a surjection of Ω_U -modules $\tilde{\varphi}: \tilde{P}_0 = \mathrm{DG}^+(Q_{0|U}) \rightarrow A|_U$, and the projective dimension

$$\mathrm{pd}_{\mathcal{O}_U} \mathrm{Gr}^{\mathrm{st}}(\ker \tilde{\varphi}) = \mathrm{pd}_{\mathcal{O}_U} \mathrm{Gr}^{\mathrm{st}} A - 1.$$

Similarly, we can make U smaller and construct a finite resolution for $A|_U$ by thick Ω_U -modules

$$0 \rightarrow \tilde{P}_n \rightarrow \cdots \rightarrow \tilde{P}_0 \rightarrow A|_U \rightarrow 0.$$

It follows that $A|_U$ is thick. Thus for any $i > 0$ the sheaf $\mathrm{Tor}_{\Omega_X^{\mathrm{gr}}}^i(\mathcal{O}_X, A)$ is trivial. $\square$

Example 5.6 Here is an example of a thick Ω_X^{gr} -module M which is not induced. Let $X = \mathrm{Spec} k[t]$, and let $\delta = k[t]/tk[t]$ be a skyscraper module. Put $M = k[t] \oplus k[t]dt/t \oplus \delta dt$, where $k[t]dt/t$ is the submodule of $k(t)dt$ generated over $k[t]$ by dt/t . The action of t is natural, and to define the action of dt one should write $k[t]dt/t$ as the sum $k[t]dt \oplus \delta$. We see that $\mathrm{Gr}^{\mathrm{st}} M \cong \Omega_X^{\mathrm{gr}} \oplus \delta \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{gr}}$. Thus M is thick and is not induced.

6 Kashivara theorem for Ω_X -modules

6.1 For an open subset $U \subset X$ we denote by $j_U: U \rightarrow X$ the corresponding open embedding. Recall that $j_U^! \cong j_U^\bullet(-) \otimes_{j_U^\bullet \mathcal{O}_X} \mathcal{O}_U$. This justifies the following notation: $M|_U = j_U^! M$, where M is an Ω_X -module.

Lemma *Let M be an Ω_X -module, and let U and V be open subsets of X . Then we have the Mayer–Vietoris triangle in $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$*

$$j_{U \cup V *} j_{U \cup V}^! M \rightarrow j_{U *} j_U^! M \oplus j_{V *} j_V^! M \rightarrow j_{U \cap V *} j_{U \cap V}^! M \rightarrow .$$

Proof Let I be an injective representative for M . Then the sequence of Ω_X -modules

$$0 \rightarrow j_{U \cup V *} I|_{U \cup V} \rightarrow j_{U *} I|_U \oplus j_{V *} I|_V \rightarrow j_{U \cap V *} I|_{U \cap V} \rightarrow 0$$

is exact since I is flabby. $\square$

It follows that coacyclicity is a local property.

Corollary *Let M be an Ω_X -module. Suppose there is a finite covering $\mathcal{U}$ of X such that for any $U \in \mathcal{U}$ the restriction $M|_U$ is coacyclic. Then M is coacyclic.*

6.2 Let $i: Z \rightarrow X$ be a closed embedding, and $j: U = X \setminus Z \rightarrow X$ be the corresponding open embedding. Define the functor of “sections with support in Z ”

$$\Gamma_Z^X: \Omega_X\text{-qcoh} \rightarrow \Omega_X\text{-qcoh}$$

as follows. Let $J_Z \subset \mathcal{O}_X$ be the ideal sheaf of Z . Then the DG-ideal sheaf of Z is the DG-ideal $J \subset \Omega_X$ generated by J_Z and dJ_Z . For an Ω_X -module M put

$$\Gamma_Z^X(M) = \varinjlim_m \mathcal{H}om_{\Omega_X}(\Omega_X/J^m, M).$$

In fact, $\Gamma_Z^X(M) \cong \varinjlim_m \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/J_Z^m, M)$. Using the argument similar to [5, III.3.2] one proves that for an injective Ω_X -module I the module $\Gamma_Z^X(I)$ is also injective. Clearly, Γ_Z^X is exact on injective Ω_X -modules. Thus we can define the derived functor $R\Gamma_Z^X: D^{\text{co}}(\Omega_X\text{-qcoh}) \rightarrow D^{\text{co}}(\Omega_X\text{-qcoh})$ using injective resolutions. When X is fixed we often write Γ_Z for Γ_Z^X .

Proposition *For any Ω_X -module M there is a functorial distinguished triangle in $D^{\text{co}}(\Omega_X\text{-qcoh})$:*

$$R\Gamma_Z(M) \rightarrow M \rightarrow j_* j^! M \rightarrow .$$

Proof We have to prove that for an injective Ω_X -module I there is an exact triple

$$0 \rightarrow \Gamma_Z(I) \rightarrow I \rightarrow j_* j^! I \rightarrow 0.$$

The left morphism is the natural inclusion, and the right morphism comes from Theorem 4.5. Since I is injective, the triple is left and right exact. Now we prove that the triple is exact in the middle. We may assume that X is affine. Let $m \in I(X)$. If the restriction of m on U is zero, then for any $f \in J_Z$ there exist $k(f)$ such that $f^{k(f)} m = 0$. Since J_Z is finitely generated there exists k such that $J^k m = 0$, i.e. $m \in \Gamma_Z(I)$. $\square$

Theorem 6.3 *Assume that S is defined over $\mathbb{Q}$, i.e. all natural numbers are invertible in S . If Z is smooth, then for any Ω_X -module M we have $i_* Ri^b(M) \cong R\Gamma_Z(M)$ in $D^{\text{co}}(\Omega_X\text{-qcoh})$.*

Proof Take an injective resolution I of M . Then $i_*i^b(I) \cong \mathcal{H}om_{\Omega_X}(i_*\Omega_Z, I)$ is a submodule of $\Gamma_Z(I)$. This gives a morphism of functors. It is enough to prove that this morphism is equivalence when X is affine, and J_Z is generated by a regular sequence $f_1, \dots, f_r$. $\square$

Lemma 6.3.1 *Let $a: Z_1 \rightarrow X$ and $Z_2 \subset Z_1$ be closed embeddings. Then*

$$R\Gamma_{Z_2}^{Z_1} \circ Ra^b \cong Ra^b \circ R\Gamma_{Z_2}^X.$$

Proof Apply both sides to I . The module $E = \Gamma_{Z_2}^X(I)$ is injective, thus $a^b(E)$ computes the right hand side. By Proposition 6.2, we have the short exact sequence of Ω_X -modules

$$0 \rightarrow E \rightarrow I \rightarrow j_*j^!I \rightarrow 0,$$

where $j: X \setminus Z_2 \rightarrow X$ is the natural open embedding. Apply the functor $\Gamma_{Z_2}^{Z_1} \circ a^b$ to this exact sequence. By Corollary 4.8, $a^b j_*j^!I \cong 0$, and it is easy to check that $\Gamma_{Z_2}^{Z_1} \circ a^b(E) \cong a^b(E)$. The lemma follows. $\square$

By the lemma, we may assume that the ideal of Z in $\mathcal{O}_X$ is generated by a single regular function f .

Let $N = \Gamma_Z(I)$, and let $N_0 \cong i_*i^b(I)$ be the submodule of N annihilated by f and df . Put

$$N_k = \{m \in N : f^k m \in N_0\}.$$

By Lemma 6.3.2 below, N_k is an Ω_X -module. Clearly, $N \cong \varinjlim_k N_k$. We claim that for $k > 0$ the module N_k/N_{k-1} is contractible. Then by Lemma 3.2(c), N is co-isomorphic to N_0 . We prove the claim.

Lemma 6.3.2 *If $m \in N_k$, then $dm \in N_k$.*

Proof We prove that $f^k dm \in N_0$, i.e. $f(f^k dm) = 0$, and $df(f^k dm) = 0$. This is clear from the relations

$$\begin{aligned} 0 &= d(f^{k+1}m) = (k+1)f^k df \cdot m + f^{k+1}dm, \\ 0 &= d(f^k df \cdot m) = -f^k df \cdot dm. \end{aligned} \quad \square$$

Lemma 6.3.3 *If $df \cdot m = 0$ for some $m \in I$, then there exists $\beta \in I$ such that $m = df \cdot \beta$.*

Proof Recall that f is regular, and I is injective. Define the morphism $\varphi: df \wedge \Omega_X \rightarrow I$ by $\varphi(df \wedge \omega) = \omega \cdot m$. This definition is correct since f is regular. We can lift φ to a morphism $\tilde{\varphi}: \Omega_X \rightarrow I$. Put $\beta = \tilde{\varphi}(1)$. $\square$

Lemma 6.3.4 *Suppose that $m \in N_k$. Then there exists $\alpha \in N_k$ such that $f^k m = f^{k-1} df \cdot \alpha$, and $f^k \alpha = 0$.*

Proof Put $m' = f^k m \in N_0$. Then $df \cdot m' = 0$, and, by Lemma 6.3.3, there exists $\beta \in I$ such that $m' = df \cdot \beta$. By assumption,

$$f df \cdot \beta = f m' = 0.$$

As before, there exists $\gamma \in I$ such that $f\beta = f df \cdot \gamma$. Put $\alpha' = \beta - df \cdot \gamma$. Then $f\alpha' = 0$, and $m' = df \cdot \alpha'$. There exists $\alpha \in I$ such that $f^{k-1}\alpha = \alpha'$. Thus $f^k\alpha = 0$, and $\alpha \in N_k$. $\square$

We define an endomorphism s on N_k/N_{k-1} as follows. Take $m \in N_k$. Then, by Lemma 6.3.4, there exists $\alpha \in N_k$ such that $f^k m = f^{k-1} df \cdot \alpha$. Let $s(m)$ be the image of α in N_k/N_{k-1} . Clearly, $s(m)$ is independent on the choice of α , and s is a morphism of Ω_X^{gr} -modules of degree -1 . Let us prove that $ds + sd = -k$. From the relation $f^k m = f^{k-1} df \cdot \alpha$ we get

$$f^k dm = f^{k-1} df(-d\alpha - km).$$

It follows that $s(dm) = -ds(m) - km$, and $-s/k$ is a contracting homotopy for N_k/N_{k-1} . $\square$

Corollary 6.4 Assume that S is defined over $\mathbb{Q}$. Let I be an injective Ω_Z -module. Then an injective hull E of $i_* I$ is co-isomorphic to $i_* I$.

Proof Note that $E = \Gamma_Z(E)$, because it is true for the graded module E^{gr} . By Theorem 6.3, $E \cong i_* i^b(E)$ in $D^{\text{co}}(\Omega_X\text{-qcoh})$. We prove that $i^b(E)$ is isomorphic to I over Ω_Z . Since I is injective, it is a direct summand of $i^b(E)$ as a graded module, i.e. $i^b(E)^{\text{gr}} \cong I^{\text{gr}} \oplus I'$ for some Ω_Z^{gr} -module I' . Thus $i_* I'$ is a submodule of E^{gr} such that $i_* I' \cap i_* I = 0$. Since E^{gr} is an injective hull of $i_* I^{\text{gr}}$, we conclude that $i_* I' = 0$. The functor i_* is full and faithful, thus $I' = 0$. $\square$

Theorem 6.5 (Kashivara theorem) Assume that S is defined over $\mathbb{Q}$. Let $D_Z^{\text{co}}(\Omega_X\text{-qcoh})$ be the full subcategory of $D^{\text{co}}(\Omega_X\text{-qcoh})$ of Ω_X -modules with support on Z , and $D_Z^{\text{abs}}(\Omega_X\text{-coh})$ be the corresponding category of coherent Ω_X -modules. Then the functor i_* gives an equivalence of $D^{\text{co}}(\Omega_Z\text{-qcoh})$ with $D_Z^{\text{co}}(\Omega_X\text{-qcoh})$, and of $D^{\text{abs}}(\Omega_Z\text{-coh})$ with $D_Z^{\text{abs}}(\Omega_X\text{-coh})$. In particular, i_* is full and faithful.

Let M be an object of $D_Z^{\text{co}}(\Omega_X\text{-qcoh})$. By Theorem 6.3 and Proposition 6.2, we have the distinguished triangle in $D^{\text{co}}(\Omega_X\text{-qcoh})$

$$i_* Ri^b(M) \rightarrow M \rightarrow j_* j^+ M \rightarrow .$$

By assumption, $j^+ M = 0$, thus $i_* Ri^b \cong \text{Id}_{D_Z^{\text{co}}(\Omega_X\text{-qcoh})}$, and by Corollary 6.4, $Ri^b i_* \cong \text{Id}_{D^{\text{co}}(\Omega_Z\text{-qcoh})}$. For a coherent Ω_Z -module M the module $i_* M$ is also coherent. Conversely, any coherent Ω_X -module M with support in Z has a finite filtration such that the associated graded module is the direct image $i_* M'$ of a coherent Ω_Z -module M' . This is enough, since $D^{\text{abs}}(\Omega_X\text{-coh})$ is a full subcategory of $D^{\text{co}}(\Omega_X\text{-qcoh})$ [11, 3.11]. $\square$

Theorem 6.6 (base change) *Assume that S is defined over $\mathbb{Q}$. Let $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ be two morphisms of smooth schemes over S . Consider the Cartesian square*

$$\begin{array}{ccc} W & \xrightarrow{\tilde{g}} & X \\ \tilde{f} \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y. \end{array}$$

Assume that $W = X \times_Y Z$ is smooth over S . Then there exists an isomorphism

$$g^! \circ f_* \cong \tilde{f}_* \circ \tilde{g}^!: D^{\text{co}}(\Omega_X\text{-qcoh}) \rightarrow D^{\text{co}}(\Omega_Z\text{-qcoh}).$$

The proof is the same as in [6, 1.7.3].

7 Trace map for a closed embedding

7.1 Čech resolution Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite open covering of X , where $I = \{0, \dots, N\}$ is a finite ordered set. Let M be an Ω_X -module. We define the Čech resolution $C^\bullet(\mathcal{U}, M)$ of M as follows:

$$C^n(\mathcal{U}, M) = \bigoplus_{J \subset I} j_{J*} j_J^+ M,$$

where J is a subset of I of order $|J| = n + 1$, and j_J is the inclusion $\bigcap_{i \in J} U_i \rightarrow X$. The closed morphism $\partial_n: C^n(\mathcal{U}, M) \rightarrow C^{n+1}(\mathcal{U}, M)$ is defined in the usual way. Put $C(\mathcal{U}, M) = \text{Tot}^\oplus C^\bullet(\mathcal{U}, M)$. The natural closed morphisms $M \rightarrow j_{i*} M|_{U_i}$ induce the natural inclusion morphism $M \rightarrow C^0(\mathcal{U}, M) \rightarrow C(\mathcal{U}, M)$.

Lemma *The closed morphism $M \rightarrow C(\mathcal{U}, M)$ is a co-isomorphism. Moreover, its cone is absolutely acyclic.*

Proof The sequence of Ω_X^{gr} -modules

$$0 \rightarrow M^{\text{gr}} \rightarrow C^0(\mathcal{U}, M)^{\text{gr}} \rightarrow \dots \rightarrow C^{|I|}(\mathcal{U}, M)^{\text{gr}} \rightarrow 0$$

is exact, because it is the Čech resolution of the graded quasi-coherent $\mathcal{O}_X$ -module M^{gr} . The Ω_X -module cone $(M \rightarrow C(\mathcal{U}, M))$ is the total object of the sequence

$$0 \rightarrow M \rightarrow C^0(\mathcal{U}, M) \rightarrow \dots \rightarrow C^{|I|}(\mathcal{U}, M) \rightarrow 0.$$

It is absolutely acyclic, since we can cut this sequence into short exact sequences. The lemma follows. $\square$

Theorem *Suppose $\mathcal{U}$ is a finite affine covering of X . Then $f_*(M)$ is co-isomorphic to $f_\bullet(C(\mathcal{U}, M))$.*

Proof Let N be an injective resolution of M . Then $j_J^+(N)$ is flabby for any $J \subset I$, thus $C(\mathcal{U}, N)$ is flabby too, and $f_*(M)$ is co-isomorphic to $f_\bullet(N)$. By the lemma, $f_\bullet(N)$ is co-isomorphic to $f_\bullet C(\mathcal{U}, N)$. There is the natural morphism $\varphi: C(\mathcal{U}, M) \rightarrow C(\mathcal{U}, N)$ with coacyclic cone. Since the morphism $g = f \circ j_J$ is affine, the functor $g_\bullet$ is exact, and $g_\bullet j_J^+(N)$ is co-isomorphic to $g_\bullet j_J^+(M)$. It follows that φ induces a morphism $f_\bullet C(\mathcal{U}, M) \rightarrow f_\bullet C(\mathcal{U}, N)$ with coacyclic cone. $\square$

7.2 Gluing morphisms in $D^{\text{co}}(\Omega_X\text{-qcoh})$ In this section we prove a counterpart of [1, 3.2.2] for Ω_X -modules.

Theorem *Let M and N be Ω_X -modules. Suppose that M is coherent. Let $\mathcal{U}$ be a finite affine covering of X . Assume that $\text{Hom}_{D^{\text{co}}(\Omega_V\text{-qcoh})}(M|_V, N|_V[j]) = 0$ for all affine $V \subset U \in \mathcal{U}$ and $j < 0$. Then*

$$\mathcal{H}om_{D^{\text{co}}(\Omega_X\text{-qcoh})}(M, N): V \mapsto \text{Hom}_{D^{\text{co}}(\Omega_V\text{-qcoh})}(M|_V, N|_V)$$

is a sheaf.

Proof It is enough to prove that sections over X can be glued of sections over the covering $\mathcal{U}$. We may assume that N is injective. Define the presheaf

$$\mathcal{H}^q: V \mapsto \text{Hom}_{D^{\text{co}}(\Omega_V\text{-qcoh})}(M|_V, N|_V[q]).$$

We have to prove that $\mathcal{H}^0$ is a sheaf. By Proposition 4.3,

$$\mathcal{H}^q(V) \cong H^q(S, f_{|V*} \mathcal{H}om_{\Omega_V}(M|_V, N|_V)),$$

where $f: X \rightarrow S$ is the morphism to the base scheme, and $f_{|V}$ is the restriction to V . By Theorem 7.1, we have the Čech spectral sequence

$$H^p(\mathcal{U}, \mathcal{H}^q) \implies H^{p+q}(S, f_* \mathcal{H}om_{\Omega_X}(M, N)) \cong \text{Hom}_{D^{\text{co}}(\Omega_X\text{-qcoh})}(M, N[p+q]).$$

By assumption, $H^p(\mathcal{U}, \mathcal{H}^q) = 0$ for $q < 0$. It follows that $H^0(\mathcal{U}, \mathcal{H}^0) \cong \text{Hom}_{D^{\text{co}}(\Omega_X\text{-qcoh})}(M, N)$. $\square$

7.3 Koszul resolution Let $i: Z \rightarrow X$ be a closed embedding of affine smooth schemes over S . Suppose that Z is a complete intersection in X . In this section we construct an explicit perfect resolution for Ω_Z over Ω_X .

Theorem *There exists a perfect Ω_X -module K and a non-canonical co-isomorphism $K[n] \rightarrow i_* \Omega_Z$.*

Lemma 7.3.1 *Suppose that the ideal of Z in $\mathcal{O}_X$ is generated by $f_1, \dots, f_n \in \mathcal{O}_X$, where $n = \dim X - \dim Z$. Let P_Z be the submodule of Ω_X generated by $df_1 \wedge \dots \wedge df_n$, and let I_Z be the submodule $f_1 P_Z + \dots + f_n P_Z$. Then P_Z is perfect, and $i_* \Omega_Z \cong P_Z / I_Z[n]$.*

Proof Locally there exist functions $z_1, \dots, z_s \in \mathcal{O}_X$ such that $dz_1, \dots, dz_s, df_1, \dots, df_n$ generate Ω_X^1 over $\mathcal{O}_X$. Thus P_Z is a direct summand in Ω_X and perfect. We have the natural surjective morphism $P_Z[n] \rightarrow i_*\Omega_Z$ given by $df_1 \wedge \dots \wedge df_n \mapsto 1$. Clearly, its kernel is I_Z . $\square$

Lemma 7.3.2 *Let $L_\bullet$ be the Koszul resolution of $\mathcal{O}_Z$ over $\mathcal{O}_X$. Take the complex of Ω_X^{gr} -modules $K_\bullet = L_\bullet \otimes_{\mathcal{O}_X} P_Z$. Then the induced morphism $\delta_m: K_m \rightarrow K_{m-1}$ is a closed morphism of Ω_X -modules.*

Proof By definition of the Koszul complex, $L_\bullet \cong \bigwedge^\bullet L_1$ is a DG-algebra over $\mathcal{O}_X$ generated by L_1 , where $L_1 \cong \bigoplus_{i=1}^n v_i \mathcal{O}_X$ for some $v_i \in L_1$ and the differential is given by $v_i \mapsto f_i$. Thus $K_0 \cong P_Z$, and $K_1 \cong \bigoplus_{i=1}^n v_i P_Z$. The differential δ_m on $K_\bullet$ comes from the formula

$$\delta_1(v_i \otimes df_1 \wedge \dots \wedge df_n) = f_i df_1 \wedge \dots \wedge df_n.$$

Now it is clear that δ_m is a closed morphism of Ω_X -modules. $\square$

Proof of Theorem 7.3 Clearly, the sequence of Ω_X -modules

$$0 \rightarrow K_n \rightarrow \dots \rightarrow K_1 \rightarrow K_0 \cong P_Z \rightarrow i_*\Omega_Z[-n] \rightarrow 0$$

is exact. Put $K = \text{Tot}^\oplus(K_\bullet)$. $\square$

Theorem 7.4 *Suppose that S is a $\mathbb{Q}$ -scheme. Then there is a natural co-isomorphism $\text{tr}: R i^b(\Omega_X) \cong \Omega_Z[2 \dim X - 2 \dim Z]$.*

Proof Apply i_* to both sides. By Theorem 6.5, it is enough to prove that

$$R\mathcal{H}om_{\Omega_X}(i_*\Omega_Z, \Omega_X) \cong i_*\Omega_Z[2 \dim X - 2 \dim Z].$$

We construct the isomorphism locally and then use Theorem 7.2.

Assume that X is affine and the ideal of Z in $\mathcal{O}_X$ is generated by $f_1, \dots, f_n \in \mathcal{O}_X$, where $n = \dim X - \dim Z$. Let $K[n]$ be the Koszul resolution for $i_*\Omega_Z$. By Theorem 4.2,

$$R\mathcal{H}om_{\Omega_X}(K[n], \Omega_X) \cong \mathcal{H}om_{\Omega_X}(K[n], \Omega_X).$$

We use the notation from the proof of Lemma 7.3.2. By definition, $K_1 \cong \bigoplus_{i=1}^n v_i P_Z$. Hence $K_n \cong P_Z$ is generated by $v_1 \wedge \dots \wedge v_n$. Moreover, $\mathcal{H}om_{\Omega_X}(P_Z, \Omega_X)$ is isomorphic to $P_Z[n]$ non-canonically. Thus the complex of Ω_X -modules $\check{K}^\bullet = \mathcal{H}om_{\Omega_X}(K_\bullet[n], \Omega_X)$ is isomorphic to $K_\bullet$. The module $\check{K}^n[n] = \mathcal{H}om_{\Omega_X}(K_n, \Omega_X) \cong P_Z$ has the generator u given by

$$v_1 \wedge \dots \wedge v_n \mapsto df_1 \wedge \dots \wedge df_n.$$

As in Lemma 7.3.1, we define the morphism $\check{K}^n[n] \rightarrow \Omega_Z$ such that u goes to $1 \in \Omega_Z$. This gives a co-isomorphism

$$\psi_X: \check{K} = R\mathcal{H}om_{\Omega_X}(i_*\Omega_Z, \Omega_X) \rightarrow i_*\Omega_Z[-2n].$$

It is clear that ψ_X does not depend on the choice of coordinates $f_1, \dots, f_n$.

We proved the theorem locally and now turn to the global situation. We have to find a co-isomorphism of $\check{K} = R\mathcal{H}om_{\Omega_X}(i_*\Omega_Z, \Omega_X)$ and $i_*\Omega_Z[-2n]$. By the local result,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}^{\mathrm{co}}(\Omega_V\text{-qcoh})}(\check{K}|_V, i_*\Omega_Z|_V[-2n+j]) \\ \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{co}}(\Omega_V\text{-qcoh})}(i_*\Omega_Z|_V, i_*\Omega_Z|_V[j]) \\ \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{co}}(\Omega_{Z \cap V}\text{-qcoh})}(\Omega_{Z \cap V}, \Omega_{Z \cap V}[j]) = 0 \end{aligned}$$

for small affine open $V \subset X$ and any $j < 0$. By Theorem 7.2, the presheaf $\mathcal{H}om_{\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})}(\check{K}, i_*\Omega_Z[-2n])$ is a sheaf. Thus we can glue local morphisms ψ_U to a global morphism $\psi \in \mathrm{Hom}_{\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})}(\check{K}, i_*\Omega_Z[-2n])$. This morphism is a co-isomorphism by Corollary 6.1. $\square$

Theorem 7.5 *Suppose that S is a $\mathbb{Q}$ -scheme. Let $i: Z \rightarrow X$ be a closed embedding of smooth schemes over S . Then the functors Ri^b and $i^!$ are naturally equivalent.*

Proof In Theorem 7.4 we construct the trace isomorphism for closed embedding

$$\mathrm{tr}: R\mathcal{H}om_{f^*\Omega_X}(\Omega_Z, f^*\Omega_X) \cong \Omega_Z[2 \dim Z - 2 \dim X].$$

By Remark 3.8, any Ω_X -module has a resolution F which is locally free over Ω_X^{gr} . We have

$$\begin{aligned} Ri^b(F) = R\mathcal{H}om_{f^*\Omega_X}(\Omega_Z, f^*(F)) &\cong R\mathcal{H}om_{f^*\Omega_X}(\Omega_Z, f^*\Omega_X) \otimes_{f^*\Omega_X} f^*(F) \\ &\cong \Omega_Z \otimes_{f^*\Omega_X} f^*(F)[2 \dim Z - 2 \dim X] = i^!(F). \end{aligned} \quad \square$$

8 Coherent Ω -modules

8.1 Recall that $\mathrm{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$ is a full subcategory of $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$. Moreover, coherent Ω_X -modules are compact objects of $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ [11, 3.1.1]. Let $\mathrm{D}^{\mathrm{coh}}(X)$ be the thick envelope of $\mathrm{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$ in $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$. We prove that an object M of $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ belongs to $\mathrm{D}^{\mathrm{coh}}(X)$ if and only if M is locally coherent.

Proposition [13, Remark 1.10] Let $\mathcal{U}$ be a finite open covering of X and let M be an object of $\mathrm{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$. Then M belongs to $\mathrm{D}^{\mathrm{coh}}(X)$ if and only if for any $U \in \mathcal{U}$ the module $M|_U$ belongs to $\mathrm{D}^{\mathrm{coh}}(U)$.

Proof We give a proof under assumption that S is defined over $\mathbb{Q}$. Take $U \in \mathcal{U}$. By the Kashiwara theorem, $\mathrm{D}^{\mathrm{co}}(\Omega_U\text{-qcoh})$ is equivalent to the Verdier quotient of $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$ by $\mathrm{D}^{\mathrm{co}}(\Omega_Z\text{-qcoh})$. Now, by Theorem [8, 2.1.5], there is an object N of

$D^{\text{coh}}(X)$ and a morphism $N \rightarrow M$ such that $N|_U \rightarrow M|_U$ is a co-isomorphism. The cone $M_1 = \text{cone}(N \rightarrow M)$ is coacyclic on U . Take $V \in \mathcal{U}$, then $M_1|_V$ belongs to $D^{\text{coh}}(V)$. We can make V smaller in such a way that $Z = V \setminus (U \cap V)$ is smooth. By the Kashivara theorem, $M_1|_{V \cup U}$ is co-isomorphic to $i_* M_Z$, where $i: Z \rightarrow V \cup U$ is the natural inclusion, and M_Z is coherent. We proved that $M|_{U \cup V}$ belongs to $D^{\text{coh}}(U \cup V)$. The proposition follows by induction. $\square$

Theorem 8.2 *Let $f: X \rightarrow Y$ be a morphism of smooth schemes over S . If f is proper, and M is a coherent Ω_X -module, then $f_*(M)$ is isomorphic to a coherent Ω_Y -module in $D^{\text{co}}(\Omega_Y\text{-qcoh})$. If f is smooth, and N is a coherent Ω_Y -module, then $f^!(N)$ is isomorphic to a coherent Ω_X -module in $D^{\text{co}}(\Omega_X\text{-qcoh})$.*

Proof The category of coherent Ω_X -modules is generated by coherent modules annihilated by Ω_X^1 . Suppose M is such a module. We have to prove that $f_*(M)$ is coherent. The direct image can be computed using Čech resolution $C^\bullet(M)$. Clearly, $f_* C^\bullet(M)$ is annihilated by Ω_Y^1 , and by Serre's theorem is quasi-isomorphic to a complex of coherent $\mathcal{O}_Y$ -modules. This quasi-isomorphism is naturally a co-isomorphism of Ω_Y -modules. The second part of the theorem follows from Lemma 4.4. $\square$

8.3 Duality Let $n = \dim_S X = \dim X - \dim S$ be a relative dimension of X over S . We define the duality functor

$$D_X: D^{\text{abs}}(\Omega_X\text{-coh}) \rightarrow D^{\text{abs}}(\Omega_X\text{-coh})$$

by $D_X = R\mathcal{H}om_{\Omega_X}(-, \Omega_X[2n])$. Take a perfect Ω_X -module P . Since Ω_X is coinduced the coherent module $\mathcal{H}om_{\Omega_X}(P, \Omega_X[2n])$ represents $R\mathcal{H}om_{\Omega_X}(P, \Omega_X[2n])$. It follows that D_X is well-defined.

Proposition $D_X^2 \cong \text{Id}_{D^{\text{abs}}(\Omega_X\text{-coh})}$.

Proof It is enough to prove that D_X^2 is equivalent to the identity functor on the category of perfect Ω_X -modules. For a perfect Ω_X -module P we have a natural transformation

$$\begin{aligned} \Phi: P &\rightarrow \mathcal{H}om_{\Omega_X}(\mathcal{H}om_{\Omega_X}(P, \Omega_X), \Omega_X) \\ &\cong \mathcal{H}om_{\Omega_X}(\mathcal{H}om_{\Omega_X}(P, \Omega_X[2n]), \Omega_X[2n]). \end{aligned}$$

In fact, Φ is an isomorphism of Ω_X -modules. We prove it locally for $P = \mathcal{O}_X$. In this case $\mathcal{H}om_{\Omega_X}(\mathcal{O}_X, \Omega_X) \cong \Omega_X^n[-n]$, and $\mathcal{H}om_{\Omega_X}(\Omega_X^n[-n], \Omega_X) \cong \mathcal{O}_X$. $\square$

Proposition 8.4 *Let M and N be coherent Ω_X -modules. Then*

$$D(M) \otimes_{\Omega_X}^D N \cong R\mathcal{H}om_{\Omega_X}(M, N).$$

Proof Choose a perfect resolution P for M and a coinduced resolution A for N , i.e. $A^{\text{gr}} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^{\text{gr}}, L)$ for some graded $\mathcal{O}_X$ -module L . Let Φ be the natural transformation

$$\mathcal{H}om_{\Omega_X}(P, \Omega_X) \otimes_{\Omega_X} A \rightarrow \mathcal{H}om_{\Omega_X}(P, A)$$

given by the formula $\varphi \otimes a \mapsto (p \mapsto \varphi(p)a)$, where $\varphi \in \mathcal{H}om_{\Omega_X}(P, \Omega_X)$, $p \in P$, and $a \in A$. We claim that Φ is an isomorphism of Ω_X -modules. Indeed, the question is local and we may assume that $P = \mathcal{O}_X$. In this case, Φ is an isomorphism since A is coinduced. The proposition follows. $\square$

Proposition 8.5 *Suppose S is a $\mathbb{Q}$ -scheme. Let $i: Z \rightarrow X$ be a closed embedding of smooth schemes over S . Then $D_X \circ i_* \cong i_* \circ D_Z$.*

Proof By Theorems 4.7 and 7.5, for any coherent Ω_Z -module M we have

$$\begin{aligned} D_X(i_*M) &= R\mathcal{H}om_{\Omega_X}(i_*M, \Omega_X[2 \dim_S X]) \\ &\cong i_* R\mathcal{H}om_{\Omega_Z}(M, i^! \Omega_X[2 \dim_S X]) \\ &\cong i_* R\mathcal{H}om_{\Omega_Z}(M, \Omega_Z[2 \dim_S Z]) = i_* D_Z(M). \end{aligned} \quad \square$$

8.6 Trace for a proper projection Suppose that the base scheme S is defined over $\mathbb{Q}$. Let $p: \mathbb{P}^n = \mathbb{P}_S^n \rightarrow S$ be the natural projection from the projective space to S . We define the trace morphism Tr_p as the composition

$$f_* \Omega_{\mathbb{P}^n}[2n] \rightarrow H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n) \rightarrow \mathcal{O}_S$$

of the natural projection to $H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n)$ with the trace morphism defined in [5, Remark 10.2].

Let $g: Y \rightarrow S$ be a smooth S -scheme. Denote by $f: X = \mathbb{P}^n \times_S Y \rightarrow Y$ and $q: X \rightarrow \mathbb{P}^n$ natural projections to Y and $\mathbb{P}^n$. By the base change formula we have the trace morphism

$$\mathrm{Tr}_f: f_* \Omega_X[2 \dim Y + 2n] \cong f_*(q^! \Omega_{\mathbb{P}^n}[2n]) \cong g^! p_* \Omega_{\mathbb{P}^n}[2n] \xrightarrow{\mathrm{Tr}_p} g^! \mathcal{O}_S \cong \Omega_Y.$$

Lemma 8.7 *Suppose $f: X \rightarrow Y$ is smooth and proper. For any $M, N \in D^{\mathrm{abs}}(\Omega_X\text{-coh})$ there is a natural morphism*

$$f_* R\mathcal{H}om_{\Omega_X}(M, N) \rightarrow R\mathcal{H}om_{\Omega_X}(f_* M, f_* N).$$

Proof By Theorem 8.2, $f_* M$ is coherent, and the right hand side is well-defined. By Theorem 4.5, we have an adjunction morphism $f^+ f_* M \rightarrow M$, and the desired morphism is the composition

$$f_* R\mathcal{H}om_{\Omega_X}(M, N) \rightarrow f_* R\mathcal{H}om_{\Omega_X}(f^+ f_* M, N) \rightarrow R\mathcal{H}om_{\Omega_X}(f_* M, f_* N). \quad \square$$

Theorem 8.8 *Let $f: X \rightarrow Y$ be a proper morphism of smooth schemes over S . Then $D_Y \circ f_* \cong f_* \circ D_X$.*

Proof The morphism f is a composition of a closed embedding and a proper projection. By Proposition 8.5, we have to prove the theorem for the projection $f: X = \mathbb{P}^n \times_S Y \rightarrow Y$. For any coherent Ω_X -module M we have the natural morphism

$$\begin{aligned} f_* D_X(M) &\cong f_* R\mathcal{H}om_{\Omega_X}(M, \Omega_X[2 \dim_S X]) \\ &\rightarrow R\mathcal{H}om_{\Omega_Y}(f_* M, f_* \Omega_X[2 \dim_S X]) \\ &\xrightarrow{\mathrm{Tr}_p} R\mathcal{H}om_{\Omega_Y}(f_* M, \Omega_Y[2 \dim_S Y]) \cong D_Y(f_* M). \end{aligned}$$

It is enough to prove that this morphism is an isomorphism when Y is affine, and $M \cong q^* \mathcal{O}_{\mathbb{P}^n}(k)$, where $q: X \rightarrow \mathbb{P}^n$ is the second projection. This is straightforward. $\square$

Theorem 8.9 *Suppose that $f: X \rightarrow Y$ is smooth and proper. Let $N \in D^{\mathrm{co}}(\Omega_X\text{-qcoh})$ and let $M \in D^{\mathrm{abs}}(\Omega_X\text{-coh})$. Then*

$$R\mathcal{H}om_{\Omega_Y}(f_* M, N) \cong f_* R\mathcal{H}om_{\Omega_X}(M, f^! N).$$

Proof This follows from the projection formula, Proposition 8.4, and Theorem 8.8,

$$\begin{aligned} f_* R\mathcal{H}om_{\Omega_X}(M, f^! N) &\cong f_*(D(M) \otimes_{\Omega_X}^D f^! N) \cong f_* D(M) \otimes_{\Omega_Y}^D N \\ &\cong D(f_* M) \otimes_{\Omega_Y}^D N \cong R\mathcal{H}om_{\Omega_Y}(f_* M, N). \end{aligned} \quad \square$$

The proof of the following result is the same as the proof of [6, 2.7.1]

Proposition 8.10 *If $f: X \rightarrow Y$ is smooth, then*

$$D_X \circ f^! \cong f^! \circ D_Y[2(\dim X - \dim Y)].$$

Equivalently, $f^+ \cong D_X \circ f^! \circ D_Y$.

9 Ω -modules and D -modules

9.1 Positselski theorem Suppose that $S = \mathrm{Spec} k$, where k is a field of characteristic zero. Let $\mathcal{D}_X$ be the sheaf of differential operators on a smooth quasi-projective algebraic variety X over k . We have an $\Omega_X\text{-}\mathcal{D}_X$ -bimodule $\mathcal{DR}_X = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ with differential given in local coordinates $x_1, \dots, x_n$ on X by the formula

$$d(\omega \otimes m) = d\omega \otimes m + \sum_{i=1}^n \omega \wedge dx_i \otimes \partial/\partial x_i m,$$

where m is a local section of $\mathcal{D}_X$ and ω is a local section of Ω_X . The following result is an easy corollary of [11, B.2, B.5].

Theorem [11, B.2, B.5] The functor

$$\mathrm{DR}_X = - \bigotimes_{\mathcal{D}_X} \mathcal{DR}_X[\dim X]: \mathcal{D}_X\text{-qcoh} \rightarrow \Omega_X\text{-qcoh}$$

induces an equivalence of the unbounded derived category of quasi-coherent left $\mathcal{D}_X$ -modules $\mathrm{D}(\mathcal{D}_X\text{-qcoh})$ and the coderived category $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$. Moreover, this functor induces an equivalence of the bounded derived category of coherent left $\mathcal{D}_X$ -modules $\mathrm{D}^b(\mathcal{D}_X\text{-coh})$ and the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\Omega_X\text{-coh})$.

Lemma 9.2 Let $\mathcal{DR}_X^{-1} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \Omega_X$ be a $\mathcal{D}_X$ - Ω_X -bimodule with differential

$$d(m \otimes \theta_1 \wedge \cdots \wedge \theta_r) = \sum_{i=1}^r m \theta_i \otimes \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_r,$$

where $\theta_1, \dots, \theta_r$ are commuting vector fields. Then there exists a natural co-isomorphism of Ω_X -modules $\mathcal{O}_X \rightarrow \mathcal{DR}_X^{-1}[\dim X]$.

Proof The morphism is induced by the natural inclusion $\mathcal{O}_X \rightarrow \mathcal{D}_X$. The filtration of $\mathcal{DR}_X^{-1}/\mathcal{O}_X$ defined in [6, 1.5.27] has acyclic associated graded complex. Each graded component is a bounded complex of $\mathcal{O}_X$ -modules and thus coacyclic. The lemma follows from Lemma 3.2(c). $\square$

9.3 Let $f: X \rightarrow Y$ be a morphism of smooth quasi-projective algebraic varieties over k . Recall the definition of the direct image f_* and the extraordinary inverse image $f^\dagger$ for $\mathcal{D}$ -modules [6]. The functor $f^\dagger$ is the derived functor of $f^\bullet(-) \otimes_{f^\bullet \mathcal{O}_Y} \mathcal{O}_X$ shifted by $\dim X - \dim Y$. Let $\mathcal{D}_{Y \leftarrow X} = \omega_X \otimes_{f^\bullet \mathcal{O}_Y} f^\bullet(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1})$ be a $\mathcal{D}_X$ - $f^\bullet \mathcal{D}_Y$ -bimodule. Then $f_* = Rf_\bullet(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L -)$. For coherent $\mathcal{D}_X$ -modules we have the duality functor $\mathrm{D}'_X(-) = R\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1})[\dim X]$. We are going to prove that the Positselski equivalence takes these functors to corresponding functors for Ω -modules. We also prove that the Positselski equivalence is a tensor functor. By definition, the tensor product of left $\mathcal{D}_X$ -modules M and N is the left $\mathcal{D}_X$ -module $M \otimes_{\mathcal{O}_X} N$ such that a vector field θ acts by the formula $\theta(m \otimes n) = \theta(m) \otimes n + m \otimes \theta(n)$.

Theorem (a) *There exists an equivalence of bifunctors from $\mathrm{D}(\mathcal{D}_X\text{-qcoh}) \times \mathrm{D}(\mathcal{D}_X\text{-qcoh})$ to $\mathrm{D}^{\mathrm{co}}(\Omega_X\text{-qcoh})$:*

$$\mathrm{DR}_X(-) \otimes_{\Omega_X}^D \mathrm{DR}_X(-) \cong \mathrm{DR}_X(- \otimes_{\mathcal{O}_X}^L -).$$

- (b) $f_*(\mathrm{DR}_X(-)) \cong \mathrm{DR}_Y(f_*(-))$.
- (c) $f^!(\mathrm{DR}_Y(-)) \cong \mathrm{DR}_X(f^\dagger(-))$.
- (d) $\mathrm{D}_X(\mathrm{DR}_X(-)) \cong \mathrm{DR}_X(\mathrm{D}'_X(-))$.

Proof (a) Let M and N be complexes of quasi-coherent $\mathcal{D}_X$ -modules. Assume that M is a complex of flat modules over $\mathcal{O}_X$. We have the following co-isomorphisms in $D^{co}(\Omega_X\text{-qcoh})$:

$$\begin{aligned} DR_X(M) \otimes_{\Omega_X}^D DR_X(N) &\cong (\Omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\Omega_X} (\Omega_X \otimes_{\mathcal{O}_X} N) \\ &\cong \Omega_X \otimes_{\mathcal{O}_X} (M \otimes_{\mathcal{O}_X} N) \cong DR_X(M \otimes_{\mathcal{O}_X}^D N). \end{aligned}$$

(b) Let L be a complex of quasi-coherent flat $\mathcal{D}_Y$ -modules. Then

$$\begin{aligned} f^!(DR_Y(L)) &\cong \Omega_X \otimes_{f^*\Omega_Y} f^\bullet(\Omega_Y \otimes_{\mathcal{O}_Y} L)[2 \dim X - \dim Y] \\ &\cong \Omega_X \otimes_{f^*\mathcal{O}_Y} f^\bullet L[2 \dim X - \dim Y] \\ &\cong \Omega_X \otimes_{\mathcal{O}_X} f^\dagger L[\dim X] \cong DR_X(f^\dagger(L)). \end{aligned}$$

(c) We give two proofs of this fact. First, the morphism f is a composition of a closed embedding $i: X \rightarrow Z$ and a smooth projection $g: Z \rightarrow Y$. For example, $Z = X \times_S Y$, and i is the graph of f . Clearly, (c) follows from (b) and Theorems 4.5 and 4.7, since the adjoint functor is unique up to equivalence. Unfortunately, this equivalence depends on the choice of g and i , and we give a more explicit second proof.

Let M be a complex of flat quasi-coherent $\mathcal{D}_X$ -modules. Then

$$\begin{aligned} DR_Y(f_*(M)) &\cong \Omega_Y \otimes_{\mathcal{O}_Y} Rf_\bullet(\mathcal{D}_Y \leftarrow_X \otimes_{\mathcal{D}_X} M)[\dim Y] \\ &\cong \Omega_Y \otimes_{\mathcal{O}_Y} Rf_\bullet(f^\bullet(\omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{f^*\mathcal{O}_Y} \omega_X \otimes_{\mathcal{D}_X} M)[\dim Y] \\ &\cong \Omega_Y \otimes_{\mathcal{O}_Y} Rf_\bullet(f^\bullet(\omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{f^*\mathcal{O}_Y} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{D}_X} M)[\dim X + \dim Y] \\ &\cong \Omega_Y \otimes_{\mathcal{O}_Y} Rf_\bullet(f^\bullet(\omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \otimes_{f^*\mathcal{O}_Y} \Omega_X \otimes_{\mathcal{O}_X} M)[\dim X + \dim Y], \end{aligned}$$

and

$$\begin{aligned} f_*(DR_X(M)) &\cong \Omega_Y \otimes_{\Omega_Y} f_*(\Omega_X \otimes_{\mathcal{O}_X} M)[\dim X] \\ &\cong \Omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\Omega_Y} f_*(\Omega_X \otimes_{\mathcal{O}_X} M)[\dim X + \dim Y], \end{aligned}$$

where we used the quasi-isomorphism of right $\mathcal{D}_X$ -modules $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X[\dim X] \rightarrow \omega_X$ (see [6, 1.5.27]), and the corresponding co-isomorphism of Ω_Y -modules $\Omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \rightarrow \Omega_Y[\dim Y]$.

Choose an affine covering $\mathcal{U}$ of X . We can compute f_* and $Rf_\bullet$ using corresponding Čech resolutions. Thus it is enough to construct the equivalence when X and Y are affine. We have to check that the natural morphism

$$\begin{aligned} \Omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y \otimes_{\Omega_Y} \Omega_X \otimes_{\mathcal{O}_X} M \\ \rightarrow \Omega_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_X \otimes_{\mathcal{O}_X} M \end{aligned}$$

commutes with differentials. This follows from definitions.

(d) The functor $\mathrm{DR}_X^{-1}(-) = \mathcal{D}\mathcal{R}_X^{-1} \otimes_{\Omega_X} -$ is quasi-inverse to DR_X . Let P be a perfect Ω_X -module. We prove that $D'_X(\mathrm{DR}_X^{-1}(P)) \cong \mathrm{DR}_X^{-1}(D_X(P))$. Using the quasi-isomorphism of left $\mathcal{D}_X$ -modules $\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \Omega_X[\dim X] \rightarrow \mathcal{O}_X$ we get

$$\begin{aligned} D'_X(\mathrm{DR}_X^{-1}(P)) &\cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} P, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1})[\dim X] \\ &\cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} P, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_X[-\dim X])[2 \dim X] \\ &\cong \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} P, \\ &\quad \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \Omega_X)[2 \dim X]. \end{aligned}$$

We have the natural transformation

$$\begin{aligned} \Phi: \mathrm{DR}_X^{-1}(D_X(P))[-2 \dim X] &\cong \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathrm{Hom}_{\Omega_X}(P, \Omega_X) \\ &\rightarrow \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} P, \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \otimes_{\mathcal{O}_X} \Omega_X) \end{aligned}$$

given over the affine subscheme $U \subset X$ by the formula $m \otimes \varphi \mapsto (d \otimes p \mapsto (m \otimes d \otimes \varphi(p)))$, where $\varphi \in \mathrm{Hom}_{\Omega_X}(P, \Omega_X)(U)$, $p \in P(U)$, $m, d \in \mathcal{D}_U \otimes_{\mathcal{O}_U} \omega_U^{-1}$. We claim that the transformation is an isomorphism of $\mathcal{D}_X$ -modules. As usual, it is enough to prove the claim locally and for $P = \mathcal{O}_X$. This is clear. $\square$

Remark The internal Hom for Ω_X -modules corresponds to the internal Hom for $\mathcal{D}$ -modules under Positselski equivalence. The last bifunctor is defined as follows:

$$R\mathrm{Hom}_{\mathcal{D}_X}(M, N) = D'_X(M) \otimes_{\mathcal{O}_X}^L N,$$

where M is coherent, and N is quasi-coherent.

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