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Rota–Baxter Operators on Pre-Lie Superalgebras

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Abstract In this paper, we study Rota–Baxter operators and super $\mathcal{O}$-operator of associative superalgebras, Lie superalgebras, pre-Lie superalgebras and L-dendriform superalgebras. Then we give some properties of pre-Lie superalgebras constructed from associative superalgebras, Lie superalgebras and L-dendriform superalgebras. Moreover, we provide all Rota–Baxter operators of weight zero on complex pre-Lie superalgebras of dimensions 2 and 3.

Keywords Rota–Baxter operator · Super $\mathcal{O}$-operator · Associative superalgebra · Lie superalgebra · Pre-Lie superalgebra · L-dendriform superalgebras

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Introduction

Rota–Baxter operators of weight $\lambda \in \mathbb{K}$ fulfil the so-called Rota–Baxter relation which may be regarded as one possible generalization of the standard shuffle relation [36,51]. They appeared for the first time in the work of the mathematician Baxter [7] in 1960 and were then intensively studied by Atkinson [6], Miller [47], Rota [50], Cartier [18], and more recently, they reappeared in the work of Guo [37] and Ebrahimi-Fard [27].

Pre-Lie algebras (called also left-symmetric algebras, Vinberg algebras, quasi-associative algebras) are a class of a natural algebraic systems appearing in many fields in mathematics and mathematical physics. They were first mentioned by Cayley in 1890 [20] as a kind of rooted tree algebra and later arose again from the study of convex homogeneous cones [53], affine manifold and affine structures on Lie groups [40], and deformation of associative algebras [34]. They play an important role in the study of symplectic and complex structures on Lie groups and Lie algebras [5,22,24,25,44], phases spaces of Lie algebras [8,42], certain integrable systems [16], classical and quantum Yang–Baxter equations [26], combinatorics [27], quantum field theory [23] and operads [19]. See [17] for a survey. Recently, pre-Lie superalgebras, the $\mathbb{Z}_2$-graded version of pre-Lie algebras also appeared in many others fields; see, for example, [19,34,52]. To our knowledge, they were first introduced by Gerstenhaber in 1963 to study the cohomology structure of associative algebras [34]. They are a class of natural algebraic appearing in many fields in mathematics and mathematical physics, especially in supersymplectic geometry, vertex superalgebras and graded classical Yang–Baxter equation. Recently, classifications of complex pre-Lie superalgebras in dimensions two and three were given by Zhang and Bai [15]. See [3,21,38,39,55] about further results.

It turns out that the construction of pre-Lie superalgebras from associative superalgebras uses Rota–Baxter operators. Let $\mathcal{A}$ be an associative superalgebra (product of x and y is denoted by xy) and R be a Rota–Baxter operator of weight λ on $\mathcal{A}$, which means that it satisfies, for any homogeneous elements x, y in $\mathcal{A}$, the identity

$$R(x)R(y) = R\Big(R(x)y + xR(y) + \lambda xy\Big). \tag{0.1}$$

If $\lambda = 0$ (resp. $\lambda = -1$), the product

$$x \circ y = R(x)y - (-1)^{|x||y|} yR(x), \quad \forall\ x, y \in \mathcal{H}(\mathcal{A}) \tag{0.2}$$

resp.

$$x \circ y = R(x)y - (-1)^{|x||y|} yR(x) - xy, \quad \forall\ x, y \in \mathcal{H}(\mathcal{A}) \tag{0.3}$$

defines a pre-Lie superalgebra (see Theorem 1.2).

The notion of dendriform algebras was introduced in 1995 by Loday [45]. Dendriform algebras are algebras with two operations, which dichotomize the notion of associative algebras. The motivation came from algebraic $\mathbb{K}$-theory, and they have been studied quite extensively with connections to several areas in mathematics and physics, including operads, homology, Hopf algebras, Lie and Leibniz algebras, combinatorics, arithmetic and quantum field theory (see [30] and the references therein). The relationship between dendriform algebras, Rota–Baxter algebras and pre-Lie algebras was given by Aguiar and Ebrahimi-Fard [2,27,28]. Bai, Liu, Guo and Ni generalized the concept of Rota–Baxter operator and introduced a new class of algebras, namely L-dendriform algebras, in [12–14]. Moreover, a close relationship among associative superalgebras, Lie superalgebras, pre-Lie superalgebras and dendriform superalgebras is given as follows in the sense of commutative diagram of categories:

$$\begin{array}{ccc} \text{Lie superalgebra} & \longleftarrow & \text{pre-Lie superalgebra} \\ \uparrow & & \uparrow \\ \text{associative superalgebra} & \longleftarrow & \text{dendriform superalgebra} \end{array}$$

Recently, the notion of Rota–Baxter operator on a bimodule was introduced by Aguiar [1]. The construction of associative, Lie, pre-Lie and L-dendriform superalgebras is extended to the corresponding categories of bimodules. See [9,29,31–33,43,46] about further results and [10,11,41,48,49] about relationships with Yang–Baxter equation.

The main purpose of this paper is to study, through Rota–Baxter operators and $\mathcal{O}$-operators, the relationship between associative superalgebras, Lie superalgebras, pre-Lie superalgebras and L-dendriform superalgebras. Moreover, we classify Rota–Baxter operators of weight zero on the complex pre-Lie superalgebras of dimensions 2 and 3.

This paper is organized as follows. In Sect. 1, we recall some definitions of associative superalgebras, Lie superalgebras and pre-Lie superalgebras and we introduce the notion of super $\mathcal{O}$-operator of these superalgebras that generalizes the notion of Rota–Baxter operators. We show that every Rota–Baxter associative superalgebra of weight $\lambda = -1$ gives rise to a Rota–Baxter Lie superalgebra. Moreover, a super $\mathcal{O}$-operator on a Lie superalgebra (of weight zero) gives rise to a pre-Lie superalgebra. As an Example of computations, we provide all Rota–Baxter operators (of weight zero) on the orthosymplectic Lie superalgebra $osp(1, 2)$. In Sect. 2, we introduce the notion of L-dendriform superalgebra and then study some fundamental properties of L-dendriform superalgebras in terms of super $\mathcal{O}$-operator of pre-Lie superalgebras. Their relationship with associative superalgebras is also described. Sections 3 and 4 are devoted to classification of all Rota–Baxter operators (of weight zero) on the complex pre-Lie superalgebras of dimension 2 and 3 with one-dimensional even part and with two-dimensional even part, respectively.

Throughout this paper, all superalgebras are finite-dimensional and are over a field $\mathbb{K}$ of characteristic zero. Let $(\mathcal{A}, \circ)$ be a superalgebra, then $L_\circ$ and $R_\circ$ denote the even left and right multiplication operators $L_\circ, R_\circ : \mathcal{A} \to End(\mathcal{A})$ defined as $L_\circ(x)(y) = (-1)^{|x||y|} R_\circ(y)(x) = x \circ y$ for all homogeneous element x, y in $\mathcal{A}$. In particular, when $(\mathcal{A}, [\ ,\ ])$ is a Lie superalgebra, we let $ad(x)$ denote the adjoint operator, that is, $ad(x)(y) = [x, y]$ for all homogeneous element x, y in $\mathcal{A}$.

1 Rota–Baxter Associative Superalgebras, Pre-Lie Superalgebras and Lie Superalgebras

Let $(\mathcal{A}, \circ)$ be an algebra over a field $\mathbb{K}$. It is said to be a superalgebra if the underlying vector space of $\mathcal{A}$ is $\mathbb{Z}_2$-graded, that is, $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, and $\mathcal{A}_i \circ \mathcal{A}_j \subset \mathcal{A}_{i+j}$, for $i, j \in \mathbb{Z}_2$. An element of $\mathcal{A}_0$ is said to be even and an element of $\mathcal{A}_1$ is said to be odd. The elements of $\mathcal{A}_j, \ \ j \in \mathbb{Z}_2$, are said to be homogenous and of parity j. The parity of a homogeneous element x is denoted by $|x|$, and we refer to the set of homogeneous elements of $\mathcal{A}$ by $\mathcal{H}(\mathcal{A})$.

We extend to graded case the concepts of $\mathcal{A}$-bimodule $\mathbb{K}$-algebra, $\mathcal{O}$-operator and extended $\mathcal{O}$-operator introduced in [13].

Definition 1.1 (1) An associative superalgebra is a pair $(\mathcal{A}, \mu)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$ and an even bilinear map $\mu : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}, \ \ (\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}, \ \ \forall \ i, j \in \mathbb{Z}_2)$ satisfying for all $x, y, z \in \mathcal{H}(\mathcal{A})$

$$x(yz) = (xy)z.$$

(2) Let $(\mathcal{A}, \mu)$ be an associative superalgebra and V be a $\mathbb{Z}_2$-graded vector space. Let $l, r : \mathcal{A} \longrightarrow End(V)$ be two even linear maps. A triple (V, l, r) is called an $\mathcal{A}$-bimodule if for all $x, y \in \mathcal{H}(\mathcal{A})$ and $v \in \mathcal{H}(V)$

$$l(xy)(v) = l(x)l(y)(v), \quad r(xy)(v) = r(y)r(x)(v), \quad l(x)r(y)(v) = r(y)l(x)(v).$$

Moreover, the quadruple (V, μ_V, l, r) is said to be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if (V, l, r) is an $\mathcal{A}$-bimodule compatible with the multiplication μ_V on V, that is, for all $x, y \in \mathcal{H}(\mathcal{A})$ and $v, w \in \mathcal{H}(V)$,

$$l(x)(\mu_V(v, w)) = \mu_V(l(x)(v), w), \quad r(x)(\mu_V(v, w)) = \mu_V(v, r(x)(w)),$$
$$\mu_V(r(x)(v), w) = \mu_V(v, l(x)(w)).$$

(3) Fix $\lambda \in \mathbb{K}$, a pair (T, T') of even linear maps $T, T' : V \longrightarrow \mathcal{A}$ is called an extended super $\mathcal{O}$-operator with modification T' of weight λ associated with the bimodule (V, l, r) if T satisfies

$$\lambda l(T'(u))v = \lambda r(T'(v))u, \tag{1.1}$$

$$T(u)T(v) = T\Big(l(T(u))v + (-1)^{|u||v|} r(T(v))u\Big) + \lambda T'(u)T'(v), \ \ \forall \ u, v \in \mathcal{H}(V). \tag{1.2}$$

(4) An even linear map $T : V \longrightarrow \mathcal{A}$ is called a super $\mathcal{O}$-operator of weight λ associated with the bimodule $\mathbb{K}$-superalgebra (V, μ_V, l, r) if it satisfies

$$T(u)T(v) = T\Big(l(T(u))v + (-1)^{|u||v|}r(T(v))u + \lambda\mu_V(u,v)\Big),$$
$$\forall\ u, v \in \mathcal{H}(V). \tag{1.3}$$

□

Notice that the notions of super $\mathcal{O}$-operator and extended super $\mathcal{O}$-operator coincide when $\lambda = 0$.

In particular, a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with the bimodule $\mathbb{K}$-algebra $(\mathcal{A}, \mu_A, L_\mu, R_\mu)$ is called a Rota–Baxter operator of weight λ on $\mathcal{A}$, that is, R satisfies the identity (0.1). We denote by a triple $(\mathcal{A}, \mu, R)$ the Rota–Baxter associative superalgebra.

We define now Rota–Baxter operators on $\mathcal{A}$-bimodules.

Definition 1.2 Let $(\mathcal{A}, \mu, R)$ be a Rota–Baxter associative superalgebra of weight zero. A Rota–Baxter operator on an $\mathcal{A}$-bimodule V (relative to R) is a map $R_V : V \longrightarrow V$ such that for all $x \in \mathcal{H}(\mathcal{A})$ and $v \in \mathcal{H}(V)$

$$R(x)R_V(v) = R_V\Big(R(x)v + xR_V(v)\Big),$$
$$R_V(v)R(x) = R_V\Big(R_V(v)x + vR(x)\Big).$$

We have similar definitions on Lie superalgebras.

Definition 1.3 (1) A Lie superalgebra is a pair $(\mathcal{A}, [\ ,\ ])$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, and an even bilinear map $[\ ,\ ] : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $([\mathcal{A}_i, \mathcal{A}_j] \subseteq \mathcal{A}_{i+j},\ \forall\ i, j \in \mathbb{Z}_2)$ satisfying for all $x, y, z \in \mathcal{H}(\mathcal{A})$,

$$[x, y] = -(-1)^{|x||y|}[y, x], \quad \text{(super-skew-symmetry)} \tag{1.4}$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]], \text{ (super-Jacobi identity)}. \tag{1.5}$$

(2) Let $(\mathcal{A}, [\ ,\ ])$ be a Lie superalgebra, V be a $\mathbb{Z}_2$-graded vector space, and $\rho : \mathcal{A} \longrightarrow End(V)$ be an even linear map. The pair (V, ρ) is said to be an $\mathcal{A}$-module or a representation of $(\mathcal{A}, [\ ,\ ])$ if for all $x, y \in \mathcal{H}(\mathcal{A})$ and $v \in \mathcal{H}(V)$,

$$\rho([x, y])(v) = \rho(x)\rho(y)v - (-1)^{|x||y|}\rho(y)\rho(x)v. \tag{1.6}$$

The triple $(V, [\ ,\ ]_V, \rho)$, where $[\ ,\ ]_V$ is a super-skew-symmetric bracket, is said to be an $\mathcal{A}$-module $\mathbb{K}$-superalgebra if, for $x \in \mathcal{H}(\mathcal{A})$ and $v, w \in \mathcal{H}(V)$,

$$\rho(x)[v, w]_V = [\rho(x)(v), w]_V + (-1)^{|v||w|}[v, \rho(x)(w)]_V.$$

(3) Let $(\mathcal{A}, [\ ,\ ])$ be a Lie superalgebra and (V, ρ) be a representation of $\mathcal{A}$. An even linear map $T : V \longrightarrow \mathcal{A}$ is called a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with an $\mathcal{A}$-module $\mathbb{K}$-superalgebra $(V, [\ ,\ ]_V, \rho)$ if T satisfies:

$$[T(u), T(v)] = T\Big(\rho(T(u))v - (-1)^{|u||v|}\rho(T(v))u + \lambda[u, v]_V\Big),\ \ \forall\ u, v \in \mathcal{H}(V).$$

In particular, a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with the bimodule $(\mathcal{A}, L_\circ, R_\circ)$ is called a Rota–Baxter operator of weight $\lambda \in \mathbb{K}$ on $(\mathcal{A}, [\ ,\ ])$, that is, R satisfies for all x, y, z in $\mathcal{H}(\mathcal{A})$

$$[R(x), R(y)] = R\Big([R(x), y] - (-1)^{|x||y|}[R(y), x] + \lambda[x, y]\Big). \tag{1.7}$$

The triple $(\mathcal{A}, [\ ,\ ], R)$ refers to a Rota–Baxter Lie superalgebra, see [54].

Definition 1.4 Let $(\mathcal{A}, [\ ,\ ], R)$ and $(\mathcal{A}', [\ ,\ ]', R')$ be two Rota–Baxter Lie superalgebras. An even homomorphism $f : (\mathcal{A}, [\ ,\ ], R) \longrightarrow (\mathcal{A}', [\ ,\ ]', R')$ is said to be a morphism of two Rota–Baxter Lie superalgebras if, for all $x, y \in \mathcal{H}(\mathcal{A})$, $f([x, y]) = [f(x), f(y)]'$ and $f \circ R = R' \circ f$.

Proposition 1.1 *Let $(\mathcal{A}, \mu, R)$ be a Rota–Baxter associative superalgebra of weight $\lambda \in \mathbb{K}$. Then the triple $(\mathcal{A}, [\ ,\ ], R)$, where $[x, y] = xy - (-1)^{|x||y|}yx$, is a Rota–Baxter Lie superalgebra of weight $\lambda \in \mathbb{K}$.*

We introduce the notion of super $\mathcal{O}$-operators of pre-Lie superalgebras and study some properties over Lie superalgebras and pre-Lie superalgebras.

Definition 1.5 Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded vector space and $\circ : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ be an even binary operation. The pair $(\mathcal{A}, \circ)$ is called a pre-Lie superalgebra if, for x, y, z in $\mathcal{H}(\mathcal{A})$, the associator

$$as(x, y, z) = (x \circ y) \circ z - x \circ (y \circ z)$$

is super-symmetric in x and y, that is, $as(x, y, z) = (-1)^{|x||y|}as(y, x, z)$, or equivalently

$$(x \circ y) \circ z - x \circ (y \circ z) = (-1)^{|x||y|}\Big((y \circ x) \circ z - y \circ (x \circ z)\Big). \tag{1.8}$$

The identity (1.8) is called pre-Lie super-identity.

Definition 1.6 Let $(\mathcal{A}, \circ)$ be a pre-Lie superalgebra.

(1) Let V be a $\mathbb{Z}_2$-graded vector space and $l, r : \mathcal{A} \longrightarrow End(V)$ be two even linear maps. The triple (V, l, r) is said to be an $\mathcal{A}$-bimodule of $(\mathcal{A}, \circ)$ if, for $x, y \in \mathcal{H}(\mathcal{A})$ and $v \in \mathcal{H}(V)$,

$$l(x)l(y)v - l(x \circ y)v = (-1)^{|x||y|}\Big(l(y)l(x)v - l(y \circ x)v\Big), \tag{1.9}$$
$$l(x)r(y)v - r(y)l(x)v = (-1)^{|x||v|}(r(x \circ y)v - r(y)r(x)v). \tag{1.10}$$

Moreover, the quadruple $(V, \circ_V, l, r)$ is said to be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if (V, l, r) is an $\mathcal{A}$-bimodule compatible with the multiplication $\circ_V$ on V, that is, for $x, y \in \mathcal{H}(\mathcal{A})$ and $v, w \in \mathcal{H}(V)$,

$$l(x)(v \circ_V w) - l(x)(v) \circ_V w = (-1)^{|x||v|}(v \circ_V l(x)(w)) - r(x)(v) \circ_V w),$$
$$r(x)(v \circ_V w) - v \circ_V r(x)(w) = (-1)^{|v||w|}(r(x)(w \circ_V v) - w \circ_V r(x)(v)).$$

(2) Let $(V, \circ_V, l, r)$ be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra. An even linear map $T : V \longrightarrow \mathcal{A}$ is called a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with $(V, \circ_V, l, r)$ if it satisfies:

$$T(u) \circ T(v) = T\Big(l(T(u))v + (-1)^{|u||v|} r(T(v))u + \lambda u \circ_V v\Big), \quad \forall \ u, v \in \mathcal{H}(V). \tag{1.11}$$

In particular, a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with the $\mathcal{A}$-bimodule $(\mathcal{A}, L_\circ, R_\circ)$ is called a Rota–Baxter operator of weight λ on $(\mathcal{A}, \circ)$, that is, R satisfies

$$R(x) \circ R(y) = R\Big(R(x) \circ y + x \circ R(y) + \lambda x \circ y\Big) \tag{1.12}$$

for all x, y, z in $\mathcal{H}(\mathcal{A})$.

Proposition 1.2 *Let $(\mathcal{A}, \circ)$ be a pre-Lie superalgebra.*

(1) The commutator

$$[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$$

defines a Lie superalgebra $(\mathcal{A}, [\ ,\ ])$ which is called the sub-adjacent Lie superalgebra of $\mathcal{A}$ and $\mathcal{A}$ is also called a compatible pre-Lie superalgebra structure on the Lie superalgebra.

(2) The map $L_\circ$ gives a representation of the Lie superalgebra $(\mathcal{A}, [\ ,\ ])$, that is,

$$L_\circ([x, y]) = L_\circ(x)L_\circ(y) - (-1)^{|x||y|} L_\circ(y)L_\circ(x).$$

Corollary 1.1 *Let $(\mathcal{A}, \circ)$ be a pre-Lie superalgebra and (V, l, r) be an $\mathcal{A}$-bimodule. Let $(\mathcal{A}, [\ ,\ ])$ be the sub-adjacent Lie superalgebra. If T is a super $\mathcal{O}$-operator associated with (V, l, r), then T is a super $\mathcal{O}$-operator of $(\mathcal{A}, [\ ,\ ])$ associated with $(V, l - r, r - l)$.*

Now, we construct pre-Lie superalgebras using super $\mathcal{O}$-operators on Lie superalgebras.

Proposition 1.3 ([54]) *Let $(\mathcal{A}, [\ ,\ ])$ be a Lie superalgebra and (V, ρ) be a representation of $\mathcal{A}$. Suppose that $T : V \longrightarrow \mathcal{A}$ is a super $\mathcal{O}$-operator of weight zero associated with (V, ρ). Then, the even bilinear map*

$$u \circ v = \rho(T(u))v, \quad \forall \ u, v \in \mathcal{H}(V)$$

defines a pre-Lie superalgebra structure on $\mathcal{A}$.

Remark 1.1 ([54]) Let $(\mathcal{A}, [\ ,\ ])$ be a Lie superalgebra and R be the super $\mathcal{O}$-operator (of weight zero) associated with the adjoint representation $(\mathcal{A}, ad)$. Then the even binary operation given by $x \circ y = [R(x), y]$, for all $x, y \in \mathcal{H}(\mathcal{A})$, defines a pre-Lie superalgebra structure on $\mathcal{A}$.

As a direct consequence, since a Rota–Baxter operator on a pre-Lie superalgebra is also a Rota–Baxter operator of its sub-adjacent Lie superalgebra, we have the following observation.

Proposition 1.4 *Let $\mathcal{A}_1 = (\mathcal{A}, \circ, R)$ be a Rota–Baxter pre-Lie superalgebra of weight zero. Then $\mathcal{A}_2 = (\mathcal{A}, *, R)$ is a Rota–Baxter pre-Lie superalgebra of weight zero, where the even binary operation is defined by*

$$x * y = R(x) \circ y - (-1)^{|x||y|} y \circ R(x).$$

Example 1.1 In this example, we calculate Rota–Baxter operators of weight zero on the Lie superalgebra $\mathfrak{osp}(1, 2)$ and give the corresponding pre-Lie superalgebras. Starting from the orthosymplectic Lie superalgebra, we consider in the sequel the matrix realization of this superalgebra.

Let $\mathfrak{osp}(1, 2) = \mathcal{A}_0 \oplus \mathcal{A}_1$ be the Lie superalgebra where $\mathcal{A}_0$ is spanned by

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and $\mathcal{A}_1$ is spanned by

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The defining relations (we give only the ones with nonzero values in the right-hand side) are

$$\begin{aligned} &[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \\ &[e_3, e_5] = e_4, \quad [e_2, e_4] = e_5, \quad [e_1, e_4] = -e_4, \quad [e_1, e_5] = e_5, \\ &[e_5, e_4] = e_1, \quad [e_5, e_5] = -2e_2, \quad [e_4, e_4] = 2e_3. \end{aligned}$$

The Rota–Baxter operators of weight zero on the Lie superalgebra $\mathfrak{osp}(1, 2)$ with respect to the homogeneous basis $\{e_1, e_2, e_3, e_4, e_5\}$ are:

$R_1(e_1) = a_1e_1 + a_2e_2 - \frac{8a_1^2a_3}{(2a_3+a_2)^2}e_3,\ R_1(e_2) = -\frac{2a_2a_1^2}{(2a_3+a_2)^2}e_1 + \frac{(2a_3-3a_2)a_1}{2(2a_3+a_2)}e_2 + \frac{2a_1^3}{(2a_3+a_2)^2}e_3,\ R_1(e_3)=a_3e_1 + \frac{(2a_3+a_2)^2}{8a_1}e_2 + \frac{a_1(a_2-6a_3)}{2(2a_3+a_2)}e_3,\ R_1(e_4)=0,\ R_1(e_5)=0,\ a_1 \neq 0,\ a_2 \neq -2a_3.$

$R_2(e_1) = a_1e_1 + a_2e_2,\ R_2(e_2) = -\frac{2a_1^2}{a_2}e_1 - \frac{3a_1}{2}e_2 + \frac{2a_1^3}{a_2^2}e_3,\ R_2(e_3) = \frac{a_2^2}{8a_1}e_2 + \frac{a_1}{2}e_3,\ R_2(e_4) = 0,\ R_2(e_5) = 0,\ a_1 \neq 0,\ a_2 \neq 0.$

$R_3(e_1) = a_1e_1 - \frac{2a_1^2}{a_3}e_3,\ R_3(e_2) = \frac{a_1}{2}e_2 + \frac{a_1^3}{2a_3^2}e_3,\ R_3(e_3) = a_3e_1 + \frac{a_3^2}{2a_1}e_2 - \frac{3a_1}{2}e_3,\ R_3(e_4) = 0,\ R_3(e_5) = 0,\ a_1 \neq 0,\ a_3 \neq 0.$

$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_3e_1 + a_4e_2,\ R_4(e_4) = 0,\ R_4(e_5) = 0.$

$R_5(e_1) = a_1e_1 - 4a_3e_2 - \frac{2a_1^2}{a_3}e_3,\ R_5(e_2) = -\frac{a_1^2}{4a_3}e_1 + a_1e_2 + \frac{a_1^3}{2a_3^2}e_3,\ R_5(e_3) = a_3e_1 - \frac{4a_3^2}{a_1}e_2 - 2a_1e_3,\ R_5(e_4) = 0,\ R_5(e_5) = 0,\ a_1 \neq 0,\ a_3 \neq 0.$
$R_6(e_1) = 0,\ R_6(e_2) = 0,\ R_6(e_3) = a_3e_1,\ R_6(e_4) = 0,\ R_6(e_5) = 0.$
$R_7(e_1) = a_1e_1 + \frac{2a_1a_5}{a_6}e_2 + a_6e_3,\ R_7(e_2) = \frac{a_6}{2}e_1 + a_5e_2 + \frac{a_6^2}{2a_1}e_3,\ R_7(e_3) = \frac{a_1a_5}{a_6}e_1 + \frac{2a_1a_5^2}{a_6^2}e_2 + a_5e_3,\ R_7(e_4) = 0,\ R_7(e_5) = 0,\ a_1 \neq 0,\ a_6 \neq 0.$
$R_8(e_1)=a_1e_1 + a_6e_3,\ R_8(e_2) = \frac{a_6}{2}e_1 + \frac{a_6^2}{2a_1}e_3,\ R_8(e_3) = 0,\ R_8(e_4) = 0,\ R_8(e_5) = 0,\ a_1 \neq 0.$
$R_9(e_1) = a_1e_1 + a_2e_2,\ R_9(e_2) = 0,\ R_9(e_3) = \frac{a_2}{2}e_1 + \frac{a_2^2}{2a_1}e_2,\ R_9(e_4) = 0,\ R_9(e_5) = 0,\ a_1 \neq 0.$
$R_{10}(e_1) = 0,\ R_{10}(e_2) = a_7e_1 + a_8e_3,\ R_{10}(e_3) = 0,\ R_{10}(e_4) = 0,\ R_{10}(e_5) = 0.$
$R_{11}(e_1) = 0,\ R_{11}(e_2) = a_5e_2 + a_8e_3,\ R_{11}(e_3) = \frac{a_5^2}{a_8}e_2 + a_5e_3,\ R_{11}(e_4) = 0,\ R_{11}(e_5) = 0,\ a_8 \neq 0.$
$R_{12}(e_1) = a_6e_3,\ R_{12}(e_2) = -\frac{a_6}{2}e_1 + a_8e_3,\ R_{12}(e_3) = 0,\ R_{12}(e_4) = 0,\ R_{12}(e_5) = 0.$
$R_{13}(e_1) = 0,\ R_{13}(e_2) = a_7e_1,\ R_{13}(e_3) = 0,\ R_{13}(e_4) = 0,\ R_{13}(e_5) = 0.$
$R_{14}(e_1) = 0,\ R_{14}(e_2) = 0,\ R_{14}(e_3) = a_4e_2,\ R_{14}(e_4) = 0,\ R_{14}(e_5) = 0.$
$R_{15}(e_1) = 0,\ R_{15}(e_2) = 0,\ R_{15}(e_3) = 0,\ R_{15}(e_4) = 0,\ R_{15}(e_5) = 0.$
$R_{16}(e_1) = a_1e_1 + a_2e_2 + \frac{a_1^2(a_2-4a_3)}{4a_3^2}e_3,\ R_{16}(e_2) = -\frac{a_1^2}{4a_3}e_1 - \frac{a_1a_2}{4a_3}e_2 + \frac{a_1^3(4a_3-a_2)}{16a_3^3}e_3,\ R_{16}(e_3) = a_3e_1 + \frac{a_2a_3}{a_1}e_2 + \frac{a_1(a_2-4a_3)}{4a_3}e_3,\ R_{16}(e_4) = 0,\ R_{16}(e_5) = 0,\ a_1 \neq 0,\ a_3 \neq 0.$
$R_{17}(e_1) = a_1e_1 - \frac{a_1^2}{a_3}e_3,\ R_{17}(e_2) = -\frac{a_1^2}{4a_3}e_1 + \frac{a_1^3}{4a_3^2}e_3,\ R_{17}(e_3) = a_3e_1 - a_1e_3,\ R_{17}(e_4) = 0,\ R_{17}(e_5) = 0,\ a_3 \neq 0.$
$R_{18}(e_1) = a_1e_1 - 4a_3e_2 - \frac{2a_1^2}{a_3}e_3,\ R_{18}(e_2) = \frac{2a_1^2}{a_3}e_1 - \frac{7a_1}{2}e_2 + \frac{a_1^3}{2a_3^2}e_3,\ R_{18}(e_3) = a_3e_1 + \frac{a_3^2}{2a_1}e_2 + \frac{5a_1}{2}e_3,\ R_{18}(e_4) = 0,\ R_{18}(e_5) = 0,\ a_1 \neq 0,\ a_3 \neq 0.$
$R_{19}(e_1) = a_1e_1 + 4a_3e_2,\ R_4(19) = -\frac{a_1^2}{4a_3}e_1 - a_1e_2,\ R_{19}(e_3) = a_3e_1 + \frac{4a_3^2}{a_1}e_2,\ R_{19}(e_4) = 0,\ R_{19}(e_5) = 0,\ a_1 \neq 0,\ a_3 \neq 0$
$R_{20}(e_1) = 0,\ R_{20}(e_2) = 0,\ R_{20}(e_3) = a_3e_1 + a_4e_2,\ R_{20}(e_4) = 0,\ R_{20}(e_5) = 0.$
$R_{21}(e_1) = -2a_3e_2,\ R_{21}(e_2) = 0,\ R_{21}(e_3) = a_3e_1 + a_4e_2,\ R_{21}(e_4) = 0,\ R_{21}(e_5) = 0.$
$R_{22}(e_1) = \frac{4a_5^2}{a_6}e_2 + a_6e_3,\ R_{22}(e_2) = -a_5e_2 - \frac{a_6^2}{4a_5}e_3,\ R_{22}(e_3) = \frac{4a_5^3}{a_6^2}e_2 + a_5e_3,\ R_{23}(e_4) = 0,\ R_{23}(e_5) = 0,\ a_5 \neq 0,\ a_6 \neq 0.$
$R_{23}(e_1) = a_2e_1,\ R_{23}(e_2) = 0,\ R_{23}(e_3) = -\frac{a_2}{2}e_2 + a_4e_2,\ R_{23}(e_4) = 0,\ R_{23}(e_5) = 0.$
$R_{24}(e_1) = 0,\ R_{24}(e_2) = a_6e_1 + a_7e_3,\ R_{24}(e_3) = 0,\ R_{24}(e_4) = 0,\ R_{24}(e_5) = 0.$
$R_{25}(e_1) = 0\ R_{25}(e_2) = 0,\ R_{25}(e_3) = a_4e_2,\ R_{25}(e_4) = 0,\ R_{25}(e_5) = 0.$
$R_{26}(e_1) = 0,\ R_{26}(e_2) = a_5e_2 + \frac{a_5^2}{a_4}e_3,\ R_{26}(e_3) = a_4e_2 + a_5e_3,\ R_{26}(e_4) = 0,\ R_{26}(e_5) = 0,\ a_4 \neq 0.$
$R_{27}(e_1) = a_1e_1,\ R_{27}(e_2) = 0,\ R_{27}(e_3) = 0,\ R_{27}(e_4) = 0,\ R_{27}(e_5) = 0.$

$R_{28}(e_1)=0,\ R_{28}(e_2)=a_7e_3,\ R_{28}(e_3)=0,\ R_{28}(e_4)=0,\ R_{28}(e_5)=0.$
$R_{29}(e_1)=a_9e_1-\frac{2a_9^2}{a_{10}}e_3,\ R_{29}(e_2)=\frac{a_9}{2}e_2+\frac{a_9^3}{2a_{10}^2}e_3,\ R_{29}(e_3)=a_{10}e_1+\frac{a_5^2}{2a_9}e_2$
$-\frac{3a_9}{2}e_3,\ R_{29}(e_4)=-a_9e_4+a_{10}e_5,\ R_{29}(e_5)=-\frac{a_9^2}{a_{10}}e_4+a_9e_5,\ a_9\neq 0,\ a_{10}\neq 0.$
$R_{30}(e_1)=-a_{10}e_2,\ R_{30}(e_2)=0,\ R_{30}(e_3)=\frac{a_{10}}{2}e_1+a_4e_2,\ R_{30}(e_4)=a_{10}e_5,\ R_{30}(e_5)=0.$
$R_{31}(e_1)=a_{11}e_3,\ R_{31}(e_2)=-\frac{a_{11}}{2}e_1+a_8e_3,\ R_{31}(e_3)=0,\ R_{31}(e_4)=0,\ R_{31}(e_5)=a_{11}e_4.$

The constants a_i are parameters.

Now, we define Rota–Baxter operators on an $\mathcal{A}$-module, where $\mathcal{A}$ is a Rota–Baxter Lie superalgebra.

Definition 1.7 Let $(\mathcal{A},[\ ,\ ],R)$ be a Rota–Baxter Lie superalgebra of weight zero. A Rota–Baxter operator on an $\mathcal{A}$-module V (relative to R) is a map $R_V:V\longrightarrow V$ such that, for all $x\in\mathcal{H}(\mathcal{A})$ and $v\in\mathcal{H}(V)$,

$$[R(x),R_V(v)]=R_V\Big([R(x),v]+[x,R_V(v)]\Big),$$
$$[R_V(v),R(x)]=R_V\Big([R_V(v),x]+[v,R(x)]\Big),$$

where the action $\rho(x)(v)$ is denoted by $[x,v]$.

Proposition 1.5 *Let $(\mathcal{A},[\ ,\ ],R)$ be a Rota–Baxter Lie superalgebra of weight zero, V an $\mathcal{A}$-module and R_V a Rota–Baxter operator on V. Define new actions of $\mathcal{A}$ on V by*

$$x\circ v=[R(x),v],\qquad v\circ x=[R_V(v),x].$$

Equipped with these actions, V is a bimodule over the pre-Lie superalgebra (Remark 1.1).

Proof Let x,y be a homogeneous elements in $\mathcal{A}$ and v in V. We have

$$\begin{aligned}
&l(x)r(y)(v)-(-1)^{|x||y|}r(y)l(x)-r(x\circ y)(v)+(-1)^{|x||y|}r(y)r(x)(v)\\
&=(-1)^{|y||v|}[R(x),[R_V(v),y]]-(-1)^{|y||v|}[R_V([R(x),v]),y]\\
&-(-1)^{|v|(|x|+|y|)}[R_V(v),[R(x),y]]+(-1)^{|v|(|x|+|y|)}[R_V([R_V(v),x]),y]\\
&=(-1)^{|y||v|}[R(x),[R_V(v),y]]-(-1)^{|y||v|}\Big([R_V([R(x),v]+[x,R_V(v)]),y]\Big)\\
&-(-1)^{|v|(|x|+|y|)}[R_V(v),[R(x),y]]\\
&=(-1)^{|y||v|}[R(x),[R_V(v),y]]-(-1)^{|y||v|}[[R(x),R_V(v)],y]\\
&-(-1)^{|v|(|x|+|y|)}[R_V(v),[R(x),y]]\\
&=0.
\end{aligned}$$

Then

$$l(x)r(y)(v)-(-1)^{|x||y|}r(y)l(x)=r(x\circ y)(v)-(-1)^{|x||y|}r(y)r(x)(v).$$

Similarly, we show that $l(x)\circ l(y)v-l(x\circ y)v=(-1)^{|x||y|}\Big(l(y)\circ l(x)v-l(y\circ x)v\Big).$ □

Now, we construct a functor from a full sub-category of the category of Rota–Baxter Lie-admissible (or associative) superalgebras to the category of pre-Lie superalgebras. The Lie-admissible algebras were studied by Albert in 1948 and Goze and Remm in 2004, they introduced the notion of G-associative algebras where G is a subgroup of the permutation group S_3 (see [35]). The graded case was studied by Ammar and Makhlouf in 2010 (see [4] for more details).

Definition 1.8 (1) A Lie-admissible superalgebra is a superalgebra $(\mathcal{A},\mu)$ in which the supercommutator bracket, defined for all homogeneous x, y in $\mathcal{A}$ by

$$[x,y]=\mu(x,y)-(-1)^{|x||y|}\mu(y,x),$$

satisfies the super-Jacobi identity (1.5).

(2) Let G be a subgroup of the permutation group $\mathcal{S}_3$. A Rota–Baxter G-associative superalgebra of weight $\lambda\in\mathbb{K}$ is a G-associative superalgebra $(\mathcal{A},\cdot)$ together with an even linear self-map $R:\mathcal{A}\longrightarrow\mathcal{A}$ that satisfies the identity

$$R(x)\cdot R(y)=R(R(x)\cdot y+x\cdot R(y)-\lambda x\cdot y),\tag{1.13}$$

for all homogeneous elements x, y, z in $\mathcal{A}$.

Theorem 1.1 *Let $(\mathcal{A},\cdot,R)$ be a Rota–Baxter Lie-admissible superalgebra of weight zero. Define an even binary operation "$*$" on any homogeneous element $x, y\in\mathcal{A}$ by*

$$x*y=R(x)\cdot y-(-1)^{|x||y|}y\cdot R(x)=[R(x),y].\tag{1.14}$$

Then $\mathcal{A}_L=(\mathcal{A},*)$ is a pre-Lie superalgebra.

Proof A direct consequence of Remark 1.1, since a Rota–Baxter operator on a Lie-admissible superalgebra is also a Rota–Baxter operator of its supercommutator Lie superalgebra. □

Theorem 1.2 *Let $(\mathcal{A},\mu,R)$ be a Rota–Baxter associative superalgebra of weight $\lambda=-1$. Define the even binary operation "$\circ$" on any homogeneous element $x, y\in\mathcal{A}$ by*

$$\begin{aligned}x\circ y&=\mu(R(x),y)-(-1)^{|x||y|}\mu(y,R(x))-\mu(x,y)\\&=R(x)y-(-1)^{|x||y|}yR(x)-xy.\end{aligned}\tag{1.15}$$

Then $\mathcal{A}_L=(\mathcal{A},\circ)$ is a pre-Lie superalgebra.

Proof For all homogeneous elements x, y, z in $\mathcal{A}$, we have

$$\begin{aligned} x \circ (y \circ z) = & R(x)(R(y)z) - (-1)^{|y||z|} R(x)(zR(y)) - R(x)(yz) \\ & - (-1)^{|x|(|y|+|z|)}(R(y)z)R(x) + (-1)^{|x|(|y|+|z|)+|y||z|}(zR(y))R(x) \\ & + (-1)^{|x|(|y|+|z|)}(yz)R(x) - x(R(y)z) + (-1)^{|y||z|}(zR(y))x + x(yz), \end{aligned}$$

and

$$\begin{aligned} (x \circ y) \circ z = & R(R(x)y)z - (-1)^{|x||y|} R(yR(x))z \\ & -R(xy)z - (-1)^{|z|(|x|+|y|)}(R(x)y)R(z) \\ & + (-1)^{|z|(|x|+|y|)+|x||y|}(yR(x))R(z) + (-1)^{|z|(|x|+|y|)}(xy)z \\ & - (R(x)y)z + (-1)^{|x||y|}(yR(x))z + (xy)z. \end{aligned}$$

Then, we obtain

$$\begin{aligned} & as_{\mathcal{A}_L}(x, y, z) - (-1)^{|x||y|} as_{\mathcal{A}_L}(y, x, z) \\ & = x \circ (y \circ z) - (x \circ y) \circ z - (-1)^{|x||y|} y \circ (x \circ z) + (-1)^{|x||y|}(y \circ x) \circ z \\ & = R(x)(R(y)z) - (-1)^{|y||z|} R(x)(zR(y)) - R(x)(yz) - (-1)^{|x|(|y|+|z|)}(R(y)z)R(x) \\ & \quad + (-1)^{|x|(|y|+|z|)+|y||z|}(zR(y))R(x) + (-1)^{|x|(|y|+|z|)}(yz)R(x) - x(R(y)z) \\ & \quad + (-1)^{|x|(|y|+|z|)+|y||z|}(zR(y))x + x(yz) \\ & \quad - R(R(x)y)z + (-1)^{|x||y|} R(yR(x))z + R(xy)z \\ & \quad + (-1)^{|z|(|x|+|y|)}(R(x)y)R(z) - (-1)^{|z|(|x|+|y|)+|x||y|}(yR(x))R(z) \\ & \quad - (-1)^{|z|(|x|+|y|)}(xy)z \\ & \quad + (R(x)y)z + (-1)^{|x||y|}(yR(x))z + (xy)z - (-1)^{|x||y|} R(y)(R(x)z) \\ & \quad + (-1)^{|x|(|y|+|z|)} R(y)(zR(x)) \\ & \quad + (-1)^{|x||y|} R(y)(xz) + (-1)^{|y||z|}(R(x)z)R(y) \\ & \quad - (-1)^{|z|(|x|+|y|)}(zR(x))R(y) - (-1)^{|y||z|}(xz)R(y) \\ & \quad + (-1)^{|x||y|} y(R(x)z) - (-1)^{|z|(|x|+|y|)}(zR(x))y \\ & \quad + (-1)^{|x||y|} y(xz) + (-1)^{|x||y|} R(R(y)x)z \\ & \quad - R(xR(y))z - (-1)^{|x||y|} R(yx)z - (-1)^{|z|(|x|+|y|)+|x||y|}(R(y)x)R(z) \\ & \quad + (-1)^{|z|(|x|+|y|)}(xR(y))R(z) \\ & \quad + (-1)^{|z|(|x|+|y|)+|x||y|}(yx)z - (-1)^{|x||y|}(R(y)x)z + (xR(y))z + (-1)^{|x||y|}(yx)z. \end{aligned}$$

The above sum vanishes by associativity and the Rota–Baxter identity (1.13) with $\lambda = -1$. □

Corollary 1.2 *Let* $(\mathcal{A}, \mu, R)$ *be a Rota–Baxter associative superalgebra of weight* $\lambda = -1$. *Then* R *is still a Rota–Baxter operator of weight* $\lambda = -1$ *on the pre-Lie superalgebra* $(\mathcal{A}, \circ)$ *defined in (1.15).*

As a consequence of Theorem 1.2 and Corollary 1.2, we have:

Proposition 1.6 *Let $(\mathcal{A}, \mu, R)$ be a Rota–Baxter associative superalgebra of weight $\lambda = -1$. Then the binary operation defined, for any homogeneous elements x, y in $\mathcal{A}$, by*

$$[x, y] = R(x)y - (-1)^{|x||y|}yR(x) - xy + xR(y) - (-1)^{|x||y|}R(y)x + (-1)^{|x||y|}yx,$$

defines a Rota–Baxter Lie superalgebra $(\mathcal{A}, [\ ,\ ], R)$ of weight $\lambda = -1$.

2 L-dendriform Superalgebras

The notion of L-dendriform algebra was introduced by Bai, Liu and Ni in 2010 (see [14]). In this section, we extend this notion to the graded case, and define L-dendriform superalgebra. Then we study relationships between associative superalgebras, L-dendriform superalgebras and pre-Lie superalgebras. Moreover, we introduce the notion of Rota–Baxter operator (of weight zero) on the $\mathcal{A}$-bimodule and we provide a construction of associative bimodules from bimodules over L-dendriform superalgebras and a construction of L-dendriform bimodules from bimodules over pre-Lie superalgebras.

2.1 L-dendriform Superalgebras and Associative Superalgebras

2.1.1 Definition and Some Basic Properties

Definition 2.1 A L-dendriform superalgebra is a triple $(\mathcal{A}, \triangleright, \triangleleft)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathcal{A}$ and two even bilinear maps $\triangleright, \triangleleft : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ satisfying, for all homogeneous elements x, y, z in $\mathcal{A}$,

$$\begin{aligned} x \triangleright (y \triangleright z) &= (x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + (-1)^{|x||y|} y \triangleright (x \triangleright z) \\ &\quad -(-1)^{|x||y|}(y \triangleleft x) \triangleright z - (-1)^{|x||y|}(y \triangleright x) \triangleright z, \qquad (2.1) \\ x \triangleright (y \triangleleft z) &= (x \triangleright y) \triangleleft z + (-1)^{|x||y|} y \triangleleft (x \triangleright z) \\ &\quad +(-1)^{|x||y|} y \triangleleft (x \triangleleft z) - (-1)^{|x||y|}(y \triangleleft x) \triangleleft z. \qquad (2.2) \end{aligned}$$

The associated bracket to a L-dendriform superalgebra is defined as $[x, y] = x \triangleright y - (-1)^{|x||y|} y \triangleleft x$.

Definition 2.2 (1) Let $(\mathcal{A}, \triangleright, \triangleleft)$ be a L-dendriform superalgebra, V be a $\mathbb{Z}_2$-graded vector space, and $l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft : \mathcal{A} \longrightarrow End(V)$ be four even linear maps. The tuple $(V, l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft)$ is an $\mathcal{A}$-bimodule if for any homogeneous elements $x, y \in \mathcal{A}$ and $u, v \in V$, the following identities are satisfied

(a) $[l_\triangleright(x), l_\triangleright(y)] = l_\triangleright([x, y])$,
(b) $[l_\triangleright(x), l_\triangleleft(y)] = l_\triangleleft(x \circ y) + (-1)^{|x||y|} l_\triangleleft(y) l_\triangleleft(x)$,
(c) $r_\triangleright(x \triangleright y)(v) = r_\triangleright(y) r_\triangleright(x)(v) + r_\triangleright(y) r_\triangleleft(x)(v) + (-1)^{|x||v|} l_\triangleright(x) r_\triangleright(y)(v) - (-1)^{|x||v|} r_\triangleright(y) l_\triangleright(x)(v) - (-1)^{|x||v|} r_\triangleright(y) l_\triangleleft(x)(v)$,

(d) $r_{\triangleright}(x \triangleleft y)(v) = r_{\triangleleft}(y) r_{\triangleright}(x)(v) - (-1)^{|x||v|}\Big(l_{\triangleleft}(x) r_{\triangleright}(y)(v) - l_{\triangleleft}(x) r_{\triangleleft}(y)(v)$
$+ r_{\triangleleft}(y) l_{\triangleleft}(x)(v)\Big)$,

(e) $l_{\triangleright}(x) r_{\triangleleft}(y)(v) - r_{\triangleleft}(y) l_{\triangleright}(x)(v) = (-1)^{|x||v|} r_{\triangleleft}(x \bullet y)(v) - (-1)^{|x||v|} r_{\triangleleft}(y) r_{\triangleleft}(x)(v)$.

where $x \circ y = x \triangleright y - (-1)^{|x||y|} y \triangleleft x$, and $x \bullet y = x \triangleright y + x \triangleleft y$.

Moreover, The tuple $(V, \triangleright_V, \triangleleft_V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ is an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra if the following identities are satisfied

(a) $l_{\triangleright}(x)(u \triangleright_V v) - (-1)^{|x||u|} u \triangleright_V l_{\triangleright}(x)(v) = l_{\triangleright}(x)(u) \triangleright_V v + l_{\triangleleft}(x)(u) \triangleright_V v - (-1)^{|x||u|} r_{\triangleright}(x)(u) \triangleright_V v - (-1)^{|x||u|} r_{\triangleleft}(x)(u) \triangleright_V v$,

(b) $l_{\triangleright}(x)(u \triangleleft_V v) - (-1)^{|x||u|} u \triangleleft_V l_{\triangleright}(x)(v) = l_{\triangleright}(x)(u) \triangleleft_V v - (-1)^{|x||u|} r_{\triangleleft}(x)(u) \triangleleft_V v + (-1)^{|x||u|} u \triangleleft_V l_{\triangleleft}(x) v$,

(c) $u \triangleright_V l_{\triangleleft}(x)(v) = r_{\triangleright}(x)(u) \triangleleft_V v - (-1)^{|x||u|} l_{\triangleleft}(x)(u \triangleright_V v) + (-1)^{|x||u|} l_{\triangleleft}(x)(u \triangleleft_V v) - (-1)^{|x||u|} l_{\triangleleft}(x)(u) \triangleleft_V v$.

(2) Let $(\mathcal{A}, \triangleright, \triangleleft)$ be a L-dendriform superalgebra and $(V, \triangleright_V, \triangleleft_V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ be an $\mathcal{A}$-bimodule $\mathbb{K}$-superalgebra. An even linear map $T : V \longrightarrow \mathcal{A}$ is called a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated with $(V, \triangleright_V, \triangleleft_V, l_{\triangleright}, r_{\triangleright}, l_{\triangleleft}, r_{\triangleleft})$ if T satisfies for any homogeneous elements u, v in V

$$T(u) \triangleright T(v) = T\Big(l_{\triangleright}(T(u))v + (-1)^{|u||v|} r_{\triangleright}(T(v))u + \lambda u \triangleright_V v\Big),$$
$$T(u) \triangleleft T(v) = T\Big(l_{\triangleleft}(T(u))v + (-1)^{|u||v|} r_{\triangleleft}(T(v))u + \lambda u \triangleleft_V v\Big).$$

In particular, a super $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ of the L-dendriform superalgebra $(\mathcal{A}, \triangleright, \triangleleft)$ associated with the bimodule $(\mathcal{A}, L_{\triangleright}, R_{\triangleright}, L_{\triangleleft}, R_{\triangleleft})$ is called a Rota–Baxter operator (of weight λ) on $(\mathcal{A}, \triangleright, \triangleleft)$, that is, R satisfies for any homogeneous elements x, y in $\mathcal{A}$

$$R(x) \triangleright R(y) = R\Big(R(x) \triangleright y + R(x) \triangleright u + \lambda x \triangleright y\Big),$$
$$R(x) \triangleleft R(y) = R\Big(R(x) \triangleleft y + R(x) \triangleleft y + \lambda x \triangleleft y\Big).$$

The following theorem provides a construction of L-dendriform superalgebras using super $\mathcal{O}$-operators of associative superalgebras.

Theorem 2.1 *Let $(\mathcal{A}, \mu)$ be an associative superalgebra and (V, l, r) be a $\mathcal{A}$-bimodule. If T is a super $\mathcal{O}$-operator of weight zero associated with (V, l, r), then there exists a L-dendriform superalgebra structure on V defined by*

$$u \triangleright v = l(T(u))v, \quad u \triangleleft v = (-1)^{|u||v|} r(T(v))u, \quad \forall\ u, v \in \mathcal{H}(V). \tag{2.3}$$

Proof For any homogeneous elements u, v and w in V, we have

$$u \triangleright (v \triangleright w) = l(T(u)) l(T(v)) w, \qquad (u \triangleright v) \triangleright w = l\Big(T(l(T(u))v)\Big) w,$$
$$(u \triangleleft v) \triangleright w = (-1)^{|u||v|} l\Big(T(r(T(v))u)\Big) w, \qquad (-1)^{|u||v|} v \triangleright (u \triangleright w) = (-1)^{|u||v|} l(T(v)) l(T(u)) w,$$

$$(-1)^{|u||v|}(v \lhd u) \rhd w = l\Big(T(r(T(u))v)\Big)w, \qquad (-1)^{|u||v|}(v \rhd u) \rhd w = (-1)^{|u||v|}l\Big(T(l(T(v))u)\Big)w.$$

Hence,

$$\begin{aligned}
&u \rhd (v \rhd w) - (u \rhd v) \rhd w - (u \lhd v) \rhd w \\
&\quad - (-1)^{|u||v|}\Big(v \rhd (u \rhd w) - (v \lhd u) \rhd w - (v \rhd u) \rhd w\Big) \\
&= l(T(u))l(T(v))w - l\Big(T(l(T(u))v)\Big)w - (-1)^{|u||v|}l\Big(T(r(T(v))u)\Big)w \\
&\quad - (-1)^{|u||v|}l(T(v))l(T(u))w + l\Big(T(r(T(u))v)\Big)w + (-1)^{|u||v|}l\Big(T(l(T(v))u)\Big)w \\
&= (-1)^{|u||v|}l(\mu(T(v), T(u))) - (-1)^{|u||v|}l(T(v))l(T(u))w \\
&\quad - l(\mu(T(u), T(v))) + l(T(u))l(T(v))w \\
&= 0,
\end{aligned}$$

and similarly, we have

$$\begin{aligned}
&u \rhd (v \lhd w) - (u \rhd v) \lhd w - (-1)^{|u||v|}v \lhd (u \rhd w) - (-1)^{|u||v|}v \lhd (u \lhd w) \\
&\quad +(-1)^{|u||v|}(v \lhd u) \lhd w = 0.
\end{aligned}$$

Therefore, $(V, \rhd, \lhd)$ is a L-dendriform superalgebra. □

A direct consequence of Theorem 2.1 is the following construction of a L-dendriform superalgebra from a Rota–Baxter operator (of weight zero) of an associative superalgebra.

Corollary 2.1 *Let $(\mathcal{A}, \mu, R)$ be a Rota–Baxter associative superalgebra of weight zero. Then, the even binary operations given by*

$$x \rhd y = \mu(R(x), y), \quad x \lhd y = \mu(x, R(y)), \quad \forall\ x, y \in \mathcal{H}(\mathcal{A})$$

defines a L-dendriform superalgebra structure on $\mathcal{A}$.

Definition 2.3 Let $(\mathcal{A}, \rhd, \lhd)$ be a L-dendriform superalgebra and $R : \mathcal{A} \longrightarrow \mathcal{A}$ be a Rota–Baxter operator of weight zero. A Rota–Baxter operator on $\mathcal{A}$-bimodule V (relative to R) is a map $R_V : V \longrightarrow V$ such that for all homogeneous elements x in $\mathcal{A}$ and v in V

$$\begin{aligned}
R(x) \rhd R_V(v) &= R_V\Big(R(x) \rhd v + x \rhd R_V(v)\Big), \\
R_V(v) \rhd R(x) &= R_V\Big(R_V(v) \rhd x + v \rhd R(x)\Big), \\
R(x) \lhd R_V(v) &= R_V\Big(R(x) \lhd v + x \lhd R_V(v)\Big), \\
R_V(v) \lhd R(x) &= R_V\Big(R_V(v) \lhd x + v \lhd R(x)\Big).
\end{aligned}$$

Proposition 2.1 *Let* $(\mathcal{A}, \mu)$ *be an associative superalgebra,* $R : \mathcal{A} \longrightarrow \mathcal{A}$ *a Rota–Baxter operator on* $\mathcal{A}$*,* V *an* $\mathcal{A}$*-bimodule and* R_V *a Rota–Baxter operator on* V*. Define a new actions of* $\mathcal{A}$ *on* V *by*

$$x \rhd v = \mu(R(x), v), \quad v \rhd x = \mu(R_V(v), x),$$
$$x \lhd v = \mu(x, R_V(v)), \quad v \lhd x = \mu(v, R(x)).$$

Equipped with these actions, V *becomes an* $\mathcal{A}$*-bimodule over the associated L-dendriform superalgebra.*

Corollary 2.2 *Let* $(V, l_{\rhd}, r_{\rhd}, l_{\lhd}, r_{\lhd})$ *be an* $\mathcal{A}$*-bimodule of a dendriform superalgebra* $(\mathcal{A}, \rhd, \lhd)$*. Let* $(\mathcal{A}, \mu)$ *be the associated associative superalgebra. If* T *is a super* $\mathcal{O}$*-operator associated with* $(V, l_{\rhd}, r_{\rhd}, l_{\lhd}, r_{\lhd})$*, then* T *is a super* $\mathcal{O}$*-operator of* $(\mathcal{A}, \mu)$ *associated with* $(V, l_{\rhd} + l_{\lhd}, r_{\rhd} + r_{\lhd})$*.*

2.2 *L*-dendriform Superalgebras and Pre-Lie Superalgebras

We have the following observation.

Proposition 2.2 *Let* $(\mathcal{A}, \rhd, \lhd)$ *be a L-dendriform superalgebra*

(1) The even binary operation $\circ : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ *given by*

$$x \circ y = x \rhd y - (-1)^{|x||y|} y \lhd x, \quad \forall \ x, y \in \mathcal{H}(\mathcal{A})$$

defines a pre-Lie superalgebra $(\mathcal{A}, \circ)$ *which is called the associated vertical pre-Lie superalgebra of* $(\mathcal{A}, \rhd, \lhd)$ *and* $(\mathcal{A}, \rhd, \lhd)$ *is called a compatible L-dendriform superalgebra structure on the pre-Lie superalgebra* $(\mathcal{A}, \circ)$*.*

(2) The even binary operation $\bullet : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ *given by*

$$x \bullet y = x \rhd y + x \lhd y, \quad \forall \ x, y \in \mathcal{H}(\mathcal{A})$$

defines a pre-Lie superalgebra $(\mathcal{A}, \bullet)$ *which is called the associated horizontal pre-Lie superalgebra of* $(\mathcal{A}, \rhd, \lhd)$ *and* $(\mathcal{A}, \rhd, \lhd)$ *is called a compatible L-dendriform superalgebra structure on the pre-Lie superalgebra* $(\mathcal{A}, \bullet)$*.*

(3) Both $(\mathcal{A}, \circ)$ *and* $(\mathcal{A}, \bullet)$ *have the same sub-adjacent Lie superalgebra* $g(\mathcal{A})$ *defined by*

$$[x, y] = x \rhd y + x \lhd y - (-1)^{|x||y|} y \rhd x - (-1)^{|x||y|} y \lhd x, \quad \forall \ x, y \in \mathcal{H}(\mathcal{A}).$$

Proof Straightforward. □

Corollary 2.3 *Let* $(V, l_{\rhd}, r_{\rhd}, l_{\lhd}, r_{\lhd})$ *be a bimodule of a L-dendriform superalgebra* $(\mathcal{A}, \rhd, \lhd)$*. Let* $(\mathcal{A}, \circ)$ *be the associated pre-Lie superalgebra. If* T *is a super* $\mathcal{O}$*-operator associated with* $(V, l_{\rhd}, r_{\rhd}, l_{\lhd}, r_{\lhd})$*, then* T *is a super* $\mathcal{O}$*-operator of* $(\mathcal{A}, \circ)$ *associated with* (V, l', r')*, where* $l' = l_{\rhd} + (-1)^{|u||v|} r_{\lhd}$ *and* $r' = l_{\lhd} + r_{\rhd}$*.*

Conversely, we can construct L-dendriform superalgebras from $\mathcal{O}$-operators of pre-Lie superalgebras.

Theorem 2.2 *Let $(\mathcal{A}, \circ)$ be a pre-Lie superalgebra and (V, l, r) be an $\mathcal{A}$-bimodule. If T is a super $\mathcal{O}$-operator of weight zero associated with (V, l, r), then there exists a L-dendriform superalgebra structure on V defined by*

$$u \rhd v = l(T(u))v, \quad u \lhd v = -r(T(u))v, \quad \forall\ u, v \in \mathcal{H}(V). \tag{2.4}$$

Therefore, there is a pre-Lie superalgebra structure on V defined by

$$u \circ v = u \rhd v - (-1)^{|u||v|} v \lhd u, \quad \forall\ u, v \in \mathcal{H}(V) \tag{2.5}$$

as the associated vertical pre-Lie superalgebra of $(V, \rhd, \lhd)$ and T is a homomorphism of pre-Lie superalgebra.

Furthermore, $T(V) = \{T(v)\ /\ v \in V\} \subset \mathcal{A}$ is a pre-Lie sub-superalgebra of $(\mathcal{A}, \circ)$ and there is a L-dendriform superalgebra structure on $T(V)$ given by

$$T(u) \rhd T(v) = T(u \rhd v), \quad T(u) \lhd T(v) = T(u \lhd v), \quad \forall\ u, v \in \mathcal{H}(V). \tag{2.6}$$

Moreover, the corresponding associated vertical pre-Lie superalgebra structure on $T(V)$ is a pre-Lie sub-superalgebra of $(\mathcal{A}, \circ)$ and T is a homomorphism of L-dendriform superalgebra.

Proof For any homogeneous elements u, v and w in V, we have

$$u \rhd (v \rhd w) = l(T(u))l(T(v))w, \qquad (u \rhd v) \rhd w = l\Big(T(l(T(u))v)\Big)w,$$

$$(u \lhd v) \rhd w = -l\Big(T(r(T(u))v)\Big)w, \qquad (-1)^{|u||v|} v \rhd (u \rhd w) = (-1)^{|u||v|} l(T(v))l(T(u))w,$$

$$(-1)^{|u||v|}(v \lhd u) \rhd w = -(-1)^{|u||v|} l\Big(T(r(T(v))u)\Big)w, \qquad (-1)^{|u||v|}(v \rhd u) \rhd w = (-1)^{|u||v|} l\Big(T(l(T(v))u)\Big)w,$$

$$u \rhd (v \lhd w) = -l(T(u))r(T(v))w, \qquad (u \rhd v) \lhd w = r\Big(T(l(T(u))v)\Big)w,$$

$$(-1)^{|u||v|} v \lhd (u \rhd w) = -(-1)^{|u||v|} r(T(v))l(T(u))w, \qquad (-1)^{|u||v|} v \lhd (u \lhd w) = (-1)^{|u||v|} r(T(v))r(T(u)),$$

$$(-1)^{|u||v|}(v \lhd u) \lhd w = (-1)^{|u||v|} r\Big(T(r(T(v))u)\Big)w.$$

Hence,

$$\begin{aligned}
& u \rhd (v \rhd w) - (u \rhd v) \rhd w - (u \lhd v) \rhd w - (-1)^{|u||v|} v \rhd (u \rhd w) \\
&\quad + (-1)^{|u||v|}(v \lhd u) \rhd w + (-1)^{|u||v|}(v \rhd u) \rhd w \\
&= l(T(u))l(T(v))w - (-1)^{|u||v|} l(T(v))l(T(u))w - l(T(l(T(u)))v)w \\
&\quad + l(T(r(T(u))v))w \\
&\quad - (-1)^{|u||v|} l(T(r(T(v))u))w + (-1)^{|u||v|} l(T(l(T(v))u))w \\
&= l(T(u))l(T(v))w - (-1)^{|u||v|} l(T(v))l(T(u))w - l\Big(T(u) \circ T(v)\Big)w
\end{aligned}$$

$$+ (-1)^{|u||v|} l\Big(T(v)\circ T(u)\Big) w$$
$$= 0,$$

and

$$\begin{aligned} & u \rhd (v \lhd w) - (u \rhd v) \lhd w - (-1)^{|u||v|} v \lhd (u \rhd w) \\ & \quad - (-1)^{|u||v|} v \lhd (u \lhd w) + (-1)^{|u||v|} (v \lhd u) \lhd w \\ & = -l(T(u))r(T(v))w + (-1)^{|u||v|} r(T(v))l(T(u))w \\ & \quad + r(T(u)\circ T(v))w - (-1)^{|u||v|} r(T(v))r(T(u)) \\ & = 0. \end{aligned}$$

Therefore, $(V, \rhd, \lhd)$ is a L-dendriform superalgebra. The other conditions follow easily. □

A direct consequence of Theorem 2.2, is the following construction of a L-dendriform superalgebra from a Rota–Baxter operator (of weight zero) of a pre-Lie superalgebra.

Corollary 2.4 *Let* $(\mathcal{A}, \circ)$ *be a pre-Lie superalgebra and* R *be a Rota–Baxter operator on* $\mathcal{A}$ (*of weight zero*). *Then even binary operations given by*

$$x \rhd y = R(x)\circ y, \quad x \lhd y = -(-1)^{|x||y|} y \circ R(x) \tag{2.7}$$

defines a L*-dendriform superalgebra structure on* $\mathcal{A}$.

Lemma 2.1 *Let* $\{R_1, R_2\}$ *be a pair of commuting Rota–Baxter operators* (*of weight zero*) *on a pre-Lie superalgebra* $(\mathcal{A}, \circ)$. *Then* R_2 *is a Rota–Baxter operator* (*of weight zero*) *on the* L*-dendriform superalgebra* $(\mathcal{A}, \rhd, \lhd)$ *defined in* (2.7) *with* $R = R_1$.

Theorem 2.3 *Let* $(\mathcal{A}, \circ)$ *be a pre-Lie superalgebra. Then there exists a compatible* L*-dendriform superalgebra structure on* $(\mathcal{A}, \circ)$ *such that* $(\mathcal{A}, \circ)$ *is the associated vertical pre-Lie superalgebra if and only if there exists an invertible super* $\mathcal{O}$*-operator* (*of weight zero*) *of* $(\mathcal{A}, \circ)$.

Proof Straightforward. □

Next, we provide a construction of a L-dendriform bimodule from a bimodule over a pre-Lie superalgebra.

Proposition 2.3 *Let* $(\mathcal{A}, \circ, R)$ *be a Rota–Baxter pre-Lie superalgebra of weight zero,* V *an* $\mathcal{A}$*-bimodule and* R_V *a Rota–Baxter operator on* V. *Define new actions of* $\mathcal{A}$ *on* V *by*

$$x \rhd v = R(x)\circ v, \quad v \rhd x = R_V(v)\circ x, \quad x \lhd v = -(-1)^{|x||v|} v \circ R(x), \quad v \lhd x = -(-1)^{|x||v|} x \circ R_V(v).$$

Equipped with actions, V *is a bimodule over the* L*-dendriform superalgebra of Corollary 2.4.*

Proof Let x, y be homogeneous elements in $\mathcal{A}$ and v in V. Then, we have

$$\begin{aligned}
&r_{\lhd}(x \bullet y)(v) - (-1)^{|x||y|} r_{\lhd}(y) r_{\lhd}(x)(v) - [l_{\rhd}(x), r_{\lhd}(y)](v) \\
&\quad = (-1)^{|v|(|x|+|y|)} v \lhd (x \rhd y + x \lhd y) - (-1)^{|v|(|x|+|y|)} (v \lhd x) \lhd y \\
&\qquad - (-1)^{|y||v|} x \rhd (v \lhd y) + (-1)^{|v|(|x|+|y|)} (x \rhd v) \lhd y \\
&\quad = -(R(x) \circ y) \circ R_V(v) + (-1)^{|x||y|} (y \circ R(x)) \circ R_V(v) \\
&\qquad - (-1)^{|x||y|} y \circ R_V(x \circ R_V(v)) \\
&\qquad - (-1)^{|x||y|} y \circ R_V(R(x) \circ v) + R(x) \circ (y \circ R_V(v)) \\
&\quad = -(R(x) \circ y) \circ R_V(v) + R(x) \circ (y \circ R_V(v)) - (-1)^{|x||y|} y \circ (R(x) \circ R_V(v)) \\
&\qquad + (-1)^{|x||y|} (y \circ R(x)) \circ R_V(v) \\
&\quad = 0.
\end{aligned}$$

Therefore,

$$[l_{\rhd}(x), r_{\lhd}(y)](v) = r_{\lhd}(x \bullet y)(v) - (-1)^{|x||y|} r_{\lhd}(y) r_{\lhd}(x)(v).$$

Similarly, we have

$$\begin{aligned}
&r_{\rhd}(x \lhd y)(v) - (-1)^{|x||y|} r_{\lhd}(y) r_{\rhd}(x)(v) - l_{\lhd}(x) r_{\rhd}(y)(v) - [l_{\lhd}(x), r_{\lhd}(y)](v) \\
&\quad = (-1)^{|v|(|x|+|y|)} v \rhd (x \lhd y) - (-1)^{|v|(|x|+|y|)+|x||y|} (v \rhd x) \lhd y \\
&\qquad - (-1)^{|y||v|} x \lhd (v \rhd y) \\
&\qquad - (-1)^{|y||v|} x \lhd (v \lhd y) + (-1)^{|y||v|} (x \lhd v) \lhd y \\
&\quad = -(-1)^{|v|(|x|+|y|)+|x||y|} R_V(v) \circ (y \circ R(x)) + (-1)^{|x|(|y|+|v|)} y \circ R_V(R_V(v) \circ x) \\
&\qquad + (-1)^{|v|(|x|+|y|)+|x||y|} (R_V(v) \circ y) \circ R(x) - (-1)^{|x|(|y|+|v|)} (y \circ R_V(v)) \circ R(x) \\
&\qquad + (-1)^{|x|(|y|+|v|)} y \circ R_V(v \circ R(x)) \\
&\quad = (-1)^{|x|(|y|+|v|)} \Big(y \circ (R_V(v) \circ R(x)) - (y \circ R_V(v)) \circ R(x) \Big) \\
&\qquad + (-1)^{|v|(|x|+|y|)+|x||y|} \Big(R_V(v) \circ y) \circ R(x) - R_V(v) \circ (y \circ R(x)) \Big) \\
&\quad = 0,
\end{aligned}$$

then

$$r_{\rhd}(x \lhd y)(v) = (-1)^{|x||y|} r_{\lhd}(y) r_{\rhd}(x)(v) + l_{\lhd}(x) r_{\rhd}(y)(v) + [l_{\lhd}(x), r_{\lhd}(y)](v).$$

The others axioms are similar. Therefore, $(V, l_{\rhd}, r_{\rhd}, l_{\lhd}, r_{\lhd})$ is a bimodule over the L-dendriform superalgebra $(\mathcal{A}, \rhd, \lhd)$. □

3 Rota–Baxter Operators on Two-dimensional Pre-Lie Superalgebras

The purpose of this section is to compute all Rota–Baxter operators (of weight zero) on the two-dimensional complex pre-Lie superalgebras given by Zhang and Bai (see

[15]). In the following, let $\mathbb{C}$ be the ground field of complex numbers and $\{e_1, e_2\}$ be a homogeneous basis of a pre-Lie superalgebra $(\mathcal{A}, \circ)$, where $\{e_1\}$ is a basis of $\mathcal{A}_0$ and $\{e_2\}$ is a basis of $\mathcal{A}_1$.

By direct computation and by help of a computer algebra system, we obtain the following results.

Proposition 3.1 *The Rota–Baxter operators* (*of weight zero*) *on two-dimensional pre-Lie superalgebras* (*associative or non-associative*) *of type* B_1, B_2 *and* B_3 *are given as follows:*

3.1 Rota–Baxter Operators on Pre-Lie Superalgebras of Type B_1

■ The pre-Lie superalgebra $(B_{1,1}, \circ)$: $e_2 \circ e_1 = e_2$ has the Rota–Baxter operator defined as

$$R_1(e_1) = a_1 e_1, \; R_1(e_2) = 0.$$

■ The pre-Lie superalgebras of type

$$\begin{cases} (B_{1,2}, \circ) \quad : \quad e_1 \circ e_1 = e_1, \quad e_2 \circ e_1 = e_2 \; \text{(associative)}. \\ \Big((B_{1,3})_k, \circ\Big) : \quad e_1 \circ e_1 = k e_1, \quad e_2 \circ e_1 = e_2, \; k \neq 0, 1. \\ (B_{1,4}, \circ) \quad : \quad e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2 \; \text{(associative)}. \\ \Big((B_{1,5})_k, \circ\Big) : e_1 \circ e_1 = k e_1, \quad e_1 \circ e_2 = k e_2, \quad e_2 \circ e_1 = (k+1) e_2, \; k \neq 0, -1. \end{cases}$$

have only the trivial Rota–Baxter operator, that is,

$$R_1(e_1) = 0, \;\; R_1(e_2) = 0.$$

3.2 Rota–Baxter Operators on Pre-Lie Superalgebras of Type B_2

■ The pre-Lie superalgebra $(B_{2,1}, \circ)$ (associative).

$$e_2 \circ e_2 = \frac{1}{2} e_1.$$

Rota–Baxter operators $\mathrm{RB}(B_{2,1})$ are:

$$R_1(e_1) = a_1 e_1, \;\; R_1(e_2) = 0.$$
$$R_2(e_1) = a_1 e_1, \;\; R_2(e_2) = 2a_1 e_2.$$

■ The pre-Lie superalgebra $(B_{2,2}, \circ)$ (associative)

$$e_1 \circ e_1 = e_1, \;\; e_1 \circ e_2 = e_2, \;\; e_2 \circ e_1 = e_2, \;\; e_2 \circ e_2 = \frac{1}{2} e_1.$$

Rota–Baxter operators $\mathrm{RB}(B_{2,2})$ is:

$$R_1(e_1) = 0, \;\; R_1(e_2) = 0.$$

3.3 Rota–Baxter Operators on Pre-Lie Superalgebras of Type B_3

■ The pre-Lie superalgebra $(B_{3,1}, \circ)$ (associative).

$$e_i \circ e_j = 0, \quad \forall\ i, j = 1, 2, 3.$$

Rota–Baxter operators $\mathrm{RB}(B_{3,1})$ are:

$R_1(e_1) = a_1 e_1,\ R_1(e_2) = a_2 e_2.$

■ The pre-Lie superalgebra $(B_{3,2}, \circ)$ (associative)

$$e_1 \circ e_1 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(B_{3,2})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1 e_2.$

■ The pre-Lie superalgebra $(B_{3,3}, \circ)$ (associative)

$$e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_2 \circ e_1 = e_2.$$

Rota–Baxter operator $\mathrm{RB}(B_{3,3})$ is:

$R_1(e_1) = 0,\ R_1(e_2) = 0.$

4 Rota–Baxter Operators on Three-dimensional Pre-Lie Superalgebras

4.1 Rota–Baxter Operators on Three-dimensional Pre-Lie Superalgebras with Two-dimensional Odd Part

We still work over the ground field $\mathbb{C}$ of complex numbers. Using the classification of the three-dimensional pre-Lie superalgebras with one-dimensional even part was given by Zhang and Bai (see [15]). The purpose of this section is to provide, using Definition 1.6, all Rota–Baxter operators (of weight zero) on these pre-Lie superalgebras by direct computation. In the following, let $\{e_1, e_2, e_3\}$ be a homogeneous basis of a pre-Lie superalgebra $(\mathcal{A}, \circ)$, where $\{e_1\}$ is a basis of $\mathcal{A}_0$ and $\{e_2, e_3\}$ is a basis of $\mathcal{A}_1$.

Proposition 4.1 *The Rota–Baxter operators (of weight zero) on three-dimensional pre-Lie superalgebras (associative or non-associative) with two-dimensional odd part of type* $C_1,\ \ C_{2_h},\ \ C_3,\ \ C_4,\ \ C_5$ *and* C_6 *are given as follows:*

4.1.1 Rota–Baxter Operators on Pre-Lie Superalgebras of Type C_1

■ The pre-Lie superalgebra $(C_{1,1}, \circ)$

$$e_2 \circ e_3 = -e_1, \quad e_3 \circ e_1 = e_2, \quad e_3 \circ e_2 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(C_{1,1})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2 + e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2e_2,\ R_2(e_3) = a_1e_2,\ \ a_2 \neq 0.$
$R_3(e_1) = a_3e_1,\ R_3(e_2) = 0,\ R_3(e_3) = a_1e_2.$

■ The pre-Lie superalgebra $\Big((C_{1,2})_k, \circ\Big)$ (associative)

$$e_1 \circ e_3 = ke_2,\ \ e_3 \circ e_1 = (k+1)e_2.$$

Rota–Baxter operators $\mathrm{RB}((C_{1,2})_k)$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_2,\ R_1(e_3) = a_2e_2,\ \ k \neq -1.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2e_2 + a_3e_3,\ \ k = -1.$
$R_3(e_1) = a_4e_1,\ R_3(e_2) = a_1e_2,\ R_3(e_3) = a_2e_2 + \frac{a_1a_4}{a_4-a_1}e_3,\ \ a_1 \neq a_4,\ \ k = -1.$

■ The pre-Lie superalgebra $(C_{1,3}, \circ)$:

$$e_1 \circ e_1 = e_1,\ \ e_3 \circ e_1 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(C_{1,3})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2 + a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_3e_2,\ R_2(e_3) = a_1e_2.$

■ The pre-Lie superalgebra $(C_{1,4}, \circ)$:

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_1 \circ e_3 = e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = e_2 + e_3.$$

Rota–Baxter operators $\mathrm{RB}(C_{1,4})$ are:

$R_1(e_1) = 0,\ R_1(e_3) = 0,\ R_1(e_2) = a_1e_2.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = 0.$

4.1.2 Rota–Baxter Operators on Pre-Lie Superalgebras of Type C_{2_h}

■ The pre-Lie superalgebra $(C_{2_h,1}, \circ)$:

$$e_1 \circ e_1 = (h+1)e_1,\ \ e_2 \circ e_1 = e_2,\ \ e_2 \circ e_3 = -e_1,\ \ e_3 \circ e_1 = he_3,\ \ e_3 \circ e_2 = e_1,\ \ h \in \mathbb{C}.$$

Rota–Baxter operators $\mathrm{RB}(C_{2_h,1})$ are:

▷ **Case 1**: If $h = 0$, we have

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2 + a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_3e_3,\ R_2(e_3) = a_2e_3.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_2e_3.$
$R_4(e_1) = 0,\ R_4(e_2) = a_3e_3,\ R_4(e_3) = a_2e_3,\ \ a_3 \neq 0.$

▷ **Case 2**: If $h \in \mathbb{C}^*$, we have

$R_5(e_1) = 0,\ R_5(e_2) = 0,\ R_5(e_3) = a_1e_2.$

$R_6(e_1) = 0,\ R_6(e_2) = a_3e_3,\ R_6(e_3) = 0,\ \ a_3 \neq 0.$
$R_7(e_1) = 0,\ R_7(e_2) = a_5e_2 + a_3e_3,\ R_7(e_3) = -\frac{a_5^2}{ha_3}e_2 - \frac{a_5}{h}e_3,\ \ a_3 \neq 0,\ \ a_5 \neq 0.$
$R_8(e_1) = 0,\ R_8(e_2) = 0,\ R_8(e_3) = 0.$

▷ **Case 3**: If $h = -1$, we have

$R_9(e_1) = 0,\ R_9(e_2) = a_3e_3,\ R_9(e_3) = 0.$
$R_{10}(e_1) = 0,\ R_{10}(e_2) = a_5e_2 + a_3e_3,\ R_{10}(e_3) = \frac{a_5^2}{a_3}e_3 + a_5e_3,\ \ a_3 \neq 0.$
$R_{11}(e_1) = a_4e_1,\ R_{11}(e_2) = 0,\ R_{11}(e_3) = a_1e_2,\ \ a_4 \neq 0.$
$R_{12}(e_1) = a_4e_1,\ R_{12}(e_2) = 0,\ R_{12}(e_3) = 0,\ \ a_4 \neq 0.$

■ The pre-Lie superalgebra $(C_{2_h,2}, \circ)$:

$$e_1 \circ e_1 = (1-h)e_1,\ \ e_1 \circ e_3 = e_2,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = e_2 + he_3,\ \ h \in \mathbb{C}.$$

Rota–Baxter operators $\mathrm{RB}(C_{2_h,2})$ are:
▷ **Case 1**: If $h = 0$, we have

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2 + a_2e_3.$

▷ **Case 2**: If $h \in \mathbb{C}^*$, we have

$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_1e_2,\ \ a_1 \neq 0.$
$R_3(e_1) = 0,\ R_3(e_2) = e_2,\ R_3(e_3) = 0.$

▷ **Case 3**: If $h = 1$, we have

$R_4(e_1) = a_3e_1,\ R_4(e_2) = 0,\ R_4(e_3) = a_1e_2,\ \ a_3 \neq 0.$
$R_5(e_1) = 0,\ R_5(e_2) = 0,\ R_5(e_3) = a_1e_2,\ \ a_1 \neq 0.$
$R_6(e_1) = 0,\ R_6(e_2) = e_2,\ R_6(e_3) = 0.$

■ The pre-Lie superalgebra $(C_{2_h,3}, \circ)$:

$$e_1 \circ e_1 = (1-h)e_1,\ \ e_1 \circ e_2 = (1-h)e_2 + e_3,\ \ e_1 \circ e_3 = (1-h)e_3,\ \ e_2 \circ e_1 = (2-h)e_2 + e_3,\ \ e_3 \circ e_1 = e_3,\ \ h \neq 1.$$

Rota–Baxter operators $\mathrm{RB}(C_{2_h,3})$ are:
▷ **Case 1**: If $h = (-1)^{\frac{1}{3}}$ or $h = -(-1)^{\frac{2}{3}}$, we have

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2e_2 + a_3e_3,\ R_2(e_3) = a_2(h-1)e_2 + a_2(h-1)e_3.$

▷ **Case 2**: If $h \neq 1$, we have

$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = 0.$
$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_1e_3,\ \ a_1 \neq 0.$
$R_5(e_1) = 0,\ R_5(e_2) = 0,\ R_5(e_3) = a_1e_3,\ \ h^3 - 2h^2 + 2h - 1 \neq 0.$

■ The pre-Lie superalgebra $(C_{2_h,4}, \circ)$:

$$e_1 \circ e_1 = (h-1)e_1,\ \ e_1 \circ e_2 = e_3,\ \ e_2 \circ e_1 = e_2 + e_3,\ \ e_3 \circ e_1 = he_3,\ \ h \neq \pm 1.$$

Rota–Baxter operators $\mathrm{RB}(C_{2_h,4})$ are:

$R_1(e_1)=0,\ R_1(e_2)=a_1e_3,\ R_1(e_3)=a_2e_3,\ \ h=0.$
$R_2(e_1)=0,\ R_2(e_2)=a_1e_3,\ R_2(e_3)=0,\ \ a_1\neq 0,\ \ h\neq 0.$
$R_3(e_1)=0,\ R_3(e_2)=0,\ R_3(e_3)=0,\ \ h\neq 0.$

■ The pre-Lie superalgebra $(C_{2_h,5},\circ)$:

$$e_1\circ e_1=(1-h)e_1,\ \ e_1\circ e_2=(1-h)e_2+e_3,\ \ e_1\circ e_3=(1-h)e_3,\ \ e_2\circ e_1=(2-h)e_2+e_3,\ \ e_3\circ e_1=e_3,\ \ h\neq\pm 1.$$

Rota–Baxter operators $\mathrm{RB}(C_{2_h,5})$ are:

$R_1(e_1)=0,\ R_1(e_2)=e_2,\ R_1(e_3)=a_1e_2-a_1e_3,\ \ h=0.$
$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=0,\ \ h\neq 0.$
$R_3(e_1)=0,\ R_3(e_2)=a_2e_2-\frac{a_3}{(h-1)^2}e_3,\ R_3(e_3)=\frac{a_2(h-1)^2}{h}e_2-\frac{a_2}{h}e_3,\ \ a_2\neq 0,\ \ h\neq 0.$
$R_4(e_1)=0,\ R_4(e_2)=a_2e_2-a_2e_3,\ R_4(e_3)=\frac{a_2}{2}e_2-\frac{a_2}{2}e_3,\ \ a_2\neq 0,\ \ h=2.$
$R_5(e_1)=0,\ R_5(e_2)=a_3e_3,\ R_5(e_3)=0,\ \ a_3\neq 0.$

■ The pre-Lie superalgebra $\Big((C_{2_h,6})_k,\circ\Big)$: $h=0\ \ or\ \ 1,\ \ k=1$ associative other cases non-associative.

$$e_1\circ e_1=ke_1,\ \ e_2\circ e_1=e_2,\ \ e_3\circ e_1=he_3,\ \ h,k\in\mathbb{C}.$$

Rota–Baxter operators $\mathrm{RB}((C_{2_h,6})_k)$ are:

▷ **Case 1**: If $h=0$, we have

$R_1(e_1)=0,\ R_1(e_2)=0,\ R_1(e_3)=a_1e_2+a_2e_3.$
$R_2(e_1)=0,\ R_2(e_2)=a_3e_3,\ R_2(e_3)=a_2e_3.$
$R_3(e_1)=0,\ R_3(e_2)=0,\ R_3(e_3)=a_2e_3.$

▷ **Case 2**: If $h\in\mathbb{C}^*$, we have

$R_4(e_1)=0,\ R_4(e_2)=0,\ R_4(e_3)=a_1e_2.$
$R_5(e_1)=0,\ R_5(e_2)=a_3e_3,\ R_5(e_3)=0.$
$R_6(e_1)=0,\ R_6(e_2)=a_4e_2+a_3e_3,\ R_6(e_3)=-\frac{a_4^2}{ha_3}e_2-\frac{a_4}{h}e_3,\ \ a_3\neq 0.$
$R_7(e_1)=0,\ R_7(e_2)=0,\ R_7(e_3)=0.$

▷ **Case 3**: If $h=k=0$, we have

$R_8(e_1)=0,\ R_8(e_2)=0,\ R_8(e_3)=a_1e_2+a_2e_3.$
$R_9(e_1)=0,\ R_9(e_2)=a_3e_3,\ R_9(e_3)=a_2e_3.$
$R_{10}(e_1)=a_5e_1,\ R_{10}(e_2)=0,\ R_{10}(e_3)=a_2e_3.$
$R_{11}(e_1)=a_5e_1,\ R_{11}(e_2)=0,\ R_{11}(e_3)=0.$
$R_{12}(e_1)=0,\ R_{12}(e_2)=a_3e_3,\ R_{12}(e_3)=a_2e_3,\ \ a_3\neq 0.$

▷ **Case 4**: If $h=1$ and $k=0$, we have

$R_{13}(e_1)=a_5e_1,\ R_{13}(e_2)=0,\ R_{13}(e_3)=a_1e_2.$
$R_{14}(e_1)=a_5e_1,\ R_{14}(e_2)=a_3e_3,\ R_{14}(e_3)=0.$
$R_{15}(e_1)=a_5e_1,\ R_{15}(e_2)=a_4e_2+a_3e_3,\ R_{15}(e_3)=-\frac{a_4^2}{a_3}e_2-a_4e_3,\ \ a_3\neq 0.$

▷ **Case 5**: If $h \neq 0, 1$ and $k = 0$, we have

$$R_{16}(e_1) = 0,\ R_{16}(e_2) = 0,\ R_{16}(e_3) = a_1 e_2,\ \ a_1 \neq 0.$$
$$R_{17}(e_1) = 0,\ R_{17}(e_2) = a_3 e_3,\ R_{17}(e_3) = 0,\ \ r_{2,2} \neq 0.$$
$$R_{18}(e_1) = 0,\ R_{18}(e_2) = a_4 e_2 + a_3 e_3,\ R_{18}(e_3) = -\frac{a_4^2}{h a_3} e_2 + \frac{a_4}{h} e_3,\ \ a_3 \neq 0.$$

■ The pre-Lie superalgebra $\left((C_{2_h,7})_k, \circ\right)$: $(C_{2_0,7})_1$ associative.

$$e_1 \circ e_1 = k e_1,\ e_1 \circ e_2 = k e_2,\ e_2 \circ e_1 = (k+1) e_2,\ e_3 \circ e_1 = h e_3,\ ,\ k \neq 0,\ h \neq \pm 1.$$

Rota–Baxter operators $\mathrm{RB}((C_{2_h,7})_k)$ are:
▷ **Case 1**: If $h = 0$, we have

$$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1 e_2 + a_2 e_3,\ \ a_1 \neq 0.$$
$$R_2(e_1) = 0,\ R_2(e_2) = a_3 e_3,\ R_2(e_3) = a_2 e_3,\ \ a_3 \neq 0.$$
$$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_2 e_3.$$

▷ **Case 2**: If $h \in \mathbb{C}^*$, we have

$$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_1 e_2,\ \ a_1 \neq 0.$$
$$R_5(e_1) = 0,\ R_5(e_2) = a_3 e_3,\ R_5(e_3) = 0,\ \ a_3 \neq 0.$$
$$R_6(e_1) = 0,\ R_6(e_2) = 0,\ R_6(e_3) = 0.$$

■ The pre-Lie superalgebra $\left((C_{2_h,8})_k, \circ\right)$: $h = -1\ \ or\ \ 0,\ \ k = 1$ associative other cases are non-associative

$$e_1 \circ e_1 = k e_1,\ \ e_1 \circ e_3 = k e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = (h+k) e_3,\ \ h \in \mathbb{C},\ \ k \neq 0.$$

Rota–Baxter operators $\mathrm{RB}((C_{2_h,8})_k)$ are:

$$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1 e_2.$$
$$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = 0.$$

■ The pre-Lie superalgebra $\left((C_{2_h)_k,9}, \circ\right)$: $h = 0\ \ or\ \ 1,\ \ k = -1$ associative other cases are non-associative

$$e_1 \circ e_1 = k e_1,\ \ e_1 \circ e_2 = k e_2,\ \ e_1 \circ e_3 = k e_3,\ \ e_2 \circ e_1 = (k+1) e_2,\ \ e_3 \circ e_1 = (h+k) e_3,\ \ h \in \mathbb{C},\ \ k \neq 0.$$

Rota–Baxter operators $\mathrm{RB}((C_{2_h,9})_k)$ are:

$$R_1(e_1) = 0,\ R_1(e_2) = a_1 e_2 + a_2 e_3,\ R_1(e_3) = -\frac{a_1^2}{a_2} e_2 - a_1 e_3,\ \ a_2 \neq 0,\ \ h = 1,\ \ k = -1.$$
$$R_2(e_1) = 0,\ R_2(e_2) = a_2 e_3,\ R_2(e_3) = 0.$$
$$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_3 e_2.$$
$$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = 0.$$

4.1.3 Rota–Baxter Operators on Pre-Lie Superalgebras of Type C_3

■ The pre-Lie superalgebra $(C_{3,1}, \circ)$:

$$e_1 \circ e_1 = 2e_1, \quad e_2 \circ e_1 = e_2, \quad e_2 \circ e_3 = -e_1, \quad e_3 \circ e_1 = e_2 + e_3, \quad e_3 \circ e_2 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(C_{3,1})$ are:

$$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2.$$
$$R_2(e_1) = 0,\ R_2(e_2) = a_2e_1+a_3e_2,\ R_2(e_3) = -\frac{a_2^2+a_2a_3}{2}e_2-(a_2+a_3)e_3,\ a_3 \neq 0.$$

■ The pre-Lie superalgebra $\Big((C_{3,2})_k, \circ\Big)$:

$$e_1 \circ e_1 = ke_1, \quad e_2 \circ e_1 = e_2, \quad e_3 \circ e_1 = e_2 + e_3.$$

Rota–Baxter operators $\mathrm{RB}((C_{3,2})_k)$ are:
▹ **Case 1**: If $k = 0$, we have

$$R_1(e_1) = a_1e_1,\ R_1(e_2) = 0,\ R_1(e_3) = a_2e_2,\ \ a_1 \neq 0.$$
$$R_2(e_1) = a_1e_1,\ R_2(e_2) = 0,\ R_2(e_3) = a_1e_2,\ \ a_1 \neq 0.$$
$$R_3(e_1) = a_1e_1,\ R_3(e_2) = 0,\ R_3(e_3) = a_2e_2,\ \ a_1 \neq 0.$$

▹ **Case 2**: If $k \in \mathbb{C}^*$, we have

$$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_2e_2.$$
$$R_5(e_1) = 0,\ R_5(e_2) = a_3e_2+a_4e_3,\ R_5(e_3) = -\frac{a_3^2+a_3a_4}{a_4}e_2-(a_3+a_4)e_3,\ a_4 \neq 0.$$
$$R_6(e_1) = 0,\ R_6(e_2) = 0,\ R_6(e_3) = 0.$$

■ The pre-Lie superalgebra $\Big((C_{3,3})_k, \circ\Big)$:

$$e_1 \circ e_3 = ke_2, \quad e_2 \circ e_1 = e_2, \quad e_3 \circ e_1 = (k+1)e_2 + e_3, \quad k \neq 0.$$

Rota–Baxter operators $\mathrm{RB}((C_{3,3})_k)$ are:

$$R_1(e_1) = a_1e_1,\ R_1(e_2) = 0,\ R_1(e_3) = a_2e_2.$$
$$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2e_2.$$
$$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = 0.$$

■ The pre-Lie superalgebra $\Big((C_{3,4})_k, \circ\Big)$:

$$e_1 \circ e_1 = ke_1, \quad e_1 \circ e_2 = ke_2, \quad e_1 \circ e_3 = ke_3, \quad e_2 \circ e_1 = (k+1)e_2, \quad e_3 \circ e_1 = e_2 + (k+1)e_3, \quad k \neq 0.$$

Rota–Baxter operator $\mathrm{RB}((C_{3,4})_k)$ is:

$$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2.$$

4.1.4 Rota–Baxter Operators on Pre-Lie Superalgebras of Type C_4

■ The pre-Lie superalgebra $(C_{4,1}, \circ)$: (associative)

$$e_2 \circ e_3 = -e_1, \quad e_3 \circ e_2 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(C_{4,1})$ are:

$R_1(e_1) = a_1 e_1,\ R_1(e_2) = a_2 e_2 + a_3 e_3,\ R_1(e_3) = a_4 e_2 - \frac{a_1 a_2 + a_3 a_4}{a_1 - a_2} e_3,\ \ a_1 \neq a_2.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_4 e_2 + a_5 e_3.$
$R_3(e_1) = a_1 e_1,\ R_3(e_2) = a_1 e_2 + a_3 e_3,\ R_3(e_3) = -\frac{a_1^2}{a_3} e_2 + a_5 e_3,\ \ a_3 \neq 0.$

■ The pre-Lie superalgebra $(C_{4,2}, \circ)$: (associative).

$$e_i \circ e_j = 0, \quad \forall\ i, j = 1, 2, 3.$$

Rota–Baxter operator $\mathrm{RB}(C_{4,2})$ is:

$R_1(e_1) = a_1 e_1,\ R_1(e_2) = a_2 e_2 + a_3 e_3,\ R_1(e_3) = a_4 e_2 + a_5 e_3.$

■ The pre-Lie superalgebra $(C_{4,3}, \circ)$: (associative)

$$e_1 \circ e_1 = e_1.$$

Rota–Baxter operator $\mathrm{RB}(C_{4,3})$ is:

$R_1(e_1) = 0,\ R_1(e_2) = a_1 e_2 + a_2 e_3,\ R_1(e_3) = a_3 e_2 + a_4 e_3.$

■ The pre-Lie superalgebra $(C_{4,4}, \circ)$: (associative)

$$e_1 \circ e_1 = e_1, \quad e_1 \circ e_3 = e_3, \quad e_3 \circ e_1 = e_3.$$

Rota–Baxter operators $\mathrm{RB}(C_{4,4})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1 e_2,\ R_1(e_3) = a_2 e_2.$
$R_2(e_1) = 0,\ R_2(e_2) = a_1 e_2 + a_3 e_3,\ R_2(e_3) = 0.$
$R_3(e_1) = 0,\ R_3(e_2) = a_1 e_2,\ R_3(e_3) = 0.$

■ The pre-Lie superalgebra $(C_{4,5}, \circ)$: (associative)

$$e_1 \circ e_1 = e_1, \quad e_1 \circ e_2 = e_2, \quad e_1 \circ e_3 = e_3, \quad e_2 \circ e_1 = e_2, \quad e_3 \circ e_1 = e_3.$$

Rota–Baxter operators $\mathrm{RB}(C_{4,5})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1 e_2.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2 e_2 + a_3 e_3,\ R_2(e_3) = -\frac{a_2^2}{a_3} e_2 - a_2 e_3,\ \ a_3 \neq 0.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = 0.$

■ The pre-Lie superalgebra $(C_{4,6}, \circ)$: (associative)

$$e_1 \circ e_3 = e_2, \quad e_3 \circ e_1 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(C_{4,6})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1e_2 + a_2e_3.$
$R_2(e_1) = a_3e_1,\ R_2(e_2) = a_4e_2,\ R_2(e_3) = a_1e_2 + \frac{a_3a_4}{a_3-a_4}e_3,\ \ a_3 \neq a_4.$

4.1.5 Rota–Baxter Operators on Pre-Lie Superalgebras of Type C_5

■ The pre-Lie superalgebra $\Big((C_{5,1})_k, \circ\Big)$: $((C_{5,1})_0$ is associative).

$$e_1 \circ e_2 = ke_3,\ \ e_2 \circ e_1 = ke_3,\ \ e_3 \circ e_3 = e_1,\ \ k = 0\ \ or\ \ 1.$$

Rota–Baxter operators $\mathrm{RB}((C_{5,1})_k)$ are:
▹ **Case 1**: If $k = 0$, we have

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_3,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_1e_3,\ R_2(e_3) = a_3e_2.$
$R_3(e_1) = 0,\ R_3(e_2) = a_4e_2,\ R_3(e_3) = a_3e_2.$
$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_2e_3.$
$R_5(e_1) = a_5e_1,\ R_5(e_2) = a_4e_2,\ R_5(e_3) = \frac{a_5a_4}{a_5-a_4}e_3,\ \ a_4 \neq a_5.$

▹ **Case 2**: If $k = 1$, we have

$R_6(e_1) = 0,\ R_6(e_2) = a_1e_3,\ R_6(e_3) = a_2e_3.$
$R_7(e_1) = 0,\ R_7(e_2) = 0,\ R_7(e_3) = a_2e_3.$
$R_8(e_1) = 0,\ R_8(e_2) = a_1e_3,\ R_8(e_3) = e_3.$
$R_9(e_1) = a_5e_1,\ R_9(e_2) = a_5e_2,\ R_9(e_3) = \frac{a_5}{2}e_3,\ \ a_5 \neq 0.$
$R_{10}(e_1) = a_5e_1,\ R_{10}(e_2) = \frac{a_5}{a_5-1}e_2,\ R_{10}(e_3) = e_3,\ \ a_5 \neq 1.$

■ The pre-Lie superalgebra $(C_{5,2}, \circ)$:

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_2 \circ e_1 = e_2,\ \ e_2 \circ e_3 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(C_{5,2})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_3,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2e_3.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_3e_2.$

■ The pre-Lie superalgebra $(C_{5,3}, \circ)$:

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2 + e_3,\ \ e_2 \circ e_1 = e_2 + e_3,\ \ e_2 \circ e_3 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(C_{5,3})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_3,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2e_3.$
$R_3(e_1) = 0,\ R_3(e_2) = a_3e_2,\ R_3(e_3) = -a_3e_2.$

■ The pre-Lie superalgebra $\Big((C_{5,4})_k, \circ\Big)$: (associative)

$$e_2 \circ e_3 = \left(\frac{1}{2} + k\right)e_1,\ \ e_3 \circ e_2 = \left(\frac{1}{2} - k\right)e_1,\ \ k \geq 0,\ \ k \neq \frac{1}{2}.$$

Rota–Baxter operators $RB((C_{5,4})_k)$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_3,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_3e_2,\ R_2(e_3) = a_4e_2.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_2e_3.$
$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = a_4e_2.$
$R_5(e_1) = 0,\ R_5(e_2) = a_1e_3,\ R_5(e_3) = 0,\ \ a_1 \neq 0.$
$R_6(e_1) = 0,\ R_6(e_2) = a_3e_2,\ R_6(e_3) = 0,\ \ a_3 \neq 0.$
$R_7(e_1) = a_5e_1,\ R_7(e_2) = 0,\ R_7(e_3) = 0.$
$R_8(e_1) = a_5e_1,\ R_8(e_2) = a_5e_2 + a_1e_3,\ R_8(e_3) = \frac{a_5^2}{a_1}e_2 + a_5e_3,\ \ a_1 \neq 0,\ \ a_5 \neq 0,\ \ k = 0.$
$R_9(e_1) = a_5e_1,\ R_9(e_2) = a_3e_2,\ R_9(e_3) = -\frac{a_5a_3}{a_5-a_3}e_3,\ \ a_3 \neq a_5.$

4.1.6 Rota–Baxter Operators on Pre-Lie Superalgebras of TypeC_6

■ The pre-Lie superalgebra $(C_{6,1}, \circ)$ (associative)

$$e_2 \circ e_2 = \frac{1}{2}e_1,\ \ e_2 \circ e_3 = -e_1,\ \ e_3 \circ e_2 = e_1.$$

Rota–Baxter operators $RB(C_{6,1})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_3,\ R_1(e_3) = a_2e_3,\ \ a_1 \neq 0.$
$R_2(e_1) = a_3e_1,\ R_2(e_2) = a_1e_3,\ R_2(e_3) = 0.$

■ The pre-Lie superalgebra $(C_{6,2}, \circ)$: (associative)

$$e_2 \circ e_2 = \frac{1}{2}e_1.$$

Rota–Baxter operators $RB(C_{6,2})$ are:

$R_1(e_1) = a_1e_1,\ R_1(e_2) = a_2e_3,\ R_1(e_3) = a_3e_3.$
$R_2(e_1) = a_1e_1,\ R_2(e_2) = 2a_1e_2 + a_2e_3,\ R_2(e_3) = a_3e_3.$

■ The pre-Lie superalgebra $(C_{6,3}, \circ)$ (associative)

$$e_1 \circ e_2 = e_3,\ \ e_2 \circ e_1 = e_3,\ \ e_2 \circ e_2 = \frac{1}{2}e_1.$$

Rota–Baxter operators $RB(C_{6,3})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_2,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = a_3e_1,\ R_2(e_2) = a_1e_3,\ R_2(e_3) = 0,\ \ a_3 \neq 0.$
$R_3(e_1) = a_3e_1,\ R_3(e_2) = 2a_3e_2 + a_1e_3,\ R_3(e_3) = \frac{2a_3}{3}e_3.$

■ The pre-Lie superalgebra $(C_{6,4}, \circ)$: (associative)

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_2 \circ e_1 = e_2,\ \ \ e_2 \circ e_2 = \frac{1}{2}e_1.$$

Rota–Baxter operators $RB(C_{6,4})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1 e_3,\ R_1(e_3) = a_2 e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2 e_3.$

4.2 Classification of Rota–Baxter Operator on Three-dimensional Pre-Lie Superalgebras with Two-Dimensional Even Part

In this section, we describe all Rota–Baxter operators of weight zero on the three-dimensional complex pre-Lie superalgebras with two-dimensional even part which were classified in [15] by Zhang and Bai. In the following, let $\{e_1, e_2, e_3\}$ be a homogeneous basis of a pre-Lie superalgebra $(\mathcal{A}, \circ)$, where $\{e_1, e_2\}$ is a basis of $\mathcal{A}_0$ and $\{e_3\}$ is a basis of $\mathcal{A}_1$. The computation is obtained using computer algebra system, and the operators are described with respect to the basis.

Proposition 4.2 *The Rota–Baxter operators (of weight zero) on three-dimensional pre-Lie superalgebras (associative or non-associative) with two-dimensional even part of type* $\widehat{A}_1,\ \widehat{A}_2,\ \widehat{A}_3,\ \widehat{A}_4,\ \widehat{A}_5,\ \widehat{A}_6,\ \widehat{A}_{7_h},\ \widehat{A}_8,\ \widehat{A}_9,\ \widehat{A}_{10_h}$ *and* $\widehat{A}_{11}$ *are given as follows:*

4.2.1 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_1$

■ The pre-Lie superalgebra $(\widehat{A}_{1,1}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_1,\quad e_2 \circ e_2 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{1,1})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = a_1 e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2} e_2,\ R_2(e_3) = a_1 e_3.$
$R_3(e_1) = 0,\ R_3(e_2) = e_1 + \frac{1}{2} e_2,\ R_3(e_3) = a_1 e_3.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{1,2})_k, \circ\right) \simeq D_1$, $(\widehat{A}_{1,2})_1$ associative.

$$e_1 \circ e_1 = e_1,\quad e_2 \circ e_2 = e_2,\quad e_3 \circ e_1 = k e_3,\quad k \in \mathbb{C}^*.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{1,2})_k)$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2} e_2,\ R_2(e_3) = 0.$
$R_3(e_1) = 0,\ R_3(e_2) = e_1 + \frac{1}{2} e_2,\ R_3(e_3) = 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{1,3})_{k_1,k_2}, \circ\right) \simeq D_1$.

$$e_1 \circ e_1 = e_1,\quad e_2 \circ e_2 = e_2,\quad e_3 \circ e_1 = k_1 e_3,\quad e_3 \circ e_2 = k_2 e_3,\quad k_1, k_2 \neq 0,\quad k_1 \leq k_2,\quad k_1 \neq -k_2.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{1,3})_{k_1,k_2})$ are:

If $\left(k_2 < 0, \quad k_1 \le k_2\right)$ or $\left((k_2 > 0, \quad k_1 < -k_2) \ or \ (-k_2 < k_1 < 0) \ or \ (0 < k_1 \le k_2)\right)$

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{1,4}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_1, \ \ e_2 \circ e_2 = e_2, \ \ \ e_2 \circ e_3 = e_3, \ \ e_3 \circ e_2 = e_3.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{1,4})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2}e_2,\ R_2(e_3) = 0.$
$R_3(e_1) = 0,\ R_3(e_2) = e_1 + \frac{1}{2}e_2,\ R_3(e_3) = 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{1,5})_{k_1,k_2}, \circ\right) \simeq D_1, k_1 = k_2 = 0 \ or \ k_1 = 1,\ k_2 = 0$ associative other cases non-associative.

$$e_1 \circ e_1 = e_1, \ \ e_2 \circ e_2 = e_2, \ \ \ e_2 \circ e_3 = e_3, \ \ e_3 \circ e_1 = k_1 e_3, \ \ e_3 \circ e_2 = k_2 e_3, \ \ k_1 \neq 0 \ \ or \ \ k_2 \neq 1.$$

Rota–Baxter operator $\mathrm{RB}((\widehat{A}_{1,5})_{k_1,k_2})$ is:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{1,6}, \circ) \simeq D_2$ (associative).

$$e_1 \circ e_1 = e_1, \ \ e_2 \circ e_2 = e_2, \ \ \ , e_2 \circ e_3 = e_3, \ \ e_3 \circ e_2 = e_3, \ \ e_3 \circ e_3 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{1,6})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2}e_2,\ R_2(e_3) = 0.$
$R_3(e_1) = 0,\ R_3(e_2) = e_1 + \frac{1}{2}e_2,\ R_3(e_3) = 0.$

4.2.2 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_2$

■ The pre-Lie superalgebra $(\widehat{A}_{2,1}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_1, \ \ e_1 \circ e_2 = e_2, \ \ \ e_2 \circ e_1 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{2,1})$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = a_2 e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2}e_1,\ R_2(e_3) = a_2 e_3.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{2,2})_k, \circ\right) \simeq D_1$, $(\widehat{A}_{2,2})_1$ is associative.

$$e_1 \circ e_1 = e_1, \ \ e_1 \circ e_2 = e_2, \ \ \ e_2 \circ e_1 = e_1, \ \ e_3 \circ e_1 = k e_3, \ \ k \neq 0.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{2,2})_k)$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = a_2 e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2} e_1,\ R_2(e_3) = a_2 e_3.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{2,3})_k, \circ\right) \simeq D_1$

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = k e_3,\ \ e_3 \circ e_2 = e_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{2,3})_k)$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = a_2 e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = \frac{1}{2} e_1,\ R_2(e_3) = 0.$
$R_3(e_1) = a_1 e_2,\ R_3(e_2) = 0,\ R_3(e_3) = a_2 e_3,\ \ k = 1.$
$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = 0,\ \ k \neq 1.$

■ The pre-Lie superalgebra $(\widehat{A}_{2,4}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_1 \circ e_3 = e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = e_3.$$

Rota–Baxter operator $\mathrm{RB}(\widehat{A}_{2,4})$ is:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{2,5})_k, \circ\right) \simeq D_1$, $(\widehat{A}_{2,5})_0$ is associative.

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_1 \circ e_3 = e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = k e_3,\ \ k \neq 1.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{2,5})_k)$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = a_2 e_3,\ \ k = 0.$
$R_2(e_1) = a_1 e_2,\ R_2(e_2) = 0,\ R_2(e_3) = 0,\ \ k \neq 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{2,6})_k, \circ\right) \simeq D_1$:

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_1 \circ e_3 = e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = k e_3,\ \ e_3 \circ e_2 = e_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{2,6})_k)$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{2,7}, \circ) \simeq D_2$ (associative).

$$e_1 \circ e_1 = e_1,\ \ e_1 \circ e_2 = e_2,\ \ e_1 \circ e_3 = e_3,\ \ e_2 \circ e_1 = e_2,\ \ e_3 \circ e_1 = e_3,\ \ e_3 \circ e_3 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{2,7})$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$

4.2.3 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_3$

■ The pre-Lie superalgebra $(\widehat{A}_{3,1}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_1.$$

Rota–Baxter operators RB$(\widehat{A}_{3,1})$ are:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = a_2 e_2,\ R_1(e_3) = a_3 e_3.$

■ The pre-Lie superalgebra $(\widehat{A}_{3,4}, \circ) \simeq D_2$: (associative).

$$e_1 \circ e_1 = e_1, \quad e_3 \circ e_3 = e_2.$$

Rota–Baxter operators RB$(\widehat{A}_{3,4})$ are

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = a_2 e_2,\ R_1(e_3) = 0.$
$R_2(e_1) = a_1 e_2,\ R_2(e_2) = a_2 e_2,\ R_2(e_3) = 2a_2 e_3.$

■ The pre-Lie superalgebras of type

$\left((\widehat{A}_{3,2})_k, \circ\right) \simeq D_1$: $e_1 \circ e_1 = e_1,\ e_3 \circ e_1 = k e_3,\ k \neq 0,$ $((\widehat{A}_{3,2})_1$ is associative).
$((\widehat{A}_{3,3})_k, \circ) \simeq D_1$: $e_1 \circ e_1 = e_1,\ e_3 \circ e_1 = k e_3,\ e_3 \circ e_2 = e_3.$
$(\widehat{A}_{3,5}, \circ) \simeq D_3$: $e_1 \circ e_1 = e_1,\ e_1 \circ e_3 = e_3,\ e_3 \circ e_1 = e_3,$ (associative).
$\left((\widehat{A}_{3,6})_k, \circ\right) \simeq D_1$: $e_1 \circ e_1 = e_1,\ e_1 \circ e_3 = e_3,\ e_3 \circ e_1 = k e_3,\ k \neq 1,$
$((\widehat{A}_{3,6})_0$ is associative).
$\left((\widehat{A}_{3,7})_k, \circ\right) \simeq D_1$: $e_1 \circ e_1 = e_1,\ e_1 \circ e_3 = e_3,\ e_3 \circ e_1 = k e_3,\ e_3 \circ e_2 = e_3.$
$(\widehat{A}_{3,8}, \circ) \simeq D_2$: $e_1 \circ e_1 = e_1,\ e_1 \circ e_3 = e_3,\ e_3 \circ e_1 = e_3,\ e_3 \circ e_3 = e_1,$
(associative).

They have the same Rota–Baxter operators:

$R_1(e_1) = a_1 e_2,\ R_1(e_2) = a_2 e_2,\ R_1(e_3) = 0.$

4.2.4 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_4$

■ The pre-Lie superalgebra $(\widehat{A}_{4,1}, \circ) \simeq D_3$ (associative).

$$e_i \circ e_j = 0, \quad \forall\ i, j = 1, 2, 3.$$

Rota–Baxter operators RB$(\widehat{A}_{4,1})$ are:

$$R_1(e_1) = a_1 e_1 + a_2 e_2,\ R_1(e_2) = a_3 e_1 + a_4 e_2,\ R_1(e_3) = a_5 e_3.$$

■ The pre-Lie superalgebra $(\widehat{A}_{4,2}, \circ) \simeq D_1$:

$$e_3 \circ e_1 = e_3.$$

Rota–Baxter operators RB$(\widehat{A}_{4,2})$ are:

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = a_3e_1 + a_4e_2,\ R_1(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{4,3}, \circ) \simeq D_2$: (associative)

$$e_3 \circ e_3 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{4,3})$ are:

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = a_3e_2,\ R_1(e_3) = 2a_3e_3.$
$R_2(e_1) = a_1e_1 + a_2e_2,\ R_2(e_2) = a_4e_1 + a_3e_2,\ R_2(e_3) = 0.$

4.2.5 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_5$

■ The pre-Lie superalgebra $(\widehat{A}_{5,1}, \circ) \simeq D_2$ (associative)

$$e_1 \circ e_1 = e_2, \quad e_3 \circ e_3 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{5,1})$ are:

$R_1(e_1) = a_1e_2,\ R_1(e_2) = a_2e_2,\ R_1(e_3) = 0.$
$R_2(e_1) = a_1e_2,\ R_2(e_2) = a_2e_2,\ R_2(e_3) = 2a_2e_3.$
$R_3(e_1) = a_3e_1 + a_1e_2,\ R_3(e_2) = \frac{a_3}{2}e_2,\ R_3(e_3) = 0.$
$R_4(e_1) = a_3e_1 + a_1e_2,\ R_4(e_2) = \frac{a_3}{2}e_2,\ R_4(e_3) = a_3e_3.$

■ The pre-Lie superalgebra $(\widehat{A}_{5,2}, \circ) \simeq D_3$ (associative).

$$e_1 \circ e_1 = e_2.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{5,2})$ are:

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = \frac{a_1}{2}e_2,\ R_1(e_3) = a_3e_3.$
$R_2(e_1) = a_2e_2,\ R_2(e_2) = a_4e_2,\ R_2(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{5,3}, \circ) \simeq D_1$:

$$e_1 \circ e_1 = e_2, \quad e_3 \circ e_1 = e_3.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{5,3})$ are:

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = \frac{a_1}{2}e_2,\ R_1(e_3) = 0.$
$R_2(e_1) = a_2e_2,\ R_2(e_2) = a_3e_2,\ R_2(e_3) = 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{5,4}, \circ) \simeq D_1$:

$$e_1 \circ e_1 = e_2\ , \quad e_3 \circ e_2 = e_3.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{5,4})$ are:

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = \frac{a_1}{2}e_2,\ R_1(e_3) = 0.$
$R_2(e_1) = a_2e_2,\ R_2(e_2) = a_3e_2,\ R_2(e_3) = 0.$

4.2.6 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_6$

■ The pre-Lie superalgebra $\left((\widehat{A}_{6,1})_k, \circ\right) \simeq (D_4)_\mu$: $k = 0$ *or* -1 associative other cases non-associative

$$e_1 \circ e_2 = -e_1, \quad e_2 \circ e_2 = -e_2, \quad e_3 \circ e_2 = ke_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{6,1})_k)$ are:
▹ **Case 1**: If $k = 0$, we have

$R_1(e_1) = a_1e_2,\ R_1(e_2) = 0,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = a_3e_1,\ R_2(e_3) = a_2e_3.$

▹ **Case 2**: If $k \in \mathbb{C}^*$, we have

$R_4(e_1) = 0,\ R_4(e_2) = a_3e_1,\ R_4(e_3) = 0.$
$R_5(e_1) = a_1e_2,\ R_5(e_2) = 0,\ R_5(e_3) = 0.$
$R_5(e_1) = a_1e_2,\ R_5(e_2) = 0,\ R_5(e_3) = 0,\ \ a_1 \neq 0.$

■ The pre-Lie superalgebra $(\widehat{A}_{6,2}, \circ) \simeq D_5$:

$$e_1 \circ e_2 = -e_1, \quad e_2 \circ e_2 = -e_2, \quad e_3 \circ e_2 = -\frac{1}{2}e_3, \quad e_3 \circ e_3 = e_1.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{6,2})$ are:

$R_1(e_1) = a_1e_2,\ R_1(e_2) = 0,\ R_1(e_3) = 0.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2e_1,\ R_2(e_3) = 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{6,3})_k, \circ\right) \simeq (D_4)_\mu$: $k = 0$ *or* -1 associative other cases are non-associative

$$e_1 \circ e_2 = -e_1, \quad e_2 \circ e_2 = -e_2, \quad e_2 \circ e_3 = -e_3, \quad e_3 \circ e_2 = ke_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{6,3})_k)$ are:

$R_1(e_1) = a_1e_2,\ R_1(e_2) = 0,\ R_1(e_3) = 0,\ \ a_1 \neq 0.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2e_1,\ R_2(e_3) = 0.$

4.2.7 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_{7_h}$

■ The pre-Lie superalgebra $(\widehat{A}_{7_h,1}, \circ) \simeq D_5$:

$$e_1 \circ e_2 = -e_1, \quad e_2 \circ e_2 = he_2, \quad e_3 \circ e_2 = -\frac{1}{2}e_3, \quad e_3 \circ e_3 = e_1, \quad h \neq -1.$$

Rota–Baxter operators $\mathrm{RB}(\widehat{A}_{7_h,1})$ are:

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_1,\ R_1(e_3) = 0.$
$R_2(e_1) = 0,\ R_2(e_2) = a_2e_2,\ R_2(e_3) = 0,\ \ h = 0.$

$R_3(e_1)=0,\ R_3(e_2)=0,\ R_3(e_3)=0,\ \ h\neq 0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{7_h,2})_k,\circ\right)\simeq(D_4)_\mu$:

$$e_1\circ e_2=-e_1,\ \ e_2\circ e_2=he_2,\ \ \ e_2\circ e_3=he_3,\ \ e_3\circ e_2=ke_3,\ \ h\neq 0,-1.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{7_h,2})_k)$ are:

$R_1(e_1)=0,\ R_1(e_2)=a_1e_1,\ R_1(e_3)=0.$
$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=0.$

■ The pre-Lie superalgebra $\left((\widehat{A}_{7_h,3})_k,\circ\right)\simeq(D_4)_\mu$:

$$e_1\circ e_2=-e_1,\ \ e_2\circ e_2=he_2,\ \ \ e_3\circ e_2=ke_3,\ \ h\neq -1.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{7_h,3})_k)$ are:

$R_1(e_1)=a_1e_2,\ R_1(e_2)=a_2e_1,\ R_1(e_3)=a_3e_3,\ \ h=-\frac{1}{2},\ \ k=0.$
$R_2(e_1)=0,\ R_2(e_2)=a_4e_2,\ R_2(e_3)=0,\ \ h=0.$
$R_3(e_1)=0,\ R_3(e_2)=a_4e_2,\ R_3(e_3)=a_3e_3,\ \ h=k=0.$
$R_4(e_1)=0,\ R_4(e_2)=0,\ R_4(e_3)=a_3e_3,\ \ h\neq 0,\ \ k=0.$
$R_5(e_1)=0,\ R_5(e_2)=a_2e_1,\ R_5(e_3)=0,\ \ h=-\frac{1}{2}.$
$R_6(e_1)=0,\ R_6(e_2)=a_2e_1,\ R_6(e_3)=a_3e_3,\ \ h\neq-\frac{1}{2},\ \ k=0.$
$R_7(e_1)=0,\ R_7(e_2)=0,\ R_7(e_3)=0,\ \ h\neq-\frac{1}{2},0.$

4.2.8 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_8$

■ The pre-Lie superalgebra $\left((\widehat{A}_{8,1})_k,\circ\right)\simeq(D_4)_\mu$:

$$e_1\circ e_1=2e_1,\ \ e_2\circ e_1=e_2,\ \ \ e_2\circ e_2=e_1,\ \ e_3\circ e_1=ke_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{8,1})_k)$ are:

$R_1(e_1)=0,\ R_1(e_2)=0,\ R_1(e_3)=a_1e_3,\ \ k=0.$
$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=0.$

4.2.9 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_9$

■ The pre-Lie superalgebra $\left((\widehat{A}_{9,1})_k,\circ\right)\simeq(D_4)_\mu$: $k=0$ *or* 1 associative others cases are non-associative.

$$e_2\circ e_1=e_1,\ \ e_2\circ e_2=e_2,\ \ \ e_3\circ e_2=ke_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{9,1})_k)$ are:
▷ **Case 1**: If $k=0$, we have

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = -\frac{a_1^2}{a_2}e_1 - a_1e_2,\ R_1(e_3) = a_3e_3,\quad a_2 \neq 0.$
$R_2(e_1) = 0,\ R_2(e_2) = a_4e_1,\ R_2(e_3) = a_3e_3.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = a_3e_3.$

▹ **Case 2**: If $k \in \mathbb{C}^*$, we have

$R_4(e_1) = 0,\ R_4(e_2) = a_4e_1,\ R_4(e_3) = 0.$
$R_5(e_1) = a_1e_1 + a_2e_2,\ R_5(e_2) = -\frac{a_1^2}{a_2}e_1 - a_1e_2,\ R_5(e_3) = 0,\quad a_2 \neq 0.$
$R_6(e_1) = 0,\ R_6(e_2) = 0,\ R_6(e_3) = 0.$

■ The pre-Lie superalgebras $\left((\widehat{A}_{9,2})_k, \circ\right) \simeq (D_4)_\mu$ and $(\widehat{A}_{9,3}, \circ) \simeq D_5$:

$$\left((\widehat{A}_{9,2})_k, \circ\right) \begin{cases} e_2 \circ e_1 = e_1 \\ e_2 \circ e_2 = e_2 \\ e_2 \circ e_3 = e_3 \\ e_3 \circ e_1 = ke_3 \end{cases}. \quad (\widehat{A}_{9,3}, \circ) \begin{cases} e_2 \circ e_2 = e_1 \\ e_2 \circ e_2 = e_2 \\ e_2 \circ e_3 = e_3 \\ e_3 \circ e_3 = \frac{1}{2}e_1 \\ e_3 \circ e_3 = e_1 \end{cases}.$$

$k = 0$ *or* 1 associative other cases are non associative.
They have the same Rota–Baxter operators RB($\mathcal{A}$), that is,

$R_1(e_1) = a_1e_1 + a_2e_2,\ R_1(e_2) = -\frac{a_1^2}{a_2}e_1 - a_1e_2,\ R_1(e_3) = 0,\quad a_2 \neq 0.$
$R_2(e_1) = 0,\ R_2(e_2) = a_3e_1,\ R_2(e_3) = 0.$
$R_3(e_1) = 0,\ R_3(e_2) = 0,\ R_3(e_3) = 0.$

4.2.10 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_{10_h}$

■ The pre-Lie superalgebra $\left((\widehat{A}_{10_h,1})_k, \circ\right) \simeq (D_4)_\mu$:

$$e_1 \circ e_2 = (h-1)e_1,\quad e_2 \circ e_1 = he_1,\quad e_2 \circ e_2 = e_1 + he_2,\quad e_3 \circ e_2 = ke_3,\qquad h \neq 0.$$

Rota–Baxter operators RB($(\widehat{A}_{10_h,1})_k$) are:
▹ **Case 1**: If $k = 0$, we have

$R_1(e_1) = 0,\ R_1(e_2) = a_1e_1,\ R_1(e_3) = a_2e_3.$
$R_2(e_1) = 0,\ R_2(e_2) = 0,\ R_2(e_3) = a_2e_3.$

▹ **Case 2**: If $k \in \mathbb{C}^*$, we have

$R_3(e_1) = 0,\ R_3(e_2) = a_1e_1,\ R_3(e_3) = 0,\quad a_1 \neq 0.$
$R_4(e_1) = 0,\ R_4(e_2) = 0,\ R_4(e_3) = 0.$

■ The pre-Lie superalgebras $\left((\widehat{A}_{10_h,2})_k, \circ\right) \simeq (D_4)_\mu$ and $(\widehat{A}_{10_h,3}, \circ) \simeq D_5$, where

$$\left((\widehat{A}_{10_h,2})_k, \circ\right) \begin{cases} e_1 \circ e_2 = (h-1)e_1 \\ e_2 \circ e_1 = he_1 \\ e_2 \circ e_2 = e_1 + he_2, \quad h \neq 0 \\ e_2 \circ e_3 = he_3 \\ e_3 \circ e_2 = ke_3 \end{cases} \quad (\widehat{A}_{10_h,3}, \circ) \begin{cases} e_1 \circ e_2 = (h-1)e_1 \\ e_2 \circ e_1 = he_1 \\ e_2 \circ e_2 = e_1 + he_2, \quad h \neq 0 \\ e_2 \circ e_3 = he_3 \\ e_3 \circ e_2 = \left(h - \frac{1}{2}\right)e_3 \\ e_3 \circ e_3 = e_1 \end{cases}.$$

They have the same Rota–Baxter operators, that is,

$$R_1(e_1)=0,\ R_1(e_2)=a_1e_1,\ R_1(e_3)=0,\ \ a_1\neq 0.$$
$$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=0.$$

4.2.11 Rota–Baxter Operators on Pre-Lie Superalgebras of Type $\widehat{A}_{11}$

■ The pre-Lie superalgebra $\left((\widehat{A}_{11,1})_k, \circ\right) \simeq (D_4)_\mu$:

$$e_1\circ e_2=-e_1,\ \ e_2\circ e_2=e_1-e_2,\ \ e_3\circ e_2=ke_3.$$

Rota–Baxter operators $\mathrm{RB}((\widehat{A}_{11,1})_k)$ are:
▹ **Case 1**: If $k=0$, we have

$$R_1(e_1)=0,\ R_1(e_2)=a_1e_1,\ R_1(e_3)=a_2e_3,\ \ a_1\neq 0.$$
$$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=a_2e_3.$$

▹ **Case 2**: If $k\in\mathbb{C}^*$, we have

$$R_3(e_1)=0,\ R_3(e_2)=a_1e_1,\ R_3(e_3)=0,\ \ a_1\neq 0.$$
$$R_4(e_1)=0,\ R_4(e_2)=0,\ R_4(e_3)=0.$$

■ The pre-Lie superalgebras $(\widehat{A}_{11,2}, \circ)\simeq D_5$ and $\left((\widehat{A}_{11,3})_k, \circ\right)\simeq (D_4)_\mu$, where

$$(\widehat{A}_{11,2}, \circ)\ :\ e_1\circ e_2=-e_1,\ e_2\circ e_2=e_1-e_2,\ e_3\circ e_2=-\tfrac{1}{2}e_3,\ \ e_3\circ e_3=e_1.$$
$$\left((\widehat{A}_{11,3})_k, \circ\right):\ e_1\circ e_2=-e_1,\ e_2\circ e_2=e_1-e_2,\ e_2\circ e_3=-e_3,\ e_3\circ e_2=ke_3.$$

They have the same Rota–Baxter operators, that is,

$$R_1(e_1)=0,\ R_1(e_2)=a_1e_1,\ R_1(e_3)=0,\ \ a_1\neq 0.$$
$$R_2(e_1)=0,\ R_2(e_2)=0,\ R_2(e_3)=0.$$

Remark 4.1 Using the above classification and Corollary 2.4, one may construct the two- and three-dimensional L-dendriform superalgebras associated with the Rota–Baxter pre-Lie superalgebras of dimension 2 and 3 (of weight zero) described above.

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