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ORIGINAL ARTICLE

Derivation rings of Lie rings

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Abstract We establish some connections between Lie rings, their derivation rings and generalized derivations rings.

Keywords Lie ring · Derivation · Generalized derivation · Lie FC-ring

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1 Introduction

Let A be a Lie ring (with addition "+" and multiplication "$[-,-]$"). An additive mapping $d : A \to A$ is called *a derivation* of A if

$$d([x, y]) = [d(x), y] + [x, d(y)]$$

for any $x, y \in A$. By Der$A$ we denote the set of all derivations of A. It is well known that Der$A$ is a Lie ring with respect to operations of the point-wise addition "+" and the point-wise Lie multiplication "$[-,-]$" defined by rules

$$(d + \delta)(r) = d(r) + \delta(r) \text{ and } [d, \delta](r) = d(\delta(r)) - \delta(d(r))$$

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for all $r \in A$ and $d, \delta \in \mathrm{Der}A$ [11]. The mapping

$$\mathrm{ad}_a : A \ni x \mapsto [a, x] \in A$$

determines a derivation ad_a of A (so-called *inner derivation* of A induced by $a \in A$). The set

$$\mathrm{ad}A = \{\mathrm{ad}_a \mid a \in A\}$$

of all inner derivations of A is an ideal in $\mathrm{Der}A$.

Let $\mathbf{N}$ be the set of positive integers,

$$[a_1, \ldots, a_n, a_{n+1}] := [[a_1, \ldots, a_n], a_{n+1}]$$

for any $n \in \mathbf{N}$ and $a_1, \ldots, a_n, a_{n+1} \in A$. If $A_i \subseteq A$, then $[A_1, \ldots, A_n]$ is a subgroup of the additive group A^+ of A generated by all $[a_1, \ldots, a_n]$, where $a_i \in A_i \quad (i = 1, \ldots, n)$. A Lie ring A is called:

- *solvable* if there exists $n \in \mathbf{N}$ such that $A^{(n)} = 0$, where $A^{(1)} := [A, A] = A'$, $A^{(2)} := [A', A'] = A''$ and $A^{(k+1)} = [A^{(k)}, A^{(k)}]$ for any $k \in \mathbf{N}$,
- *abelian* if $A' = 0$,
- *perfect* if $A' = A$,
- *complete* if its derivations are inner and the center $Z(A) = 0$ is zero (i.e. A is centerless) [13].

Many authors have been investigated the structure of derivation algebra $\mathrm{Der}L$ and its relations with the structure of a (finite or infinite dimensional) Lie algebra L (see e.g. [6,7,9,12,23–26,28,30,31] and others). In this way Leger [16] has investigated Lie algebras L over a field of characteristic 0 such that $\mathrm{Der}L = \mathrm{ad}L$. Luks [20] has constructed an example of a Lie algebra over a field of characteristic zero with only inner derivations which is not complete. Su and Zhu [27, Theorem 1.1] have proved that the Lie algebra of all derivations of a centerless perfect Lie algebra (over any field) is complete. Augolopoulos [1] has constructed a class of complete Lie algebras over the complex numbers field that are not semisimple. Interesting results about complete Lie algebras were obtained by Meng and Wang (see [21], where further references can be found).

An additive mapping $F : A \to A$ is called *a generalized derivation of A associated with a derivation* $\delta \in \mathrm{Der}A$ (in the sence of Brešar [4]) if

$$F([x, y]) = [F(x), y] + [x, \delta(y)]$$

for any $x, y \in A$. The set of all generalized derivations of a Lie ring A we denote by

$$\mathrm{GDer}A.$$

We will write $(F, \delta) \in \mathrm{GDer}A$ *if and only if* F *is a generalized derivation of* A *associated with* $\delta \in \mathrm{Der}A$. Since $(\delta, \delta) \in \mathrm{GDer}A$ for any $\delta \in \mathrm{Der}A$, we conclude that

$$\mathrm{ad}A \subseteq \mathrm{Der}A \subseteq \mathrm{GDer}A.$$

A generalized derivation F of A associated with an inner derivation $\mathrm{ad}_a \in \mathrm{ad}A$ is called *a generalized inner derivation* of A. By

$$\mathrm{IGDer}A$$

we denote the set of all generalized inner derivations of A. Another types of generalized derivations was introduced in [8–10,17,29,32] and others.

We shall use the following notation. *Let* $D = \mathrm{Der}A$, $G = \mathrm{GDer}A$, Δ *be a nonempty subset of* D (respectively G). If $I \subseteq A$ and $\delta(I) \subseteq I$ for all $\delta \in \Delta$, then we say that I is Δ*-closed* in A. If I is a Δ-closed ideal of A, then it is called a Δ*-ideal* of A. Moreover, $Z(A) := \{z \in A \mid za = az \text{ for all } a \in A\}$ is the center of A.

Let $X \in \{0, \Delta\}$. By a 0-ideal of R we mean an ideal of R. An ideal Y of a Lie ring A is called:

- X*-semiprime* if, for any X-ideal B of A, the condition $[B, B] \subseteq Y$ implies that $B \subseteq Y$,
- X*-prime* if, for any X-ideals B, C of A, the condition $[B, C] \subseteq Y$ implies that $B \subseteq Y$ or $C \subseteq Y$,
- X*-simple* if $[A, A] \nsubseteq Y$ and, for any proper X-ideal B of A, it is true that $B \subseteq Y$,
- X*-primary* if, for any X-ideals B, C of A, the condition $[B, C] \subseteq Y$ implies that $B \subseteq Y$ or
$$[\underbrace{C, \ldots, C}_{m \text{ times}}] \subseteq Y \tag{1}$$
for some $m \in \mathbf{N}$.

In particular, if the zero ideal 0 of A is a X-simple (respectively X-prime, X-semiprime or X-primary) Lie ring and $X = 0$, then A is called *simple* (respectively *prime*, *semiprime* or *primary*). Moreover, A/Y is semiprime (respectively prime, simple or primary) if so is Y. Every Δ-prime Lie ring is Δ-semiprime and every Δ-simple Lie ring is Δ-prime.

The purpose of this paper is to study relationships between Lie rings A, their derivation rings $\mathrm{Der}A$ (in particular, inner derivation rings $\mathrm{IDer}A$) and generalized derivation rings $\mathrm{GDer}A$. In many cases every derivation of a simple Lie algebra is inner (see e.g. [11,24,34,35] and others). Our first result is the following

Proposition 1 *Let A be a Lie ring. Then we have:*

(1) *if D is a simple Lie ring, then one of the following holds:*
  (a) *A is abelian,*
  (b) $D = \mathrm{ad}A$, $A/Z(A)$ *is a simple Lie ring and either*
    (i) $A = A'$ *is simple, or*
    (ii) $A = A' + Z(A)$ *and* A' *is the smallest noncentral ideal of A,*
(2) *if A is a D-simple Lie ring, then* $A = A'$, $Z(A) = 0$ *and D is complete.*

We say that a subring S is *of finite index* in A if its additive subgroup S^+ has finite index $|A : S| := |A^+ : S^+|$ in A^+. By analogy with group theory [22], we will say that a Lie ring A is *a Lie FC-ring* if, for any $a \in A$, the centralizer

$$C_A(a) = \{z \in A \mid [z, a] = 0\}$$

is of finite index in A. If A is FC, then for every $a \in A$ there exists an ideal I_a of A such that $[a, I_a] = 0$ (Corollary 4). In the proof of this result we use the following

Proposition 2 *Let A be a Lie ring. If S is its subring of finite index, then there exists an ideal I of A such that $I \leq S$ and $|A : I| < \infty$.*

Since an inner derivation $\mathrm{ad}_a : A^+ \to A^+$ is an endomorphism of the additive group A^+ for any $a \in A$,

$$[A, a] \ni [x, a] \mapsto x + C_A(a) \in A/C_A(a)$$

is an additive group isomorphism and the kernel $\mathrm{Ker}\ \mathrm{ad}_a = C_A(a)$, we conclude that A is FC if and only if the image $\mathrm{Im}\ \mathrm{ad}_a$ is finite for any $a \in A$. If the set $\mathrm{IDer}A$ is finite (or equivalently $|A : Z(A)| < \infty$ by Proposition 3), then A is FC (Lemma 6), the commutator ideal A' is finite and there exists a solvable ideal S of A such that $S \leq A'$, $A' = A'' + S$ and A'/S is a direct summand of A/S (Corollary 6).

Derivations are very important in the study of structures of Lie algebras. Lie algebras with semisimple (in particular, simple) derivation algebras have been discussed by Hochschild [9], Block [3], de Ruiter [5], Elduque and Montaner [7], Walcher [34] and others. For example, it was proved in [9, Theorem 4.4] that a finite dimensional Lie algebra L over a field of characteristic 0 is semisimple if and only if its derivation algebra $\mathrm{Der}L$ is semisimple. In this way we prove the next result.

Theorem 1 *Let A be a Lie ring. If A is a D-prime (respectively D-semiprime) Lie ring, then D is prime (respectively semiprime).*

An additive mapping $T : A \to A$ is called *a multiplier* of A if

$$T([x, y]) = [T(x), y]$$

for all $x, y \in A$. Then we have

$$T([x, y]) = T(-[y, x]) = -[T(y), x] = [x, T(y)].$$

The set of all multipliers of A we denote by

$$\mathrm{M}(A).$$

Obviously, for any $T \in \mathrm{M}(A)$, $(T, 0) \in \mathrm{GDer}A$ and so

$$\mathrm{M}(A) \subseteq \mathrm{IGDer}A.$$

We obtain the following

Theorem 2 *Let A be a Lie ring. If* $\mathrm{GDer}A/\mathrm{M}(A)$ *is a prime (respectively semiprime, simple or primary) Lie ring, then* $Z(A)$ *is a* G*-prime (respectively* G*-semiprime,* G*-simple or* G*-primary) ideal of* A.

Any unexplained terminology is standard as in [13–15,22,33].

2 Inner derivations

We first give some information about inner derivation rings.

Lemma 1 *Let A be a Lie ring. Then we have:*

(i) *if* Δ *is a subring of* $\mathrm{Der}A$*,* S *is a* Δ*-closed additive subgroup of* A *and* $\mathrm{ad}_S A = \{\mathrm{ad}_t \mid t \in S\}$*, then* $[\mathrm{ad}_S A, \Delta] \subseteq \mathrm{ad}_S A$ *(where* $\mathrm{ad}_A A = \mathrm{ad}A$*),*
(ii) *if* K *is an ideal of* A*, then* $\mathrm{ad}_K A$ *is an ideal of* $\mathrm{ad}A$*,*
(iii) *if* Φ *is an ideal of* $\mathrm{Der}A$*, then*

$$\nabla_\Phi = \{x \in A \mid \mathrm{ad}_x \in \Phi\}$$

is a D*-ideal of* A*,*
(iv) *if* Φ *is an ideal of* $\mathrm{ad}A$*, then* ∇_Φ *is an ideal of* A*,*

Proof (i) If $s \in S$ and $\delta \in \Delta$, then $\delta(s) \in S$ and so

$$[\delta, \mathrm{ad}_s] = \mathrm{ad}_{\delta(s)} \in \mathrm{ad}_S A.$$

(ii) Since K is an $(\mathrm{ad}A)$-closed additive subgroup of A, the result follows from part (i).
(iii) If $x, y \in \nabla_\Phi$, $a \in A$ and $\delta \in D$, then $\mathrm{ad}_{x-y} = \mathrm{ad}_x - \mathrm{ad}_y$, $\mathrm{ad}_{[a,x]} = [\mathrm{ad}_a, \mathrm{ad}_x]$, $\mathrm{ad}_{\delta(x)} = [\delta, \mathrm{ad}_x] \in \Phi$ and consequently $x - y$, $[a, x]$, $\delta(x) \in \nabla_\Phi$.
(iv) By the same argument as in part (iii).

□

Lemma 2 *Let A be a Lie ring and Δ a subring of* $\mathrm{Der}A$*. Then we have:*

(i) *if* B *is a* Δ*-ideal of* A*, then* $\{\delta \in \Delta \mid \delta(A) \subseteq B\}$ *is an ideal of* Δ*,*
(ii) *if* B *is an ideal of* A*, then* $\{\delta \in \mathrm{ad}A \mid \delta(A) \subseteq B\}$ *is an ideal of* $\mathrm{ad}A$*,*
(iii) *if* Φ *is an ideal of* $\mathrm{ad}A$*, then* $\{a \in A \mid \delta(a) = 0 \text{ for all } \delta \in \Phi\}$ *is an ideal of* A*,*
(iv) *if* B *is a* Δ*-ideal of* A*, then* $\{x \in A \mid \mathrm{ad}_x(A) \subseteq B\}$ *is a* Δ*-ideal of* A*,*
(v) *if* B *is a* Δ*-ideal of* A*, then the centralizer* $C_A(B) = \{x \in A \mid \mathrm{ad}_x(B) = 0\}$ *of* B *in* A *is a* Δ*-ideal of* A*,*
(vi) [19, Proposition 2.2] *there exists the Lie ring isomorphism*

$$\mathrm{ad}A \ni \mathrm{ad}_a \mapsto a + Z(A) \in A/Z(A),$$

(vii) *if* B *is a* Δ*-ideal of* A*, then the center* $Z(B)$ *is a* Δ*-ideal of* A*.*

Proof By routine calculations. □

Lemma 3 *Let A be a Δ-semiprime Lie ring, B its nonzero Δ-ideal, where $\emptyset \neq \Delta \subseteq D$. Then the following are true:*

(i) *A is nonabelian,*
(ii) *$Z(A) = 0$, i.e. A is centerless,*
(iii) *$B \cap C_A(B) = 0$,*
(iv) *$Z(B) = 0$,*
(v) *$C_A(A') = 0$,*
(vi) *if A is Δ-prime, then $C_A(B) = 0$.*

Proof (i)–(ii) Evident.
(iii) Inasmuch as $B \cap C_A(B)$ is a Δ-ideal of A in view of Lemma 3 (v) and

$$\left[B \bigcap C_A(B), B \bigcap C_A(B)\right] = 0,$$

we deduce that $B \cap C_A(B) = 0$.
(iv) In view of Lemma 2 (vii), $[A, Z(B)]$ is a Δ-ideal of A and $[A, Z(B)] \subseteq Z(B)$. However,

$$[[A, Z(B)], [A, Z(B)]] = 0$$

and therefore $[A, Z(B)] = 0$. Then we find that $Z(B) \subseteq Z(A) = 0$.
(v) It is easy to see that A' is nonzero,

$$[z, A] \subseteq A' \bigcap C_A(A') = 0$$

for any $z \in C_A(A')$ and so $z \in Z(A)$. Hence $z = 0$ by (ii).
(vi) It holds in view of Lemma 2 (v). □

Corollary 1 *Let A be a Lie ring. Then we have:*

(i) *if A is simple, then* $\mathrm{ad}A$ *is a simple Lie ring,*
(ii) *if* $\mathrm{ad}A$ *is simple, then $A/Z(A)$ is a simple Lie ring and*

$$A = A' + Z(A) \tag{2}$$

(and then A' is the smallest noncentral ideal of A),
(iii) *if A is a prime (respectively semiprime) Lie ring, then so is* $\mathrm{ad}A$,
(iv) *if $A/Z(A)$ is a primary Lie ring, then so is* $\mathrm{ad}A$,
(v) *if* $\mathrm{ad}A$ *is prime (respectively semiprime or primary), then so is $A/Z(A)$.*

Proof (i) and (iii)–(v) It follows in view of Lemma 2 (vi).
(ii) Obviously that A is nonabelian and therefore $A' \neq 0$. Lemma 2 (vi) implies that the quotient Lie ring $A/Z(A)$ is simple. Using the fact that

$$\mathrm{ad}_{A'} A = [\mathrm{ad}A, \mathrm{ad}A] = \mathrm{ad}A \tag{3}$$

and $A' \nsubseteq Z(A)$ we deduce that A satisfies Eq. (2). □

Corollary 2 *Let A be a semiprime Lie ring. Then* $\mathrm{ad}A$ *is simple if and only if so is A.*

Let p be a prime,

$$F(A) := \{a \in A \mid a \text{ is of finite order in the additive group } A^{+}\}$$

the torsion part and

$$F_p(A) := \{a \in F(A) \mid p^n a = 0 \text{ for some nonnegative integer } n\}$$

the torsion p-part of a Lie ring A.

Remark 1 If A is a Δ-prime Lie ring, then one of the following holds:

(i) $F(A) = 0$,
(ii) $pA = 0$ for some prime p.

Indeed, if $F(A)$ is nonzero, then $F_p(A) \neq 0$ for some prime p. From $pA \neq 0$ it follows that

$$\Omega_1 := \{a \in F_p(A) \mid pa = 0\} \neq 0$$

and $[pA, \Omega_1] = 0$, a contradiction. Hence $pA = 0$.

Lemma 4 *Let A be a centerless Lie ring. If Φ is an ideal of* $\mathrm{Der}A$*, then*

$$\Phi \bigcap \mathrm{ad}A = 0 \Leftrightarrow \Phi = 0.$$

Proof In fact, if $\Phi \cap \mathrm{ad}A = 0$, then $0 = [d, \mathrm{ad}_a] = \mathrm{ad}_{d(a)}$ and therefore $d(a) \in Z(A)$ for any $d \in \Phi$ and $a \in A$. Consequently $d = 0$. □

Lemma 5 *If A is a D-simple Lie ring, then*

$$\mathrm{ad}A = [\mathrm{ad}A, \mathrm{ad}A] \tag{4}$$

is the smallest nonzero ideal of D (and so A is perfect).

Proof It is easy to see that A' is a nonzero D-ideal of A, $Z(A) = 0$ and therefore $A' = A$ by Corollary 1. Let Φ be a nonzero ideal of D. By Lemma 4,

$$\Phi_1 := \Phi \bigcap \mathrm{ad}A \neq 0.$$

Then $\nabla_{\Phi_1} \neq 0$ is a D-ideal of A by Lemma 1(iii) and consequently $\nabla_{\Phi_1} = A$. Then $\mathrm{ad}A \subseteq \Phi$, Eqs. (4) and (3) are true and $\mathrm{ad}A$ is the smallest nonzero ideal of D. □

Corollary 3 *Let A be a Lie ring. Then the following hold:*

(1) $[\mathrm{ad}A, \mathrm{ad}A]$ *is a simple (respectively semiprime, prime or primary) Lie ring if and only if so is*

$$A' \Big/ \left(A' \bigcap Z(A) \right),$$

(2) *if* A *is semiprime and* $[\mathrm{ad}A, \mathrm{ad}A]$ *is a simple Lie ring, then* A' *is the smallest nonzero ideal of* A.

Proof (1) If $a, b \in A$, then the rule

$$[\mathrm{ad}A, \mathrm{ad}A] \ni [\mathrm{ad}_a, \mathrm{ad}_b] \mapsto [a, b] + \left(A' \bigcap Z(A) \right) \in A' \Big/ \left(A' \bigcap Z(A) \right)$$

induces a Lie ring isomorphism.

(2) Let I be a nonzero ideal of A. Then $0 \neq [I, I] \subseteq A'$. Since $\mathrm{ad}_I A$ is a nonzero ideal of the Lie ring $\mathrm{ad}A$, we deduce that

$$\mathrm{ad}_{[I,I]} A = [\mathrm{ad}_I A, \mathrm{ad}_I A] = [\mathrm{ad}A, \mathrm{ad}A] = \mathrm{ad}_{A'} A.$$

Moreover, $Z(A) = 0$ and therefore $A' = [I, I] \subseteq I$. Hence A' is the smallest nonzero ideal of A.

□

3 Lie *FC*-rings

Proposition 2 is analogous with the Lewin result [18, Lemma 1].

Proof of Proposition 2. Since every Lie ring is a Leibniz ring, Proposition 2 follows from [2, Proposition 5.2]. We prove it here in order to have the paper more self-contained. Suppose that $|A : S| = n$ for some $n \in \mathbf{N}$ and the quotient group

$$A^+/S^+ = \{a_1 + S^+, \ldots, a_n + S^+\}$$

for some elements $a_1, \ldots, a_n \in A$. Let $s \in S$. The rule

$$g_s : A^+/S^+ \ni a + S^+ \mapsto [a, s] + S^+ \in A^+/S^+$$

determines an endomorphism g_s of the additive group A^+/S^+. Since A^+/S^+ is finite, its endomorphism ring $\mathrm{End}(A^+/S^+)$ is the ones. Then the group homomorphism

$$g : S^+ \ni s \mapsto g_s \in \mathrm{End}(A^+/S^+)$$

has the kernel $K_g := \{s \in S \mid [A, s] \subseteq S\}$ of finite index in S. The rule

$$\varphi_{(i_1,\ldots,i_k)} : K_g \ni w \mapsto \varphi_{(w,i_1,\ldots,i_k)} \in \mathrm{End}(A^+/S^+),$$

where $k \in \mathbf{N}$, $(i_1, \ldots, i_k) \in \mathbf{N}^k$ and

$$\varphi_{(w,i_1,\ldots,i_k)} : A^+/S^+ \ni r + S^+ \mapsto [[\ldots[[w, a_{j_{i_1}}], a_{j_{i_2}}], \ldots, a_{j_{i_k}}], r] + S^+ \in A^+/S^+$$

is an endomorphism of A^+/S^+, determines a group homomorphism. Then the set

$$\{\varphi_{(w,i_1,\ldots,i_k)} \mid w \in K_g,\ k \in \mathbf{N} \text{ and } (i_1, \ldots, i_k) \in \mathbf{N}^k\}$$

is finite, every kernel Ker $\varphi_{(w,i_1,\ldots,i_k)}$ is of finite index in K_g and therefore we deduce that

$$I := \bigcap_{\substack{w \in K_g \\ (i_1,\ldots,i_k) \in \mathbf{N}^k}} \operatorname{Ker} \varphi_{(w,i_1,\ldots,i_k)}$$

is of finite index in K_g (and consequently in A). Moreover, $I \leq S$ and

$$[I, \underbrace{A, \ldots, A}_{k \text{ times}}] \leq S$$

for any $k \in \mathbf{N}$. Hence

$$I_0 := I + \sum_{k=1}^{\infty} [I, \underbrace{A, \ldots, A}_{k \text{ times}}]$$

is an ideal of finite index in A such that $I_0 \leq S$. □

Corollary 4 *If A is a Lie FC-ring, then, for every $a \in A$, there exists an ideal I_a of finite index in A such that $[a, I_a] = 0$.*

The next proposition is an analogue of [2, Theorem 5.2].

Proposition 3 *Let A be a Lie ring. Then the set* IDer$A$ *is finite if and only if $|A : Z(A)| < \infty$.*

Proof (⇒) Suppose that IDer$A = \{\mathrm{ad}_{u_i} \mid i = 1, \ldots, m\}$ for some $m \in \mathbf{N}$ and $u_1, \ldots, u_m \in A$. If $x \in A$, then there exists $s = s(x) \in \mathbf{N}$ such that $1 \leq s \leq m$ and $\mathrm{ad}_x = \mathrm{ad}_{u_s}$. Hence $x \in u_s + Z(A)$ and $A/Z(A) = \{u_i + Z(A) \mid i = 1, \ldots, m\}$ is finite.

(⇐) Since

$$A/Z(A) = \{a_1 + Z(A), \ldots, a_n + Z(A)\} \tag{5}$$

for some $n \in \mathbf{N}$ and $a_1, \ldots, a_n \in A$ and, for every $x \in A$, there exists $i = i(x)$ $(1 \leq i \leq n)$ such that $x \in a_i + Z(A)$, we see that $\mathrm{ad}_x = \mathrm{ad}_{a_i}$. Thus

$$\mathrm{IDer}A = \{\mathrm{ad}_x \mid x \in A\} = \{\mathrm{ad}_{a_i} \mid i = 1, \ldots, n\}$$

is finite. □

Corollary 5 *Let A be a Lie ring. Then the following hold:*

(1) *if* $|A : Z(A)| < \infty$*, then the commutator ideal* A' *is finite,*
(2) *if* IDer$A$ *is finite, then the commutator ideal* A' *is finite.*

Proof For a proof, see [2, Lemma 5.12]. □

Lemma 6 *Let A be a Lie ring. Then the following hold:*

(1) *if* A' *is finite, then A is FC,*
(2) *if* IDer$A$ *is finite, then A is FC.*

Proof (1) Let $a \in A$. Since ad_a is an endomorphism of A^+ and $Z(A) \leq C_A(x) = \mathrm{Ker}\ \mathrm{ad}_a$, we conclude that

$$A/C_A(a) = A/\mathrm{Ker}\ \mathrm{ad}_a \cong [A, a] \leq A'$$

is finite for any $a \in A$. Hence A is FC.

(2) follows immediately from part (1). □

Lemma 7 *If A is a finitely generated Lie FC-ring, then its commutator ideal* A' *is finite.*

Proof Suppose that A is generated by some elements $x_1, \ldots, x_n \in A$. Inasmuch as $|A : C_A(x_i)| < \infty$ for $i = 1, \ldots, n$ and

$$Z(A) = \bigcap_{i=1}^{n} C_A(x_i),$$

we have that $Z(A)$ is of finite index in A and, by Corollary 5, A' is finite. □

Lemma 8 *If F is a finite ideal of a Lie ring A, then* $|A : C_A(F)| < \infty$.

Proof Suppose that $F = \{x_1, \ldots, x_n\}$. Then

$$[x_i, A] \cong A/C_A(x_i)$$

for any $x_i \in F$ what implies that

$$\bigcap_{i=1}^{n} C_A(x_i) \leq C_A(F)$$

and the result follows. □

Recall that M is *a minimal ideal* of a Lie ring A if $M \neq 0$ and, for any ideal I of A, the implication

$$0 \leq I \leq M \Rightarrow I = 0 \text{ or } I = M$$

holds. If M is a minimal nonzero ideal of A, then $[M, M] = M$ (i.e. M is perfect) or $[M, M] = 0$.

Lemma 9 *If M is a perfect minimal ideal of a Lie ring A, then the quotient Lie ring $A/C_A(M)$ is prime.*

Proof If B, C are ideals of A such that $[B, M] \neq 0$ and $[C, M] \neq 0$, then $[B, M] = M = [C, M]$ and $M \leq [B, C]$. This yields that $[[B, C], M] \neq 0$ and so $A/C_A(M)$ is prime. □

Lemma 10 *If F is a finite ideal of a Lie ring A, then the following hold:*

(1) *if F is a perfect minimal ideal, then $A = F \oplus C_A(F)$ is a direct sum of ideals,*
(2) *if F does not contain nonzero nonabelian ideal of A, then there exist perfect minimal ideals $B_1, \ldots, B_k$ of A such that $B_i \leq F$ $(i = 1, \ldots k)$, $A = B_1 \oplus \cdots \oplus B_k \oplus C$ is a direct sum of ideals and $F \cap C$ is solvable,*
(3) *F contains a solvable ideal S of A such that $F = F' + S$ and the quotient Lie ring $A/S = (F/S) \oplus K$ is a direct sum for some its ideal K.*

Proof (1) By Lemmas 9 and 8, $K := A/C_A(F)$ is a finite prime (and therefore simple) Lie ring. Since K is perfect and $F \nsubseteq C_A(F)$, we deduce that

$$(F + C_A(F))/C_A(F) = A/C_A(F)$$

and the result follows.

(2) It is easy to see that F contains a minimal ideal B_1 of A and B_1 is nonabelian. By part (1), $A = B_1 \oplus C_A(B_1)$ is a direct sum of ideals. Since F is finite, we obtain the assertion by finite number of steps.
(3) Suppose that S is an ideal generated by all solvable ideals of A that are contained in F. Then F/S is a finite semiprime Lie ring (and consequently it is a direct sum of finitely many nonabelian minimal ideals of A/S) in view of part (1). This gives that $F = F' + S$. The rest it follows from part (2).

□

Corollary 6 *Let A be a Lie ring. If* IDer$A$ *is finite, then the commutator ideal A' is finite and there exists a solvable ideal S of A such that $S \leq A'$, $A' = A'' + S$ and $A/S = (A'/S) \oplus K$ is a direct sum of ideals for some abelian ideal K.*

Proof By Proposition 3 and Corollary 5, A' is finite and so the result holds by Lemma 10. □

4 Generalized derivations

Let

$$\mathrm{CDer} A := \{h \in \mathrm{Der} A \mid h(A) \subseteq Z(A)\}$$

be the set of all *central derivations* of A. The structural properties of a Lie algebra L with central inner derivations (i.e. $\mathrm{ad} L \subseteq \mathrm{CDer} L$) was studied by Tôgô [30].

Lemma 11 *Let A be a Lie ring. Then:*

(i) $\mathrm{GDer}A$ *is a Lie ring,*
(ii) $F(Z(A)) \subseteq Z(A)$ *for any* $F \in \mathrm{GDer}A$*,*
(iii) $\mathrm{CDer}A$ *is an ideal of* $\mathrm{GDer}A$*,*
(iv) $\mathrm{GDer}A = \mathrm{M}(A) + \mathrm{Der}A$*, where* $\mathrm{M}(A)$ *is an ideal of* $\mathrm{GDer}A$*, and*

$$\mathrm{M}(A) \bigcap \mathrm{Der}A \subseteq \mathrm{CDer}A,$$

(v) $\mathrm{IGDer}A = \mathrm{M}(A) + \mathrm{ad}A$*, where* $\mathrm{M}(A)$ *is an ideal of* $\mathrm{IGDer}A$*, and*

$$\mathrm{M}(A) \bigcap \mathrm{ad}A \subseteq \mathrm{CDer}A,$$

(vi) *if B is a D-closed ideal of A, then*

$$\mathrm{I_B GDer}A := \{F \in \mathrm{GDer}A \mid F \text{ is associated with some } \mathrm{ad}_a, \text{ where } a \in B\}$$

(in particular, $\mathrm{M}(A) = \mathrm{I_O GDer}A = \mathrm{I_{Z(A)} GDer}A \subseteq \mathrm{IGDer}A := \mathrm{I_A GDer}A$*) is an ideal of* $\mathrm{GDer}A$*,*
(vii) $C(A) := \{k \in \mathrm{M}(A) \mid k(A) \subseteq Z(A)\}$ *is an ideal of* $\mathrm{GDer}A$*,*
(viii) *if* $(F, \delta), (F, \mu) \in \mathrm{GDer}A$*, then* $\delta + \mathrm{CDer}A = \mu + \mathrm{CDer}A$*,*
(ix) *if* $(F, \mathrm{ad}_a), (F, \mathrm{ad}_b) \in \mathrm{GDer}A$ *for some* $a, b \in A$*, then* $[a - b, A] \subseteq Z(A)$*.*

Proof Assume that $(F, \delta), (H, d) \in G$, $T \in \mathrm{M}(A)$, $h \in \mathrm{CDer}A$ and $x, y \in A$.

(i) We see that $(F - H, \delta - d) \in G$,

$$\begin{aligned}[F, H]([x, y]) &= F([H(x), y] + [x, d(y)]) - H([F(x), y] + [x, \delta(y)]) \\ &= [[F, H](x), y] + [x, [\delta, d](y)]\end{aligned}$$

and so $([F, H], [\delta, d]) \in G$.
(ii) Evident.
(iii) Since $h(A) \subseteq Z(A)$ for $h \in \mathrm{CDer}A$, we have that $[F, h](A) \subseteq Z(A)$, i.e. $[F, h] \in \mathrm{CDer}A$.
(iv) The equality

$$[F, T]([x, y]) = [[F, T](x), y]$$

implies that $[F, T] \in \mathrm{M}(A)$ and so $\mathrm{M}(A)$ is an ideal of G. Moreover,

$$(\delta - F)([x, y]) = [\delta(x), y] + [x, \delta(y)] - [F(x), y] - [x, \delta(y)] = [(\delta - F)(x), y]$$

and thus $\delta - F \in \mathrm{M}(A)$. If $h \in D \cap \mathrm{M}(A)$, then

$$[h(x), y] = h([x, y]) = [h(x), y] + [x, h(y)].$$

From this it follows $[x, h(y)] = 0$ and therefore $h(A) \subseteq Z(A)$.
(v) By the same argument as in (iv).

(vi) If $(K, \mathrm{ad}_a), (M, \mathrm{ad}_b) \in \mathrm{I_B GDer}A$, then $(K - M, \mathrm{ad}_{a-b}) \in \mathrm{I_B GDer}A$ and

$$[F, K]([x, y]) = F([K(x), y] + [x, \mathrm{ad}_a(y)]) - K([F(x), y] + [x, \delta(y)]) = \\ = [[F, K](x), y] + [x, \mathrm{ad}_{\delta(a)}(y)] \quad (6)$$

that is $([F, K], \mathrm{ad}_{\delta(a)}) \in \mathrm{I_B GDer}A$.

(vii) If $k \in C(A)$, then

$$[F, k]([x, y]) = [[F, k](x), y] = 0$$

and consequently $[F, k] \in C(A)$.

(viii)–(ix) If $(F, \delta), (F, \mu) \in G$ for some $\delta, \mu \in D$, then

$$[x, \delta(y)] = [x, \mu(y)]$$

and therefore $[x, (\delta - \mu)(y)] = 0$. This means that $(\delta - \mu)(A) \subseteq Z(A)$ and the result follows.

□

Corollary 7 *Let A be a Lie ring. Then the following hold:*

(1) *if* $Z(A) = 0$, *then*

$$\mathrm{GDer}A = \mathrm{M}(A) + \mathrm{Der}A,\ \ \mathrm{IGDer}A = \mathrm{M}(A) + \mathrm{ad}A \text{ and } \mathrm{M}(A) \bigcap \mathrm{Der}A = 0,$$

(2) *if A is a simple (respectively semiprime or prime) ring, then the Lie rings* $\mathrm{GDer}A/\mathrm{M}(A)$ *and* $\mathrm{Der}A$ *are isomorphic.*

Proof (1) If $Z(A) = 0$, then $\mathrm{CDer}A = 0$ and the result holds by Lemma 11 (iv) and (v).

(2) Since $Z(A)$ is an ideal of A, we deduce that $Z(A) = 0$. The rest follows in view of part (1).

□

Let $\Phi \subseteq \mathrm{GDer}A$, $\Gamma \subseteq \mathrm{Der}A$,

$$T_\Phi = \{d \in \mathrm{Der}A \mid \text{ there is } H \in \Phi \text{ that is associated with } d \in \mathrm{Der}A\},$$
$$U_\Gamma = \{H \in \mathrm{GDer}A \mid H \text{ is associated with some } d \in \Gamma\}$$

and

$$\Sigma_\Phi = \{a \in A \mid \text{ there exists } H \in \Phi \text{ that is associated with } \mathrm{ad}_a\}.$$

Lemma 12 *Let A be a Lie ring. Then the following hold:*

(i) *if* Φ *is an ideal of* $\mathrm{GDer}A$, *then* T_Φ *is an ideal of* $\mathrm{Der}A$,

(ii) *if* Γ *is an ideal of* $\mathrm{Der}A$, U_Γ *is a nonzero ideal of* $\mathrm{GDer}A$ *(in particular,* $U_0 = \mathrm{M}(A)$*),*
(iii) *if* Φ *is an ideal of* $\mathrm{IGDer}A$ *(respectively* $\mathrm{GDer}A$*), then* Σ_Φ *is an ideal (respectively a D-ideal) of* A.

Proof For a proof, see [2, Lemma 5.7]. □

Lie algebras L with abelian derivation algebras $\mathrm{Der}L$ was studied, in particular, in [29].

Lemma 13 *Let* A *be a Lie ring and* $(F, d) \in \mathrm{GDer}A$. *Then we have:*

(i) *if* $F = 0$*, then* $d(A) \subseteq Z(A)$*,*
(ii) *if* $d(A) \subseteq Z(A)$*, then* $F \in \mathrm{M}(A)$*,*
(iii) *if* $\mathrm{GDer}A$ *is an abelian Lie ring, then* $\mathrm{Der}A$ *is abelian,*
(iv) *if* $A \neq 0$*, then* $\mathrm{M}(A) \neq 0$.

Proof For a proof, see [2, Lemma 5.4]. □

Lemma 14 *Let* A *be a Lie ring and* $(M, \mathrm{ad}_a) \in \mathrm{IGDer}A$. *Then the following hold:*

(i) *if* $M = 0$*, then* $[a, A] \subseteq Z(A)$*,*
(ii) *if* $[a, A] \subseteq Z(A)$*, then* $M \in \mathrm{M}(A)$*,*
(iii) *if* $\mathrm{IGDer}A$ *is an abelian Lie ring, then* $\mathrm{ad}A$ *is abelian,*
(iv) *if* A *is abelian, then* $\mathrm{IGDer}A = \mathrm{M}(A)$.

Proof For a proof, see [2, Lemmas 5.4 and 5.5]. □

Lemma 15 *Let* A *be a Lie ring,* B *its ideal. Then:*

(i) *if* Φ *is an ideal of* $\mathrm{GDer}A$*, then* $\Phi \cap \mathrm{IGDer}A = 0$ *implies that* $[\delta(A), A] \subseteq Z(A)$ *for any* $\delta \in T_\Phi$*,*
(ii) *the following conditions are equivalent:*
  (a) $\mathrm{I_B GDer}A \subseteq \mathrm{M}(A)$*,*
  (b) $B \subseteq Z(A)$*,*
  (c) $\mathrm{ad}_B A = 0$*,*
(iii) *there exist Lie ring isomorphisms:*
  (d)

$$\mathrm{Der}A/\mathrm{CDer}A \ni \delta + \mathrm{CDer}A \mapsto \delta + \mathrm{M}(A) \in \mathrm{GDer}A/\mathrm{M}(A),$$

  (e)

$$\mathrm{ad}A\Big/\left(\mathrm{ad}A \bigcap \mathrm{CDer}A\right) \ni \mathrm{ad}_a + \left(\mathrm{ad}A \bigcap \mathrm{CDer}A\right) \mapsto \mathrm{ad}_a + \mathrm{M}(A) \in \mathrm{IGDer}A/\mathrm{M}(A).$$

Proof (i) If $(F, \delta) \in \Phi$ and $(H, \mathrm{ad}_a) \in \mathrm{IGDer}A$, then $([F, H], \mathrm{ad}_{\delta(a)}) \in \mathrm{IGDer}A$ and so $\mathrm{ad}_{\delta(a)} \in \mathrm{CDer}A$ by Lemma 14 (i).

(ii)–(iii) are evident. □

Corollary 8 *Let A be a Lie ring. If* $\mathrm{ad}A$ *is a semiprime (respectively prime or simple) Lie ring, then* $\mathrm{IGDer}A/\mathrm{M}(A)$ *is a semiprime (respectively prime, or simple) Lie ring.*

Proof If $a \in A$ and $\mathrm{ad}_b \in \mathrm{CDer}A \cap \mathrm{ad}A$, then $[a, b] \in Z(A)$ and $[\mathrm{ad}_a, \mathrm{ad}_b] = \mathrm{ad}_{[a,b]} = 0$. Then $\mathrm{CDer}A \cap \mathrm{ad}A = 0$ because $\mathrm{ad}A$ is semiprime (respectively prime or simple) and so $\mathrm{ad}A$ is isomorphic to $\mathrm{IGDer}A/\mathrm{M}(A)$ by Lemma 15 (iii). □

Lemma 16 *Let A be a nonnilpotent Lie ring. If A is primary, then the quotient Lie ring so is* $\mathrm{IGDer}A/\mathrm{M}(A)$.

Proof Assume that Φ, Λ are ideals of $\mathrm{IGDer}A$ such that $[\Phi, \Lambda] = 0$. By Lemma 14 (i), $[[\Sigma_\Phi, \Sigma_\Lambda], A] \subseteq Z(A)$. Since A is nonnilpotent primary (and therefore $Z(A) = 0$), we deduce that $[\Sigma_\Phi, \Sigma_\Lambda] = 0$. This implies that $\Sigma_\Phi = 0$ (and then $\Phi \subseteq \mathrm{M}(A)$) or

$$[\underbrace{\Sigma_\Lambda, \ldots, \Sigma_\Lambda}_{m \text{ times}}] = 0$$

(and consequently

$$[\underbrace{\Lambda, \ldots, \Lambda}_{m \text{ times}}] \subseteq \mathrm{M}(A)$$

for some positive integer m. Hence $\mathrm{IGDer}A/\mathrm{M}(A)$ is a primary Lie ring. □

5 Proofs

Proof of Proposition 1. (1) Let D be a simple Lie ring. Then D and A are nonzero. Since $\mathrm{ad}A$ is an ideal of D, we deduce that $\mathrm{ad}A = 0$ (and then A is abelian) or $\mathrm{ad}A = D$. Assume that $\mathrm{ad}A = D$ and K is arbitrary noncentral ideal of A. Then

$$0 \neq \mathrm{ad}_K A = \mathrm{ad}A$$

by Lemma 1 (ii) and so $A = K + Z(A)$. This means that $\overline{A} := A/Z(A) \cong \mathrm{ad}A$ is a simple Lie ring. Then it is nonabelian and therefore $\overline{A}' \neq \overline{0}$. Consequently $\overline{A}' = \overline{A}$ and $A = A'$ is simple or Eq. (2) follows.

(2) Let A be a D-simple Lie ring. Then $0 \neq A' = A$ and $Z(A) = 0$. By the same argument, as in the proof of Theorem 1.1 (i) from [27], $D = \mathrm{ad}A$ is complete. □

Proof of Theorem 1. (a) Let A be a D-prime Lie ring. Then $Z(A) = 0$. Assume that Φ, Λ are nonzero ideals of D such that $[\Phi, \Lambda] = 0$. By Lemma 4,

$$\Phi_1 := \Phi \bigcap \mathrm{ad}A \neq 0 \text{ and } \Lambda_1 := \Lambda \bigcap \mathrm{ad}A \neq 0$$

and ∇_{Φ_1}, ∇_{Λ_1} are nonzero. Since

$$\mathrm{ad}_{[\nabla_{\Phi_1}, \nabla_{\Lambda_1}]}A = [\mathrm{ad}_{\nabla_{\Phi_1}} A, \mathrm{ad}_{\nabla_{\Lambda_1}} A] = [\Phi_1, \Lambda_1] = 0,$$

we see that

$$[\nabla_{\Phi_1}, \nabla_{A_1}] \subseteq Z(A) = 0.$$

By Lemma 1 (iii), ∇_{Φ_1} and ∇_{A_1} are D-ideals of A and we obtain a contradiction. Hence D is prime.

(b) If A is a D-semiprime Lie ring, then we can obtain that D is semiprime by the same argument as in part (a). □

Proof of Theorem 2. (a) Assume that $G/\mathrm{M}(A)$ is a prime Lie ring and B, C are G-ideals of A such that

$$[B, C] \subseteq Z(A). \tag{7}$$

Then $\mathrm{I_B GDer} A$, $\mathrm{I_C GDer} A$ are ideals of G by Lemma 11 (vi) and

$$[\mathrm{I_B GDer} A, \mathrm{I_C GDer} A] \subseteq \mathrm{M}(A) \tag{8}$$

in view of Eq. (6). Then, by the primeness of $G/\mathrm{M}(A)$, $\mathrm{I_B GDer} A \subseteq \mathrm{M}(A)$ or $\mathrm{I_C GDer} A \subseteq \mathrm{M}(A)$ what implies that $B \subseteq Z(A)$ or $C \subseteq Z(A)$ by Lemma 15 (iii), and hence $Z(A)$ is G-prime.

(b) If $G/\mathrm{M}(A)$ is a semiprime Lie ring, then we can prove by the same argument as in case (a).

(c) Assume that $G/\mathrm{M}(A)$ is a simple Lie ring and B is a G-ideal of A. Then $\mathrm{I_B GDer} A$ is an ideal of G and consequently

$$\mathrm{I_B GDer} A \subseteq \mathrm{M}(A)$$

(and so $B \subseteq Z(A)$ by Lemma 15 (ii)) or

$$G/\mathrm{M}(A) = \mathrm{I_B GDer} A/\mathrm{M}(A).$$

In the second case we have $\mathrm{M}(A) \neq \mathrm{I_B GDer} A = \mathrm{IGDer} A = \mathrm{GDer} A$. Then $\mathrm{ad}_B A = \mathrm{ad} A$ what gives that $A = B + Z(A)$. This means that $Z(A)$ is G-simple.

(d) Let $G/\mathrm{M}(A)$ be a primary Lie ring and B, C be G-ideals of A such that Eq. (7) is true. Then Eq. (8) is true (and so $\mathrm{I_B GDer} A \subseteq \mathrm{M}(A)$) or

$$[\underbrace{\mathrm{I_C GDer} A, \ldots, \mathrm{I_C GDer} A}_{m \text{ times}}] \subseteq \mathrm{M}(A)$$

for some positive integer m). Then $B \subseteq Z(A)$ or

$$[\underbrace{C, \ldots, C}_{m \text{ times}}] \subseteq Z(A)$$

and consequently $Z(A)$ is a G-primary ideal of A. □

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