Dynamic Behavior of Leptospirosis Disease with Saturated Incidence Rate

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Abstract Leptospirosis is a tropical disease and found almost in globe. Human as well as mammals are mostly infected from this disease. This work presents a mathematical study of the leptospirosis disease with saturated incidence rate. Initially, we present the model formulation and their fundamental properties. Then we find the local stability of the disease free and endemic equilibrium. The disease free equilibrium is stable both locally and globally when $R_0 < 1$. Further, we find that endemic equilibrium is stable locally and globally if $R_0 > 1$. The numerical results are shown for analytical results.

Keywords Leptospirosis · Saturated incidence · Basic reproduction number · Global stability · Numerical results

Mathematics Subject Classification 92D25 · 49J15 · 93D20

Introduction

Leptospirosis is a worldwide zoonotic disease which is commonly occurs in tropical and subtropical regions. The magnitude of the problem in the tropics and subtropics may be largely not only because of their climatic and environmental conditions, but also likely relationship

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with the caused leptospira contaminated environment, such as local agricultural practices and poor housing and disposal, all what will be many sources of infection [1,2]. In tropical countries, people are always risky, especially if their contact with potentially infected areas, rodents and pets in a wet and hot climate. In template climate, the cattle, pits and dogs are the carriers for important source of leptospirosis [3]. Valuable information about the risk factor of leptospirosis disease can be found in [4].

To better understand the epidemiology of an infectious disease, mathematical modeling has played an important role [5–8]. These models provide us the quantitative descriptions of the complicated, non linear process of disease transmission and help us to obtain inside into the dynamics of the disease and we are able to make such decision for public health policy. Many mathematical models [9–11] have been proposed to represent the compartmental dynamics of both susceptible, infected and recovered human and vector population.

Various mathematical studies have been proposed on leptospirosis disease. Here, we give an overview. Triampo et al. presented a mathematical model for the transmission of leptospirosis disease in [12]. In their work, they considered a number of leptospirosis disease in Thailand and shown their numerical simulations. Zaman [13] considered the real data presented in [12] to studied the dynamical behavior and role of optimal control theory of this disease. Pongsumpun et al. [14] developed mathematical model to study the behavior of leptospirosis disease. In their work, they represent the rate of change for both the vector (rats) and human population. They divided the human population further into two main groups Juveniles and adults. Moreover, we refer the reader to see [15–17] and the references therein.

In epidemic models a variety of non linear incidence rate have been used [18–20]. Liu et al. [21] showed that the case where hosts can exhibit prolonged immunity to infection is not linear incidence rate of infection could greatly expand the breadth dynamics caused by this disease. The saturated incidence was first time used by Capasso and Serio [20] in cholera epidemic model.

In this work, we present analysis of nonlinear incidence in leptospirosis epidemic model. Here, we study the model presented in [22]. In [22], the authors used the standard optimal control technique Pontryagin's Maximum Principle and used three control variables for possible eradication of the infection in the community. We study the model [22], to obtain its stability analysis on behalf of reproduction number. First, we find the threshold/ basic reproduction number for the model and then show that the Disease free equilibrium is locally as well as globally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$. Further, we find that an endemic equilibrium is locally as well as globally asymptotically stable for $R_0 > 1$. The global stability of an endemic equilibrium is shown by the geometric approach method. We solve, the proposed model numerically and the results are presented in the form of plots for justification purpose. Finally, the conclusion and references are presented.

This paper is organized as follows: section "Mathematical Model" is devoted to the mathematical formulation of the model. We discuss the equilibria and local stability of both the disease-free and endemic equilibrium in section "Equilibria and Local Stability". In section "Global Stability", we show the global stability of both the disease-free and endemic equilibrium. In section "Numerical Simulations" the theoretical results are demonstrated in the form of numerical graphics. Finally, we conclude our work by conclusion in section "Conclusion".

Mathematical Model

In this section, we present the model formulation of the leptospirosis disease with their interaction with human population. The human population denoted by $N_h(t)$, is subdivided

into three compartments; $S_h(t)$ -Susceptible individuals, $I_h(t)$ -infected individuals (those who have been infected) and $R_h(t)$ -recovered individuals(those who recovered from infection or removed), so

$$N_h(t) = S_h(t) + I_h(t) + R_h(t). \quad (1)$$

The population of susceptible is increased by recruitment of the individuals (which is assumed susceptible), at the rate of b_h . The susceptible population decreased by following the effective contact with the infected individuals and vectors (in the I_h and I_v) at a rate $\lambda(t)$, where

$$\lambda(t) = \left(\frac{\beta_2 I_v}{1 + \alpha_1 I_v} + \frac{\beta_1 I_h}{1 + \alpha_2 I_h} \right). \quad (2)$$

In (2), β_2 and β_1 are the effective contact rate (capable of leading to infection). The population of susceptible is further decreased with the natural death μ_h , while increased with the number of individuals, which are susceptible again at a rate of λ_h . So, we can write, the rate of change of the susceptible population given by

$$\frac{dS_h}{dt} = b_h - \mu_h S_h - S_h \lambda(t) + \lambda_h R_h. \quad (3)$$

The population of infected individuals is generated by the infection of susceptible individuals (at a rate $\lambda(t)$). The population of infected individuals is decreased by the natural mortality rate μ_h , disease induced death rate δ_h and from the recovery rate of infection γ_h . So, we can write their rate of change for infected population given by

$$\frac{dI_h}{dt} = S_h \lambda(t) - (\mu_h + \delta_h + \gamma_h) I_h. \quad (4)$$

The population of recovered individuals is generated by the rate of recovery γ_h (from infected class), while decreased by the natural death rate μ_h and λ_h (those individuals who susceptible again). Thus, their rate of change can be expressed as follows:

$$\frac{dR_h}{dt} = \gamma_h I_h - (\mu_h + \lambda_h) R_h. \quad (5)$$

The vector population is denoted by $N_v(t)$, is subdivided into two classes, $S_v(t)$ -susceptible vector and $I_v(t)$ -infected vector, so

$$N_v(t) = S_v(t) + I_v(t). \quad (6)$$

The population of susceptible vector is generated by the recruitment of the vector at a rate of b_v . The population of susceptible is decreased by following the effective contact with infected individuals I_h (at a rate of $\bar{\lambda}(t)$), where $\bar{\lambda}(t) = \beta_3 I_h / 1 + \alpha_2 I_h$ and β_3 is the effective contact rate that leads to infection. Also, the population of susceptible vector is decreased by the natural death rate at a rate γ_v . Thus, the rate of change for the susceptible vector is given by

$$\frac{dS_v}{dt} = b_v - \gamma_v S_v - \bar{\lambda}(t) S_v. \quad (7)$$

The population of infected vector is generated by the infection of susceptible vector at rate of $\bar{\lambda}(t)$, while decreased by the natural death γ_v and disease related death rate δ_v . Their rate of change can be written as:

$$\frac{dI_v}{dt} = \bar{\lambda}(t) S_v - \gamma_v I_v - \delta_v I_v. \quad (8)$$

Fig. 1 The flow diagram of human and vector interaction

The complete transfer flow diagram of human and vector population is depicted in Fig. 1. Thus, the system for the transmission dynamics of the leptospirosis disease is given by the following nonlinear system of differential equations:

$$\begin{aligned}
 \frac{dS_h}{dt} &= b_h - \mu_h S_h - S_h \lambda(t) + \lambda_h R_h, \quad S_h(0) = S_0 \geq 0, \\
 \frac{dI_h}{dt} &= S_h \lambda(t) - (\mu_h + \delta_h + \gamma_h) I_h, \quad I_h(0) = I_0 \geq 0, \\
 \frac{dR_h}{dt} &= \gamma_h I_h - (\mu_h + \lambda_h) R_h, \quad R_h(0) = R_0 \geq 0, \\
 \frac{dS_v}{dt} &= b_v - \gamma_v S_v - \bar{\lambda}(t) S_v, \quad S_v(0) = S_0 \geq 0, \\
 \frac{dI_v}{dt} &= \bar{\lambda}(t) S_v - \gamma_v I_v - \delta_v I_v, \quad I_v(0) = I_0 \geq 0,
 \end{aligned} \tag{9}$$

Subject to nonnegative initial conditions.

One infected human will shed leptospirosis in their urine during and after the illness of the period, and thus may pose a risk of infection to others but only in a certain ways. Chief social interaction is perfectly safe, as airborne bacteria. Saliva is not consider high risky, because the bacteria cannot be tolerate the acidity of the human mouth for a long time, so even we advise against the risk of sharing food, cups or cutlery very small. Items that can dry out between uses, such as towels, also very low risk once they dry, but handle very blood soaked clothing, wet bed or similar may present risks, for detail see the reference [23]. Let $N(h) = S_h + I_h + R_h$, shows the total dynamics of human population at time t , is given by

$$\frac{dN_h}{dt} = b_h - \mu_h N_h - \delta_h I_h \leq b_h - \mu_h N_h,$$

$N(v) = S_v + I_v$, represent the total population of vector at time t , given by

$$\frac{dN_v}{dt} = b_v - N_v\gamma_v - \delta_v I_v \leq b_v - N_v.$$

The feasible region for the system (9) is

$$\Pi = \left((S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^5, \quad \left(N_h \leq \frac{b_h}{\mu_h}, N_v \leq \frac{b_v}{\gamma_v} \right) \right).$$

It is easy to prove that the set Π is positively invariant with respect to system (9).

Lemma 2.1 *Let the initial data $S_h(0) > 0$, $I_h(0) > 0$, $R_h(0) > 0$, $S_v(0) > 0$ and $I_v(0) > 0$. Then the solution of $S_h(t)$, $I_h(t)$, $R_h(t)$, $S_v(t)$ and $I_v(t)$ corresponding to model (9), are non-negative $\forall$ time $t > 0$.*

Proposition 2.1 *Let $(S_h, I_h, R_h, S_v, I_v)$, be the solution of the system (9) with non-negative initial conditions and closed set*

$\Pi = ((S_h, I_h, R_h, S_v, I_v) \in \mathbb{R}_+^5, N_h \leq \frac{b_h}{\mu_h}, N_v \leq \frac{b_v}{\gamma_v})$, *then Π is positively invariant and attracting under the flow described by system (9).*

Proof Consider the Lyapunov function

$$M(t) = (N_h(t), N_v(t)) = (S_h + I_h + R_h, S_v + I_v). \quad (10)$$

The time derivative of Eq. (10) is

$$\frac{dM}{dt} = (b_h - \mu_h N_h - \delta_h I_h, \quad b_v - \gamma_v N_v - \delta_v I_v). \quad (11)$$

Now it is easy to prove that

$$\begin{aligned} \frac{dN_h}{dt} &\leq b_h - \mu_h N_h \leq 0 \quad \text{for} \quad N_h \geq \frac{b_h}{\mu_h}, \\ \frac{dN_v}{dt} &\leq b_v - \gamma_v N_v \leq 0 \quad \text{for} \quad N_v \geq \frac{b_v}{\gamma_v}. \end{aligned} \quad (12)$$

Thus, it follows that $\frac{dM}{dt} \leq 0$ which implies that Π is positively invariant set. Also a standard comparison theorem [24] is used to show that

$$0 \leq (N_h, N_v) \leq ((N_h(0)e^{-\mu_h t} + \frac{b_h}{\mu_h}(1 - e^{-\mu_h t}), \quad N_v(0)e^{-\gamma_v t} + \frac{b_v}{\gamma_v}(1 - e^{-\gamma_v t})).$$

Thus as $t \rightarrow \infty$, $0 \leq (N_h, N_v) \leq (\frac{b_h}{\mu_h}, \frac{b_v}{\gamma_v})$ and so Π is an attracting and positive invariant set. $\square$

Equilibria and Local Stability

To obtain the endemic equilibria of the system (9), setting the left side of the system (9) equal to zero, we get

$$\begin{cases} S_h^* = \frac{P_2(1+\alpha_2 I_h^*)(P_1(\gamma_v(1+\alpha_2 I_h^*)+\beta_3 I_h^*)+\alpha_1 \beta_3 b_v I_h^*)}{\beta_2 \beta_3 b_v(1+\alpha_2 I_h^*)+\beta_1(P_1(\gamma_v(1+\alpha_2 I_h^*)+\beta_3 I_h^*)+\alpha_1 \beta_3 b_v I_h^*)}, \\ R_h^* = \frac{\gamma_h I_h^*}{(\mu_h+\delta_h)}, \\ S_v^* = \frac{b_v(1+\alpha_2 I_h^*)}{(\gamma_v(1+\alpha_2 I_h^*)+\beta_3 I_h^*)}, \\ I_v^* = \frac{\beta_3 b_v I_h^*}{(\gamma_v+\delta_v)(\gamma_v(1+\alpha_2 I_h^*)+\beta_3 I_h^*)}, \end{cases}$$

where $P_1 = (\gamma_v + \delta_v)$, $P_2 = (\mu_h + \delta_h + \gamma_h)$, $P_3 = (\mu_h + \lambda_h)$.

The disease free equilibrium of the system (9) at E_0 is

$$S_h^0 = \frac{b_h}{\mu_h}, \quad S_v^0 = \frac{b_v}{\gamma_v}.$$

The basic reproduction number R_0 for system (9) is

$$R_0 = \frac{b_h \beta_2 \beta_3 b_v + \beta_1 (\gamma_v + \delta_v) b_h \gamma_v}{\mu_h (\gamma_v + \delta_v) (\mu_h + \delta_h + \gamma_h) \gamma_v}.$$

Disease Free Local Stability

Theorem 3.1 For $R_0 < 1$, the DFE of the model (9) at E_0 is stable locally asymptotically if $((\mu_h + \delta_h + \gamma_h) - \beta_1 b_h / \mu_h) > 0$, otherwise unstable.

Proof To show the local stability of the disease free equilibrium, we setting the left side of the system (9), equating to zero, we get the following jacobian matrix J_o

$$J_o = \begin{pmatrix} -\mu_h & -\beta_1 S_h^o & \lambda_h & 0 & -\beta_2 S_h^o \\ 0 & \beta_1 S_h^o - (\mu_h + \delta_h + \gamma_h) & 0 & 0 & \beta_2 S_h^o \\ 0 & \gamma_h & -(\mu_h + \lambda_h) & 0 & 0 \\ 0 & -\beta_3 S_v^o & 0 & -\gamma_v & 0 \\ 0 & \beta_3 S_v^o & 0 & 0 & -(\gamma_v + \delta_v) \end{pmatrix}.$$

The characteristics equation to the Jacobian matrix J_o is follows as

$$(\mu_h + \lambda)(\gamma_v + \lambda)(\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3) = 0, \quad (13)$$

where,

$$\begin{aligned} a_1 &= (\mu_h + \lambda_h) + (\gamma_v + \delta_v) + (\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 b_h}{\mu_h}, \\ a_2 &= (\mu_h + \lambda_h) \left((\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 b_h}{\mu_h} \right) + (\mu_h + \lambda_h)(\gamma_v + \delta_v) \\ &\quad + (\mu_h + \delta_h + \gamma_h)(\gamma_v + \delta_v)(1 - R_0), \\ a_3 &= (\mu_h + \lambda_h)(\mu_h + \delta_h + \gamma_h)(\gamma_v + \delta_v)(1 - R_0). \end{aligned} \quad (14)$$

Clearly, the two roots of the characteristics equation (14) are $-\mu_h$ and $-\gamma_v$ have negative real parts. The other roots can be determined from the cubic terms of the characteristics equation (14). Direct calculations by using (15), we get

$$\begin{aligned} a_1 a_2 - a_3 &= \left[(\mu_h + \lambda_h) + (\gamma_v + \delta_v) + (\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 b_h}{\mu_h} \right] \\ &\quad \times \left[(\mu_h + \lambda_h) \left\{ (\gamma_v + \delta_v) + \left((\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 b_h}{\mu_h} \right) \right\} \right] \\ &\quad + (\gamma_v + \delta_v) \left((\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 b_h}{\mu_h} \right) (\gamma_v + \delta_v) (\mu_h + \delta_h + \gamma_h) (1 - R_0) > 0. \end{aligned}$$

It follows from Routh-Hurwitz criteria, $a_i > 0$ for $i = 1, 2, 3$ and $a_1 a_2 - a_3 > 0$. Thus, all the eigenvalues of the system (9) have negative real parts if $R_0 < 1$ and $((\mu_h + \delta_h + \gamma_h) - \beta_1 b_h / \mu_h) > 0$. Therefore, the system (9) at the disease free equilibrium E_0 is locally asymptotically stable, if $R_0 < 1$ and $((\mu_h + \delta_h + \gamma_h) - \beta_1 b_h / \mu_h) > 0$. $\square$

The stability of the Disease free equilibrium of the system (9) for $R_0 < 1$ means that the disease dies out from the community. In this case the endemic equilibrium does not exist but we are interested to find the properties of the model about the endemic equilibrium point E_1 , when $R_0 > 1$. In the following, we show that the endemic equilibrium point E_1 of the model (9) is locally asymptotically stable, when $R_0 > 1$.

Local Stability of Endemic Equilibrium

In this subsection, we discuss the local stability of the model (9) at E_1 , by setting the left hand side of the system (9), equal to zero, we get the Jacobian matrix J^* in the following theorem.

Theorem 3.2 For $R_0 > 1$, the endemic equilibrium point E_1 of the system (9) is locally asymptotically stable if condition (18) and the terms under braces are positive, otherwise unstable.

Proof The Jacobian matrix J^* of the system (9) at E_1 is given by

$$J^* = \begin{bmatrix} -\mu_h - \frac{\beta_2 I_v^*}{1+\alpha_1 I_v^*} - \frac{\beta_1 I_h^*}{1+\alpha_2 I_h^*} & -\frac{\beta_1 S_h^*}{(1+\alpha_2 I_h^*)^2} & \lambda_h & 0 & -\frac{\beta_2 S_h^*}{(1+\alpha_1 I_v^*)^2} \\ \frac{\beta_2 I_v^*}{1+\alpha_1 I_v^*} + \frac{\beta_1 I_h^*}{1+\alpha_2 I_h^*} & \frac{\beta_1 S_h^*}{(1+\alpha_2 I_h^*)^2} - (\mu_h + \delta_h + \gamma_h) & 0 & 0 & \frac{\beta_2 S_h^*}{(1+\alpha_1 I_v^*)^2} \\ 0 & \gamma_h & -(\mu_h + \lambda_h) & 0 & 0 \\ 0 & -\frac{\beta_3 S_v^*}{(1+\alpha_2 I_h^*)^2} & 0 & -\gamma_v - \frac{\beta_3 I_h^*}{1+\alpha_2 I_h^*} & 0 \\ 0 & \frac{\beta_3 S_v^*}{(1+\alpha_2 I_h^*)^2} & 0 & \frac{\beta_3 I_h^*}{1+\alpha_2 I_h^*} & -(\gamma_v + \delta_v) \end{bmatrix}. \quad (15)$$

The characteristics equation of (15) is given by

$$\lambda^5 + A_1 \lambda^4 + A_2 \lambda^3 + A_3 \lambda^2 + A_4 \lambda + A_5 = 0, \quad (16)$$

where

$$A_1 = B_4 + \underbrace{(\mu_h + \lambda_h)}_{\text{positive}} - \underbrace{(B_1 + B_3)}_{\text{positive}} + (\gamma_v + \delta_v),$$

$$A_2 = B_4(\gamma_v + \delta_v) + B_5 B_6 + \underbrace{((\mu_h + \lambda_h) - (B_1 + B_3))}_{\text{positive}} (B_4 + (\gamma_v + \delta_v)) \\ + \underbrace{[(B_2 B_7 - B_1 B_3) - (B_3 + B_1)(\mu_h + \lambda_h)]}_{\text{positive}},$$

$$A_3 = \underbrace{(\mu_h + \lambda_h)[B_2 B_7 - B_1 B_3] - B_2 \gamma_h \lambda_h}_{\text{positive}} + \underbrace{[(B_2 B_7 - B_1 B_3) - (B_3 + B_1)(\mu_h + \lambda_h)]}_{\text{positive}} \\ \times [B_4 + (\gamma_v + \delta_v)] + \underbrace{[(\mu_h + \lambda_h) - (B_1 + B_3)]}_{\text{positive}} B_4 (\gamma_v + \delta_v) + [(2\mu_h + \lambda_h) + \gamma_v] B_5 B_6,$$

$$\begin{aligned}
A_4 &= \underbrace{\{(\mu_h + \lambda_h)[B_2 B_7 - B_1 B_3] - B_2 \gamma_h \lambda_h\}}_{\text{brace}} [B_4 + (\gamma_v + \delta_v)] \\
&\quad + B_4 (\gamma_v + \delta_v) \underbrace{[(B_2 B_7 - B_1 B_3) - (B_3 + B_1)(\mu_h + \lambda_h)]}_{\text{brace}} \\
&\quad + [(2\mu_h + \lambda_h)\gamma_v + \mu_h(\mu_h + \lambda_h)] B_5 B_6, \\
A_5 &= B_4 (\gamma_v + \delta_v) \underbrace{\{(\mu_h + \lambda_h)[B_2 B_7 - B_1 B_3] - B_2 \gamma_h \lambda_h\}}_{\text{brace}} + B_5 B_6 \mu_h (\mu_h + \lambda_h) \gamma_v, \quad (17)
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \mu_h + \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_2 I_v}{1 + \alpha_1 I_v}, \quad B_2 = \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_2 I_v}{1 + \alpha_1 I_v}, \\
B_3 &= \frac{\beta_1 S_h^*}{(1 + \alpha_2 I_h)^2} - (\mu_h + \delta_h + \gamma_h), \\
B_4 &= (\gamma_v + \frac{\beta_3 I_h}{1 + \alpha_2 I_h}), \quad B_5 = \frac{\beta_3 S_v}{(1 + \alpha_2 I_h)^2}, \quad B_6 = \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2}, \\
B_7 &= \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2}, \quad B_8 = \frac{\beta_3 I_h}{(1 + \alpha_2 I_h)}
\end{aligned}$$

The Routh-Hurwitz criteria for (16) is follows as

$$\begin{aligned}
H_5 &= \begin{vmatrix} A_1 & A_3 & A_5 & 0 & 0 \\ 1 & A_2 & A_4 & 0 & 0 \\ 0 & A_1 & A_3 & A_5 & 0 \\ 0 & 1 & A_2 & A_4 & 0 \\ 0 & 0 & A_1 & A_3 & A_5 \end{vmatrix} = A_1 A_4 A_5 (A_2 A_3 - A_1 A_4) - A_1 A_2^2 A_5^2 \\
&\quad + A_1 A_4 A_5^2 - A_4 A_5 (A_3^2 - A_1 A_5) + A_2 A_3 A_5^2 - A_5^2 > 0 \quad (18)
\end{aligned}$$

The eigenvalues of the characteristics equations (16) have negative real parts if $A_i > 0$ for $i = 1, 2, 3, 4, 5$, $H_5 > 0$ and $R_0 > 1$ and the terms under braces in positive. Thus, it follows from Routh-Hurtwiz criteria that the system (9) at the endemic equilibrium point E_1 is locally asymptotically stable, if $R_0 > 1$ and the terms under braces are positive and the conditions (18) are satisfied. $\square$

Global Stability

In this section, we investigate the global stability of the system (9) at the disease free equilibrium E_0 and the endemic equilibrium at E_2 . In the following theorem, we first show the disease free global stability for the case ($\lambda_h = 0$).

Theorem 4.1 *The system (9) at E_0 is globally asymptotically stable if $R_0 < 1$, otherwise unstable.*

Proof In order to show this result, we construct the following lyapanove function:

$$\begin{aligned}
L(t) &= (\gamma_v + \delta_v) \left(S_h - S_h^0 - S_h^0 \log \frac{S_h}{S_h^0} \right) + (\gamma_v + \delta_v) I_h + \frac{\beta_2 b_h}{\mu_h} \left(S_v - S_v^0 - S_v^0 \log \frac{S_v}{S_v^0} \right) \\
&\quad + \frac{\beta_2 b_h}{\mu_h} I_v. \quad (19)
\end{aligned}$$

Taking the time derivative of (19) along the solutions of system (9), we obtain

$$\begin{aligned} L'(t) = & (\gamma_v + \delta_v) \left(\frac{S_h - S_h^0}{S_h} \right) [b_h - \mu_h S_h - \left(\frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_2 I_v}{1 + \alpha_1 I_v} \right) S_h] \\ & + (\gamma_v + \delta_v) \left[\left(\frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_2 I_v}{1 + \alpha_1 I_v} \right) S_h - (\mu_h + \delta_h + \gamma_h) I_h \right] \\ & + \frac{\beta_2 b_h}{\mu_h} \left(\frac{S_v - S_v^0}{S_v} \right) [b_v - \gamma_v S_v - \frac{\beta_3 S_v I_h}{1 + \alpha_2 I_h}] + \frac{\beta_2 b_h}{\mu_h} \left[\frac{\beta_3 S_v I_h}{1 + \alpha_2 I_h} \right. \\ & \left. - (\gamma_v + \delta_v) I_v \right]. \end{aligned} \quad (20)$$

Making use of $S_h^0 = \frac{b_h}{\mu_h}$ and $S_v^0 = \frac{b_v}{\gamma_v}$ in equation (20), and taking some arrangements, we get

$$\begin{aligned} L'(t) = & -\mu_h (\gamma_v + \delta_v) \frac{(S_h - S_h^0)^2}{S_h} - \gamma_v \frac{\beta_2 b_h}{\mu_h} \frac{(S_v - S_v^0)^2}{S_v} \\ & - (\gamma_v + \delta_v) (\mu_h + \delta_h + \gamma_h) \alpha_2 \frac{I_h^2}{1 + \alpha_2 I_h} \\ & - \frac{(\gamma_v + \delta_v) \beta_2 b_h \alpha_1}{\mu_h} \frac{I_v^2}{1 + \alpha_1 I_v} - \frac{I_h}{1 + \alpha_2 I_h} (\gamma + \delta_v) (\mu_h + \delta_h + \gamma_h) (1 - R_0). \end{aligned} \quad (21)$$

$L'(t)$ is negative if $R_0 < 1$ and $L'(t) = 0$ if $S_h = S_h^0, S_v = S_v^0, I_h = I_v = 0$. Hence, the largest compact invariant set $(S_h, I_h, R_h, S_v, I_v) \in \Pi : L'(t) = 0$ is the singleton set E_0 , where E_0 is the disease free equilibrium. Thus, by Principle [25], E_0 is globally asymptotically stable in Π . $\square$

To prove that the endemic equilibrium point E_1 is globally asymptotically stable, we reduce the system (9), by using $R_h = N_h - S_h - I_h$ in first equation of the system (9) and $S_v = \frac{b_v - (\gamma_v + \delta_v) I_v}{\gamma_v}$ in fifth equation of the system (9), we obtain the following reduced system:

$$\begin{aligned} \frac{dS_h}{dt} = & b_h - \mu_h S_h - S_h \left(\frac{\beta_2 I_v}{1 + \alpha_1 I_v} + \frac{\beta_1 I_h}{1 + \alpha_2 I_h} \right) + \lambda_h (N_h - S_h - I_h), \\ \frac{dI_h}{dt} = & S_h \left(\frac{\beta_2 I_v}{1 + \alpha_1 I_v} + \frac{\beta_1 I_h}{1 + \alpha_2 I_h} \right) - (\mu_h + \delta_h + \gamma_h) I_h, \\ \frac{dI_v}{dt} = & \frac{\beta_3 I_h (b_v - (\gamma_v + \delta_v) I_v)}{\gamma_v (1 + \alpha_2 I_h)} - (\gamma_v + \delta_v) I_v, \end{aligned} \quad (22)$$

subject to nonnegative initial conditions

$$S_h = S_h(0) \geq 0, \quad I_h = I_h(0) \geq 0, \quad I_v = I_v(0) \geq 0.$$

The endemic equilibrium of the system (22) is denoted by E_2 .

To prove the global stability of the endemic equilibrium, we first give the following lemma.

Lemma *If the model $\frac{dx}{dt} = g(x)$, where $g : D \rightarrow \mathbb{R}^n$, posses a unique equilibrium x^* and also a compact absorbing set exists for x^* , then x^* is stable globally asymptotically given that the function $P(x)$ and a Lozinskii measure ℓ exist such that $q = \lim_{t \rightarrow \infty} \sup \sup_{x \in \frac{1}{t} \int_0^t \ell(B(x(s, x))) ds} < 0$ [26]. The symbols P , ℓ and B will stated in the following result.*

Theorem 4.2 *If $R_0 > 1$, then the system (22) is globally asymptotically stable at E_2 .*

Proof At E_2 , the Jacobian matrix $J(E_2)$ is given by

$$J(E_2) = \begin{pmatrix} -\mu_h - \frac{\beta_2 I_v}{1+\alpha_1 I_v} - \frac{\beta_1 I_h}{1+\alpha_2 I_h} - \lambda_h & -\frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - \lambda_h & -\frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} \\ \frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h}{1+\alpha_2 I_h} & \frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - (\mu_h + \delta_h + \gamma_h) & \frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} \\ 0 & \frac{\beta_3(b_v - (\gamma_v + \delta_v)I_v)}{\gamma_v(1+\alpha_2 I_h)^2} & -\frac{\beta_3 I_h(\gamma_v + \delta_v)}{\gamma_v(1+\alpha_2 I_h)} - (\gamma_v + \delta_v) \end{pmatrix}.$$

The second additive compound matrix of $J(E_2)$ is denoted by $J^{[2]}$ given by

$$J^{[2]} = \begin{pmatrix} A_{11} & \frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} & \frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} \\ \frac{\beta_3(b_v - (\gamma_v + \delta_v)I_v)}{\gamma_v(1+\alpha_2 I_h)^2} & A_{22} & -\frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - \lambda_h \\ 0 & \frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h}{1+\alpha_2 I_h} & A_{33} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1+\alpha_1 I_v} - \frac{\beta_1 I_h}{1+\alpha_2 I_h} + \frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - (\mu_h + \delta_h + \gamma_h), \\ A_{22} &= -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1+\alpha_1 I_v} - \frac{\beta_1 I_h}{1+\alpha_2 I_h} - \frac{\beta_3(\gamma_v + \delta_v)I_h}{\gamma_v(1+\alpha_2 I_h)} - (\gamma_v + \delta_v), \\ A_{33} &= \frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - P_2 - \frac{\beta_3(\gamma_v + \delta_v)I_h}{\gamma_v(1+\alpha_2 I_h)} - (\gamma_v + \delta_v), \end{aligned} \quad (23)$$

Consider the function

$$P = P(S_h, I_h, I_v) = \text{diag}\left(1, \frac{I_h}{I_v}, \frac{I_h}{I_v}\right) \quad (24)$$

with P^{-1} is

$$P^{-1} = \text{diag}\left(1, \frac{I_v}{I_h}, \frac{I_v}{I_h}\right).$$

And

$$P_f = \text{diag}\left(0, \frac{I_v I'_h - I'_v I_h}{I_v^2}, \frac{I_v I'_h - I'_v I_h}{I_v^2}\right).$$

$P_f P^{-1}$ is

$$P_f P^{-1} = \text{diag}\left(0, \frac{I'_h}{I_h} - \frac{I'_v}{I_v}, \frac{I'_h}{I_h} - \frac{I'_v}{I_v}\right),$$

And $P_f J^{[2]} P^{-1}$ is

$$P J^{[2]} P^{-1} = \begin{pmatrix} A_{11} & \frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} \frac{I_v}{I_h} & \frac{\beta_2 S_h}{(1+\alpha_1 I_v)^2} \frac{I_v}{I_h} \\ \frac{\beta_3(b_v - (\gamma_v + \delta_v)I_v)}{\gamma_v(1+\alpha_2 I_h)^2} \frac{I_h}{I_v} & A_{22} & -\frac{\beta_1 S_h}{(1+\alpha_2 I_h)^2} - \lambda_h \\ 0 & \frac{\beta_2 I_v}{1+\alpha_1 I_v} + \frac{\beta_1 I_h}{1+\alpha_2 I_h} & A_{33} \end{pmatrix}.$$

So, we write

$$B = P_f P^{-1} + P J^{[2]} P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - (\mu_h + \delta_h + \gamma_h), \\ B_{12} &= \max \left\{ \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2} \frac{I_v}{I_h}, \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2} \frac{I_v}{I_h} \right\}, \\ B_{21} &= \left(\frac{I_h}{I_v} \frac{\beta_3 (b_v - (\gamma_v + \delta_v) I_v)}{\gamma_v (1 + \alpha_2 I_h)^2}, 0 \right)^T, \\ B_{22} &= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} T_{11} &= \frac{I'_h}{I_h} - \frac{I'_v}{I_v} - \mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} - (\gamma_v + \delta_v), \\ T_{12} &= -\frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - \lambda_h, \\ T_{21} &= \frac{\beta_2 I_v}{1 + \alpha_1 I_v} + \frac{\beta_1 I_h}{1 + \alpha_2 I_h}, \\ T_{22} &= \frac{I'_h}{I_h} - \frac{I'_v}{I_v} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} - (\gamma_v + \delta_v) - (\mu_h + \delta_h + \gamma_h). \end{aligned}$$

Suppose the vector $(\hat{u}, \hat{v}, \hat{w})$ in R^3 and its norm $\|\cdot\|$ will be defined as

$$\|(\hat{u}, \hat{v}, \hat{w})\| = \max\{|\hat{u}|, |\hat{v}| + |\hat{w}|\}.$$

Suppose μB represent Lozinski measure with the above defined norm. So as described in [26], we choose

$$\mu(B) \leq \sup(g_1, g_2),$$

where

$$g_1 = \mu(B_{11}) + |B_{12}|, \quad g_2 = |B_{21}| + \mu(B_{22}),$$

$|B_{21}|$ and $|B_{12}|$ are the matrix norm associated with vector ℓ and μ represent the Lozinski measure with respect to ℓ norm, then

$$\begin{aligned} \mu(B_{11}) &= -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - (\mu_h + \delta_h + \gamma_h), \\ |B_{12}| &= \max \left\{ \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2} \frac{I_v}{I_h}, \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2} \frac{I_v}{I_h} \right\}, \end{aligned} \quad (25)$$

Therefore,

$$\begin{aligned} g_1 &= \mu(B_{11}) + |B_{12}| \\ &= -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} + \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)^2} \frac{I_v}{I_h} - (\mu_h + \delta_h + \gamma_h), \\ &\leq -\mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)} + \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)} \frac{I_v}{I_h} - (\mu_h + \delta_h + \gamma_h), \\ &\leq \frac{I'_h}{I_h} - \mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h}, \end{aligned}$$

using system (22),

$$\frac{I'_h}{I_h} = \frac{I_v}{I_h} \frac{\beta_2 S_h}{(1 + \alpha_1 I_v)} + \frac{\beta_1 S_h}{1 + \alpha_2 I_h} - (\mu_h + \delta_h + \gamma_h),$$

we get

$$g_1 \leq \frac{I'_h}{I_h} - \mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h}.$$

Again

$$|B_{21}| = \frac{I_h \beta_3 (b_v - (\gamma_v + \delta_v) I_v)}{I_v \gamma_v (1 + \alpha_2 I_h)^2},$$

$$\begin{aligned} \mu(B_{22}) &= \sup \left\{ \frac{I'_h}{I_h} - \frac{I'_v}{I_v} - \mu_h - \lambda_h - \frac{\beta_2 I_v}{1 + \alpha_1 I_v} - \frac{\beta_1 I_h}{1 + \alpha_2 I_h} - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} \right. \\ &\quad \left. - (\gamma_v + \delta_v) + \frac{\beta_2 I_v}{1 + \alpha_1 I_v} + \frac{\beta_1 I_h}{1 + \alpha_2 I_h}, \frac{I'_h}{I_h} - \frac{I'_v}{I_v} + \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} \right. \\ &\quad \left. - (\gamma_v + \delta_v) - (\mu_h + \delta_h + \gamma_h) - \frac{\beta_1 S_h}{(1 + \alpha_2 I_h)^2} - \lambda_h \right\}, \\ &= \frac{I'_h}{I_h} - \frac{I'_v}{I_v} - \mu_h - \lambda_h - (\gamma_v + \delta_v) - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} - (\mu_h + \delta_h + \gamma_h) - \lambda_h. \end{aligned}$$

So

$$\begin{aligned} g_2 &= \mu(B_{22}) + |B_{21}|, \\ &= \frac{I'_h}{I_h} - \frac{I'_v}{I_v} - \mu_h - \lambda_h - (\gamma_v + \delta_v) - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} \\ &\quad - (\mu_h + \delta_h + \gamma_h) - \lambda_h + \frac{I_h \beta_3 (b_v - (\gamma_v + \delta_v) I_v)}{I_v \gamma_v (1 + \alpha_2 I_h)^2}, \\ &\leq \frac{I'_h}{I_h} - \frac{I'_v}{I_v} - \mu_h - \lambda_h - (\gamma_v + \delta_v) - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} \\ &\quad - (\mu_h + \delta_h + \gamma_h) - \lambda_h + \frac{I_h \beta_3 (b_v - (\gamma_v + \delta_v) I_v)}{I_v \gamma_v (1 + \alpha_2 I_h)^2}, \\ &\leq \frac{I'_h}{I_h} - \mu_h - \lambda_h - \frac{\beta_3 (\gamma_v + \delta_v) I_h}{\gamma_v (1 + \alpha_2 I_h)} - (\mu_h + \delta_h + \gamma_h) - \lambda_h, \end{aligned}$$

we used in above g_2 , the third equation of the system (22),

$$\frac{I'_v}{I_v} = \frac{I_h \beta_3 (b_v - (\gamma_v + \delta_v) I_v)}{I_v \gamma_v (1 + \alpha_2 I_h)^2} - (\gamma_v + \delta_v).$$

So,

$$\mu B \leq \sup(g_1, g_2) \leq \frac{I'_h}{I_h} - \mu$$

then,

$$q = \frac{1}{t} \int_0^t \mu B ds \leq \frac{1}{t} \int_0^t \left(\frac{I'_h}{I_h} - \mu \right) ds = \frac{1}{t} \ln \frac{I_h(t)}{I_h(0)} - \mu.$$

Table 1 Parameter values used in numerical simulation

Notation	Parameter description	Value	References
b_h	Recruitment rate for human population	1.2	Assumed
β_1	Transmission rate for human population	0.04	Assumed
β_2	Transmission rate for vector population	0.04	Assumed
β_3	Transmission rate between S_v and I_h	0.04	Assumed
μ_h	Natural mortality rate of human population	4.6×10^{-5}	[12]
λ_h	Proportionality constant	2.85×10^{-3}	[12]
δ_h	Disease death rate for human population	1.0×10^{-3}	[28]
γ_v	Natural mortality rate of vector population	1.8×10^{-3}	[12]
δ_v	Disease death rate for vector population	0.04	Assumed
α_1	The rate at which the infection force saturates effect	0.83	Assumed
α_2	The rate at which the infection force saturates	0.83	Assumed
b_v	Recruitment rate for vector population	1.3	Assumed
γ_h	Recovery rate of the infection	2.7×10^{-3}	[28]

Implies that $q \leq -\frac{\mu}{2} < 0$. Thus the result [27], implies that the positive equilibrium point E_2 of the system (22) is globally asymptotically stable. $\square$

Numerical Simulations

In this section, we present the numerical solution of the system (9). The parameter values used in the numerical solution are shown in Table 1. Figures 2 and 3 shows the population behavior of human and vector population respectively. Figures 4, 5, 6, 7 and 8 represents Susceptible human, infected human, recovered human, susceptible vector and infected vector with the parameters α_1 and α_2 . Figure 2 represent the human population, the dashed line represent the behavior of susceptible human, the bold line shows the population behavior of infected individuals, the dot dashed line shows the population behavior of recovered human. Figure 3 shows the population of behavior of vector, the susceptible vector is represented by bold line and the dashed line shows the population of infected vector. The plot shows the susceptible human population with saturation effect for the values of α_1 and α_2 in Fig. 4. The graph of susceptible human decreases with the saturated effect, for the values of $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$. The dotted line shows the population of susceptible human without saturation effect and the dashed line shows the saturation effect of human and the saturation effect of vector is represented by a dot dashed line. Due the effect of α_1 and α_2 the population of

Fig. 2 Population behavior of human

Fig. 3 Population behavior of vector

susceptible individuals change sharply. The plot shows the infected human population with the saturated effect of the values α_1 and α_2 in Fig. 5. The graph of infected human decreases with the saturated effect, for the values of $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$. The dotted line shows the population of infected human without saturated effect and the dashed line shows the saturated effect. Due the effect of α_1 and α_2 the population of infected individuals changes. The plot shows the recovered human population with saturated effect for the values of α_1 and α_2 in Fig. 6. The graph of recovered human decreases with the saturated effect, for the values for $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$. The dotted line shows the population of recovered human without saturated effect and the dashed line shows the saturated effect of human and the saturated effect of vector is represented by a dot dashed line. Due the effect of α_1 and α_2 the population of recovered individuals increases. The plot shows the vector population with saturated effect for the values of α_1 and α_2 in Fig. 7. The graph of susceptible vector

Fig. 4 Represents the graph of susceptible human with the effect of α_1 and α_2 with the parameters $\mu_h = 0.31$, $b_h = 0.99$, $\alpha_1 = 0.83$, $\alpha_2 = 0.083$, $\delta_v = 0.32$, $\beta_1 = 0.91$, $b_v = 290$, $\gamma_h = 0.71$, $\gamma_v = 0.8$, $\beta_3 = 0.22$, $\delta_h = 0.61$, $\lambda_h = 0.71$, $\beta_2 = 0.92$

Fig. 5 Represents the graph of infected human with the effect of α_1 and α_2 with the parameters $\mu_h = 0.31$, $b_h = 0.99$, $\beta_2 = 0.92$, $\alpha_1 = 0.83$, $\alpha_2 = 0.083$, $\beta_1 = 0.91$, $\delta_v = 0.32$, $b_v = 290$, $\gamma_h = 0.71$, $\gamma_v = 0.8$, $\beta_3 = 0.22$, $\delta_h = 0.61$, $\lambda_h = 0.71$

decreases with the saturated infection rate, for the values for $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$. The dotted line shows the population of susceptible human without saturated effect and the dashed line shows the saturated effect of human and the saturated effect of vector is represented by a dot dashed line. Due the effect of α_1 and α_2 the population of susceptible individuals decreases. The plot shows the infected vector population with the effect of α_1 and α_2 in Fig. 8. The graph of infected vector decreases with the with the saturation factor $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$. The dotted line shows the population of infected vector without saturation effect and the dashed line shows the saturation effect of human and the saturated parameter effect of vector is represented by a dot dashed line. Due the effect of α_1 and α_2 the population

Fig. 6 Represents the graph of recovered human with the effect of α_1 and α_2 with the parameters $\mu_h = 0.31$, $b_h = 0.99$, $\beta_1 = 0.91$, $\alpha_1 = 0.83$, $\alpha_2 = 0.083$, $\delta_v = 0.32$, $b_v = 290$, $\gamma_h = 0.71$, $\beta_3 = 0.22$, $\gamma_v = 0.8$, $\beta_2 = 0.92$, $\delta_h = 0.61$, $\lambda_h = 0.71$

Fig. 7 Represents the graph of susceptible vector with the effect of α_1 and α_2 with the parameters $\mu_h = 0.31$, $b_h = 0.99$, $\alpha_1 = 0.83$, $\beta_3 = 0.22$, $\alpha_2 = 0.083$, $\delta_v = 0.32$, $\beta_1 = 0.91$, $b_v = 290$, $\gamma_h = 0.71$, $\gamma_v = 0.8$, $\beta_2 = 0.92$, $\delta_h = 0.61$, $\lambda_h = 0.71$

of infected vector decreases. The suggested parameters $\alpha_1 = 0.83$ and $\alpha_2 = 0.83$ can reduce the infection in the leptospirosis disease interaction with human.

Conclusion

The dynamic behavior of leptospirosis disease with saturated incidence has been presented successfully. The mathematical results obtained with their mathematical interpretation. The stability of the disease free and endemic equilibrium is completely described by the basic

Fig. 8 Represents the graph of susceptible vector with the effect of α_1 and α_2 with the parameters $\mu_h = 0.31$, $b_h = 0.99$, $\alpha_1 = 0.83$, $\beta_3 = 0.22$, $\alpha_2 = 0.083$, $\delta_v = 0.32$, $b_v = 290$, $\gamma_h = 0.71$, $\gamma_v = 0.8$, $\beta_1 = 0.91$, $\delta_h = 0.61$, $\lambda_h = 0.71$, $\beta_2 = 0.92$

reproduction number R_0 . We found, when $R_0 < 1$, the disease free equilibrium at E_0 is stable locally as well as globally. Further, the endemic equilibrium is obtained. We obtained that, when $R_0 > 1$, the endemic equilibrium is stable both locally and globally. Moreover, the numerical results for the model is obtained and briefly discussed. The parameters α_1 and α_2 effect have been discussed in numerical solution.

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