Semi-analytical Investigation of Momentum and Heat Transfer of a Non-Newtonian Fluid Flow for Specific Turbine Cooling Application Using AGM

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Abstract In this paper, a non-Newtonian fluid flow in an axisymmetric channel with porous wall for specific turbine cooling application has been considered. The purpose of this article is based on solving the nonlinear differential equations of momentum and heat transfer of the mentioned problem by utilizing a new and innovative method in semi-analytical field which is called Akbari–Ganji’s method. Meanwhile, relationships between power law index, Reynolds, Prandtl and Nusselt numbers have been investigated. Results have been compared with numerical method (Runge–Kutte 4th) to achieve conclusions based on not only accuracy of the solution but also simplicity of their procedures which would have remarkable effects on the time devoted for solving process. Moreover, results are presented for various values of constant parameters and different steps of trial function due to the aim of comparison and prove that proposed solution is very accurate, simple and also have efficient convergence.

Keywords Akbari–Ganji’s method (AGM) · Heat transfer · Non-Newtonian fluid flow · Nusselt number · Porous wall

List of symbols

A, B	Symmetric kinematic matrices
C	Species heat
C_n	Blade-wall temperature coefficients
$\frac{\delta v_m}{\delta x_n}$	Velocity gradients
$\frac{\delta a_m}{\delta x_n}$	Acceleration gradients

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x_k	General coordinates
$f(\eta)$	Velocity function
F	Transformation of f
$\bar{k}$	Fluid thermal conductivity
N	Power law index in temperature distribution
Nu	Nusselt number
K_r	Rotation parameter
AGM	Akbari–Ganji’s method
p	Pressure
Pr	Prandtl number
Re	Reynolds number
T	Fluid temperature
$q_n(\eta)$	Temperature function
τ_{ij}	Stress tensor component
V	Injection velocity
u_r, u_z	Velocity components in r and z directions, respectively
η	Dimensionless coordinates in z direction
r, θ, z	Cylindrical coordinate symbols
ϕ_k	Viscosity coefficients
φ	Dissipation function
ρ	Fluid density

Introduction

Non-Newtonian fluid flow and heat transfer problems that are pertinent to their applications have attracted many researchers all around the world due to their technical and industrial applications such as polymer processing, electronic packing, drag reduction and cooling problems. For instance, a model of cooling turbine blades internally by continuous injection through an interior baffle was analyzed by Deburge and Han [1] in order to increase the resistance of blades against the hot steam around the blades. With respect to their considered analytical model and also their physical assumption that was based on incompressible and laminar flow, they would be able to determine the velocity and the temperature fields. Also it is noteworthy to mention that Kurtcebe and Erim [2] extended the work of Deburge and Han [1] to investigate the influence of non-Newtonian viscoelastic fluid flow on the cooling of the turbine disk. Therefore, it is obvious that non-Newtonian fluid flow could be utilized in various fields of study which some of the excellence of the previous research areas could be mentioned as [3–5]. It is also noteworthy to mention some of the recent investigations in similar area which have the same analogy and methodology and are done by several researchers around the world [6–8].

Because of the importance of mathematical formulation and analyzation in solving engineering problems, many researches from different fields devote their time to expand relevant knowledge to obtain better understanding for the mentioned case. For instance, one of the most significant feature of this ever-increasing trend could be mentioned in analytical, semi-analytical and numerical technics in solving nonlinear differential equations. The main reason of widespread usage of analytical and semi-analytical methods is that by utilizing these approaches, researches would obtain a unique function in their solving procedures which could be used in other fields such as designing a control system for heat transfer

or fluid mechanics appliances. Therefore, for the purpose of achieving the afore-mentioned fact, many researchers have tried to reach acceptable solution for differential equations due to their nonlinearity by utilizing analytical and semi-analytical methods such as: Perturbation Method [9], Homotopy Perturbation Method [10–12], Variational Iteration Method [13, 14], Optimal Variational Iteration Method using Adomian's Polynomials (OVIMAP) [15], Homotopy Analysis Method [14, 16, 17], Parameterized Perturbation Method (PPM) [18], Collocation Method (CM) [19], Adomian Decomposition Method [20, 21], Variation of Parameters Method (VPM) [22] and so on.

It is worthwhile to mentioned the fact that each proposed method has their own merits which in general they could be categorized into two perspectives due to their solving procedures. We would call these two groups as: Iterate-Base and Trial Function-Base Methods. In Iterate-Base Method such as: HPM, VIM, ADM and etc., the important factor which affect the solving procedures is the number of iterations. Although in these methods we may assume a trial functions, which are based on our independent functions, however, in order to achieve solution in each step we are eligible to solve previous steps at first. According to mentioned explanations, it would be crystal clear that if the iteration results in higher steps could not be obtained by related software (such as Maple or Matlab), we would definitely face problem which will interrupt our solving procedures. Also, these methods usually take more time for obtaining solutions.

On the contrary, in Trial Function-Base Method such as: CM, LSM, Akbari–Ganji's Method (AGM) [23–30] and etc., the main factor which affect the solving procedures is considered trial function. In this methods, we assume an efficient trial function base on the problem's boundary or initial conditions which contains different constant coefficients. Afterwards, according to the basic idea of each method, we are obligated for solving the constant coefficients. In most cases the constant coefficients would be obtained easily by solving set of rudimentary polynomials. Although in these methods, number of terms in our trial function could be referred as needed iterations, nonetheless, it's essential to mention the fact that utilized constants will obtain simultaneously in solving procedures. So, in these methods the iteration problems have been eliminated.

In this article attempts have been made in order to obtain approximate solutions of the nonlinear momentum and heat transfer equations of non-Newtonian fluid flow in an axisymmetric channel with porous wall for turbine cooling applications. We have utilized a new and innovative semi-analytical method calling Akbari–Ganji's Method which was developed by Akbari and Ganji in 2014 for the first time. Since this method has been investigated by many authors to solve highly nonlinear equations in different aspects of engineering problems such as: Fluid Mechanics, Nonlinear Vibration Problems, Heat Transfer Applications, Nanofluids and etc. Some of the excellence of proposed method could be referred as [23–30].

Mathematical Formulation

Flow Analysis

Simultaneous development of flow and heat transfer for a non-Newtonian viscoelastic fluid flow on the turbine disc for cooling purposes have been considered in this study. Figure 1 represents schematic of the proposed problem in which the r -axis is parallel to the surface of disk and the z -axis is normal to it. The porous disc of the channel is $z = +L$. As it is depicted, although the wall that coincides with the r -axis is heated, due to the external gas

Fig. 1 Schematic diagram of the considered physical system

flow, non-Newtonian fluid is injected uniformly from the other perforated wall in order to cool the heated wall.

For a steady, axisymmetric, non-Newtonian fluid flow the following equations could be written in cylindrical coordinates. The continuity equation:

$$\frac{\partial(ru_r)}{\partial r} + \frac{\partial(ru_z)}{\partial z} = 0 \quad (1)$$

And the momentum equations:

$$u_r \frac{\partial(u_r)}{\partial r} + u_z \frac{\partial(u_r)}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{\rho} \left[\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) + \frac{\partial \tau_{rz}}{\partial z} \right] \quad (2)$$

$$u_r \frac{\partial(u_z)}{\partial r} + u_z \frac{\partial(u_z)}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} \left[\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r} \tau_{rz} + \frac{\partial \tau_{zz}}{\partial z} \right] \quad (3)$$

Here ρ , P are the density, pressure and also it is noteworthy to mention that τ_{rr} , τ_{rz} , τ_{zr} , τ_{zz} are the components of stress matrix and u_r , u_z are the velocity components in the r and z directions. The analytical model under consideration leads to the following boundary conditions:

$$z = 0 \rightarrow u_r = u_z = 0 \quad (4)$$

$$z = L \rightarrow u_r = 0, u_z = -V \quad (5)$$

In which V is the injection velocity. For particular class of viscoelastic and viscoinelastic fluids Rivlin [31] represented that if the stress components τ_{ij} at a point x_k ($k = 1, 2, 3$) and time t are assumed to be polynomials in the velocity gradient $\frac{\delta v_m}{\delta x_n}$ ($m, n = 1, 2, 3$) and the acceleration gradients $\frac{\delta a_m}{\delta x_n}$ ($m, n = 1, 2, 3$), and if in addition the medium is assumed to be isentropic the stress matrix can be demonstrate in the following form:

$$\|\tau_{ij}\| = \phi_0 I + \phi_1 A + \phi_2 B + \phi_3 A^2 + \dots \quad (6)$$

Here I is the unit matrices, A and B are symmetric kinematic matrixes defined by :

$$A = \left\| \frac{\delta v_i}{\delta x_j} + \frac{\delta v_j}{\delta x_i} \right\|, B = \left\| \frac{\delta a_i}{\delta x_j} + \frac{\delta a_j}{\delta x_i} + 2 \frac{\delta v_m}{\delta x_i} \frac{\delta v_m}{\delta x_j} \right\| \quad (7)$$

Moreover, ϕ_k ($k = 0, 1, 2, 3$) are polynomials in the invariants of A , B , A^2 . This study is restricted to second order fluids for which ϕ_k ($k = 0, 1, 2, 3$) are constant and ϕ_k ($k = 4, 5, \dots$) are zero. So that the stress components are as follows:

$$\tau_{rr} = \phi_1 A_{rr} + \phi_2 A_{rr}^2 + \phi_3 B_{rr} \quad (8)$$

$$\tau_{zz} = \phi_1 A_{zz} + \phi_2 A_{zz}^2 + \phi_3 B_{zz} \quad (9)$$

$$\tau_{\theta\theta} = \phi_1 A_{\theta\theta} + \phi_2 A_{\theta\theta}^2 + \phi_3 B_{\theta\theta} \quad (10)$$

$$\tau_{rz} = \phi_1 A_{rz} + \phi_2 A_{rz}^2 + \phi_3 B_{rz} \quad (11)$$

In order to reach a solution of the afore-mentioned problem depicted in Fig. 1 in the case of axially symmetric flow, it would be much more convenient to define a stream function so that the continuity equation is satisfied:

$$\psi = Vr^2 f(\eta) \quad (12)$$

Where $\eta = \frac{z}{L}$ and the velocity components can be derived as:

$$u_r = \frac{Vr}{L} f'(\eta) \quad (13)$$

$$u_z = -2Vf(\eta) \quad (14)$$

Using Eqs. (12)–(14) the equations of motion reduce to:

$$f'^2 - 2ff'' = -\frac{L^2}{\rho V^2 r} \frac{\partial P}{\partial r} + \frac{\phi_1}{\rho VL} f''' + \frac{\phi_2}{\rho L^2} (f'^2 - 2f'f''') + \frac{\phi_3}{\rho L^2} (f'^2 - 2ff^{iv}) \quad (15)$$

$$4ff' = -\frac{L^2}{\rho V^2} \frac{\partial P}{\partial z} - 2\frac{\phi_1}{\rho VL} f'' + 2\frac{\phi_2}{\rho L^2} \left(14f'f'' + \frac{r^2}{L} f''f''' \right) + 4\frac{\phi_3}{\rho L^2} \left(11f'f'' + ff''' + \frac{r^2}{L} f''f''' \right) \quad (16)$$

The pressure term can be eliminated by differentiating Eq. (15) with respect to z and Eq. (16) with respect to r and subtracting the resulting equations. Therefore, the following equations would be obtained:

$$-2ff''' = \frac{f^{iv}}{Re} - K_1 (4f''f''' + 2f'f^{iv}) - K_2 (4f''f''' + 2f'f^{iv} + 2ff^{iv}) \quad (17)$$

where $K_1 = \frac{\phi_2}{\rho L^2}$, $K_2 = \frac{\phi_3}{\rho L^2}$ is the injection Reynolds number. For $K_2 = 0$ the equation turned to:

$$f^{iv} + 2Re ff''' - K_1 Re (4f''f''' + 2f'f^{iv}) = 0 \quad (18)$$

The boundary conditions are:

$$f(0) = 0, f'(0) = 0, f(1) = 1, f'(1) = 0 \quad (19)$$

Heat Transfer Analysis

The energy equation for the present problem with viscous dissipation in non-dimensional form is given by:

$$\rho C \left(u_r \frac{\partial T}{\partial r} + u_z \frac{\partial T}{\partial z} \right) = \bar{k} \nabla^2 T + \varphi \quad (20)$$

$$\varphi = \tau_{rr} \frac{\partial u_r}{\partial r} + \tau_{\theta\theta} \frac{u_r}{r} + \tau_{zz} \frac{\partial u_z}{\partial z} + \tau_{rz} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \quad (21)$$

Here u_r, u_z are the velocity components in the r and z directions and V is the injection velocity; $\rho, P, T, C, \bar{k}$ are the density, pressure, temperature, specific heat, and heat conduction coefficient of fluid, respectively. φ is the dissipation function.

Letting the blade wall $z = 0$ temperature distribution be $T_w = T_0 + \sum_{n=0}^{\infty} C_n \left(\frac{r}{L} \right)^n$ and assuming the fluid temperature to have the form of [1]:

$$T = T_0 + \sum_{n=0}^{\infty} C_n \left(\frac{r}{L} \right)^n q_n(\eta) \quad (22)$$

Where T_0 is the temperature of the incoming coolant $z = L$ and neglecting dissipation effect, the following equations and boundary conditions are obtained:

$$q_n'' - Pr.Re (f' q_n - 2f q_n') = 0, \quad (n = 0, 2, 3, 4, \dots) \quad (23)$$

$$q_n(0) = 1, \quad q_n(1) = 0 \quad (24)$$

Application of Akbari–Ganji's Method (AGM)

According to mentioned coupled system of nonlinear differential equations and also by considering the basic idea of the method, we rewrite Eqs. (18) and (23) in the following order:

$$F(\eta) = f^{iv} + 2Re f f''' - K_1 Re (4f'' f''' + 2f' f^{iv}) = 0 \quad (25)$$

$$\Theta(\eta) = q_n'' - Pr.Re (f' q_n - 2f q_n') = 0, \quad (n = 0, 2, 3, 4, \dots) \quad (26)$$

Due to the basic idea of AGM, we have utilized a proper trial function as solution of the considered differential equation which is a finite series of polynomials with constant coefficients, as follows:

$$f(\eta) = \sum_{i=0}^9 a_i \eta^i = a_0 + a_1 \eta^1 + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4 + a_5 \eta^5 + a_6 \eta^6 + a_7 \eta^7 + a_8 \eta^8 + a_9 \eta^9 \quad (27)$$

$$\theta(\eta) = \sum_{i=0}^{12} b_i \eta^i = b_0 + b_1 \eta^1 + b_2 \eta^2 + b_3 \eta^3 + b_4 \eta^4 + b_5 \eta^5 + b_6 \eta^6 + b_7 \eta^7 + b_8 \eta^8 + b_9 \eta^9 + b_{10} \eta^{10} + b_{11} \eta^{11} + b_{12} \eta^{12} \quad (28)$$

Applying Boundary Conditions

In AGM, the boundary conditions are applied in order to calculate constant coefficients of Eqs. (27) and (28) according to the following approaches:

(a) Applying the boundary conditions on Eqs. (27) and (28) are expressed as follows:

$$u = u(B.C) \quad (29)$$

Where BC is the abbreviation of boundary condition. According to the above explanations, the boundary conditions are applied on Eqs. (27) and (28) in the following form:

$$f(0) = 0 \rightarrow a_0 = 0 \quad (30)$$

$$f(1) = 0 \rightarrow a_9 + a_8 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 = 0 \quad (31)$$

$$f'(0) = 0 \rightarrow a_1 = 0 \quad (32)$$

$$f'(1) = 0 \rightarrow 9a_9 + 8a_8 + 7a_7 + 6a_6 + 5a_5 + 4a_4 + 3a_3 + 2a_2 + a_1 = 0 \quad (33)$$

$$\theta(0) = 1 \rightarrow b_0 = 1 \quad (34)$$

$$\theta(1) = 0 \rightarrow b_{12} + b_{11} + b_{10} + b_9 + b_8 + b_7 + b_6 + b_5 + b_4 + b_3 + b_2 + b_1 + b_0 = 0 \quad (35)$$

(b) Applying the boundary conditions on the main differential equations, which in this case study are Eqs. (25) and (26), and also on their derivatives is done after substituting Eqs. (27) and (28) into the main differential equations as follows:

$$F(f(\eta)) \rightarrow F(f(B.C)) = 0, \quad F'(f(B.C)) = 0, \dots \quad (36)$$

$$\Theta(\theta(\eta)) \rightarrow \Theta(\theta(B.C)) = 0, \quad \Theta'(\theta(B.C)) = 0, \dots \quad (37)$$

The boundary conditions are applied on the above differential equation. In fact, due to the excellence of AGM from other methods, we have to reach to set of polynomials in the processes of solution that are exactly in the same number as the overall used constant coefficients in our trial functions. Therefore, we would be able to obtain these only by simple and mundane calculations. Since in the proposed problem we have assumed two trial functions which contain 23 constant coefficients and we have 6 equations according to Eqs. (30)–(35), we have to create 17 additional equations from Eqs. (36) and (37) in order to achieve a set of polynomials which contains of 23 equations and 23 constants.

According to the above explanations we have created additional equations Eqs. (36) and (37) in the following order:

- I. 6 equations have been created by calculating obtained equations from $F(0) = 0, F(1) = 0, F'(0) = 0, F'(1) = 0, F''(0) = 0, F''(1) = 0$
- II. 11 equations have been created by calculating obtained equations from $\Theta(0) = 0, \Theta(1) = 0, \Theta'(0) = 0, \Theta'(1) = 0, \Theta''(0) = 0, \Theta''(1) = 0, \Theta'''(0) = 0, \Theta'''(1) = 0, \Theta''''(0) = 0, \Theta''''(1) = 0, \Theta'''''(0) = 0, \Theta'''''(1) = 0$

The mentioned equations in (I) and (II) subsections are too large to be displayed graphically. So by utilizing the above procedures we have obtained a set of polynomials containing 23 equations and 23 constants which by solving them we would be able to obtain Eqs. (27) and (28). For instance, when $Re = 0.5, k = 0.1, Pr = 2.5, n = 0$, by substituting obtained

Fig. 2 **a** Convergence figure for $f(\eta)$ in varies terms of AGM. **b** Convergence figure for $q_n(\eta)$ in varies terms of AGM

constant coefficients from mentioned procedures Eqs. (27) and (28) could easily be yielded as follows:

$$f(\eta) = -0.03009057971\eta^7 + 0.09356884058\eta^6 + 0.002083333333\eta^4 - 2.227989130\eta^3 + 3.162427536\eta^2 \quad (38)$$

$$\theta(\eta) = -0.06730148307\eta^{12} + 0.4123487273\eta^{11} - 0.9074026402\eta^{10} + 0.6454071134\eta^9 + 0.4475609223\eta^8 - 0.6806457857\eta^7 + 0.0002382034452\eta^6 - 0.3821145744\eta^5 + .09039613610\eta^4 + 1 - 1.372051844\eta \quad (39)$$

Result and Discussion

As could be detected, AGM has properly and efficiently applied through the considered problem. So in this section we have presented some of the detailed investigations through the mentioned problem using different concepts to support our work.

Figure 2 is divided into two parts which both of them represent the difference between obtained solution by AGM and numerical method (Runge–Kutte 4th) in which we have introduced error percentage as follow:

$$\% \text{ Error} = \left| \frac{u(\eta)_{NM} - u(\eta)_{AGM}}{u(\eta)_{NM}} \right| \times 100 \quad (40)$$

where $u(\eta)_{NM}$ is value obtained by numerical method (Runge–Kutte 4th) and $u(\eta)_{AGM}$ is value obtained by AGM. It is worthwhile to mention the fact that both Fig. 2a and b depict the convergence issue. It could be deducted that while we utilized more terms for our proposed AGM solution, the obtained error in $\eta = 0.5$ would decrease.

Moreover, comparison between AGM and numerical results have been done for different values of active parameters in Fig. 3. The effects of these parameters on the velocity ($f(\eta)$) and the temperature ($q_n(\eta)$) are shown. The obtained results represent that AGM has enough accuracy and efficiency so it would be applicable for solving nonlinear equations of coupled system.

Fig. 3 a, b Comparison between numerical results and AGM solution for $f(\eta)$ and $q_n(\eta)$

Fig. 4 Velocity component profiles a $f(\eta)$ and b $f'(\eta)$ for variable Re at $k_1 = 0.01$

Due to the importance of the Reynolds number, which is a dimensionless number that illustrates the correlation of inertial forces to viscous forces, Fig. 4a–b and Fig. 5 have been considered. Figure 4 a–b represent the effect of Reynolds number on velocity profiles ($f(\eta)$, $f'(\eta)$) where they have directly correlated. Additionally, effects of Reynolds number and injection Reynolds number on skin friction ($f''(0)$) have been shown in Fig. 5 in which it is crystal clear that by increasing of both Re and k_1 parameters, the skin friction would increase substantially.

Afterwards, effect of power law index (n) and the Reynolds number on temperature profile are represented. Figure 6 illustrates that by increasing the Reynolds number and power law index (n), the thermal boundary layer would decrease. It is noteworthy to mention the fact that the Reynolds number changing in the value of the Reynolds number play a major role in the obtained results.

Finally, in Fig. 7 the effects of considered parameters on Nusselt number, which is equal to the dimensionless temperature gradient at the surface and provides a measure of the convection heat transfer occurring at the surface, have been discussed. Therefore, Nusselt number on the horizontal axis is plotted against Reynolds number on the vertical axis then due to Prandtl number and power law index, the various profiles are depicted. This figure demonstrates that when the Prandtl number or power law index amounts increase in the

Fig. 5 Skin friction under the effect of k_1 and Re

Fig. 6 Temperature profile (q_n) at $K_1 = 0.01$

constant value of the Reynolds number, the Nusselt number would climb dramatically. Due to the fact that the Nusselt number and the thermal boundary thickness layer are negatively correlated; it is obvious that by rising the Nusselt number, with respects to its relationship with other factors, we would obtain the better cooling efficiency in industrial applications. It is also noteworthy to mention the fact that one could refer to previous articles in this field and

Fig. 7 Nusselt number for **a** variable n at $K_1 = 0.01$, $Pr = 1$ **b** variable Pr at $K_1 = 0.1$, $n = 2$

obtain even more details and results throughout other analytical and semi-analytical methods [4].

Conclusion

In this study, AGM has been utilized in order to solve nonlinear differential equations of momentum and heat transfer of a non-Newtonian fluid flow for specific turbine cooling application. Comparisons have been done among AGM and numerical method (Runge–Kutte 4th) by different parameters values. Obtained data have been revealed that AGM solutions are in a very excellence agreement with numerical solutions. Moreover, convergence figures represent that by applying more terms of AGM we would be able to obtain more accurate solutions. Therefore, it would be obvious that AGM is a convenient and qualified method for solving nonlinear problems due to its accuracy, efficiency.

Furthermore, according to achieved results, it is crystal clear that Nusselt number is an increasing function of each Reynolds number, Prandtl number and power law index. Therefore, when these properties increase during the pertinent procedures, Nusselt number would rise substantially. Finally, due to the reverse relation between Nusselt number and the thermal boundary layer thickness, we would be able to achieve more appropriate cooling characteristics in the industrial applications.

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