Impact of Torsional Load on a Penny-Shaped Crack in an Elastic Layer Sandwiched Between Two Elastic Half-Space

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Abstract Sudden impact of torsional load on a penny-shaped crack in an elastic layer sandwiched between two elastic half-spaces has been analyzed in this paper. The axi-symmetric mixed boundary value problem is reduced to the problem of solving a pair of dual integral equations by using Hankel and Laplace transforms. The problem is further reduced to Fredholm integral equation of second kind which is solved numerically. Taking numerical inversion of Laplace transform, stress intensity factor at the tip of the crack is calculated and plotted for different values of the parameters.

Keywords Penny-shaped crack · Hankel and Laplace transforms · Fredholm integral equation · Stress intensity factor

Introduction

Advanced composite materials are multi-phased nonhomogeneous materials with anisotropic properties. This complicates the stress analysis for fracture, particularly if the loading is time-dependent, because the crack geometry involves sharp edges.

An effective approach for finding dynamic stresses in a nonhomogeneous composite containing a crack has been developed by Sih and Chen [1] by utilizing both the Laplace and Fourier transforms.

The dynamic problem of torsional impact are important in view of construction technology and fabrication process. Eason [2], Ghosh [3], Shail [4] have considered the sudden torsional impact problem in half-space. Shabuya [5] discussed the problem of torsional impact of a

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thick elastic plate. The torsional oscillations of a rigid circular disc attached to an elastic layer bonded to an elastic half-space has been considered by Keer et al. [6]. S.Itou [7] analyzed the problem of transient dynamic stress intensity factors around a crack in a nonhomogeneous interfacial layer between two dissimilar elastic half-planes. Das et al. [8] solved the problem of determining the stress intensity factor for an interfacial crack between two orthotropic half planes bonded to a dissimilar orthotropic layer with a punch. They reduced the problem to a system of simultaneous integral equations which are solved by Chebyshev polynomials. The problem of two perfectly bonded dissimilar orthotropic strip with an interfacial crack is studied by Li [9]. Shear wave interaction with a pair of rigid strips in elastic strip was analyzed by Pramanick et al. [10]. Wu et al. [11] considered the torsional vibration problem of rigid circular plate on transversely isotropic saturated soil. Morteza et al. [12, 13] considered the vibration problem of rigid circular disc on transversely isotropic media. Recently Matbuly [14] considered the problem of mode *III* crack perpendicular to the interface of a bi-strip composite.

The model of the penny-shaped crack has been analysed as early as 1970's by F. Erdogan and K. Arin [15, 16]. Then Ueda et al. [17, 18] discussed the problem of torsional impact response of a penny-shaped interface crack. A penny-shaped interface crack of two bonded dissimilar transversely isotropic elastic half-spaces and dissimilar nonhomogeneous elastic layers under axially symmetric torsion have been considered by Saxena et al. [19, 20]. The contact problem for an open penny-shaped crack under normally tension-compression wave has been analysed by Menshykov et al. [21]. Li et al. [22] solved the problem of Coulomb traction on a penny-shaped crack in a three dimensional piezoelectric body. Dovzhik [23] considered the problem of fracture of a half-space compressed along a penny-shaped crack located at a short distance from the surface and Lee [24] discussed the problem of penny-shaped crack in a plate of finite thickness subjected to a uniform shearing stress. A penny-shaped crack subjected to uniform symmetric heat flux has been resolved by Yang et al. [25] and semi-analytical solution for penny-shaped crack in a soft inhomogeneous layer has been solved by Aizikovitch et al. [26]. Analysis of crack problems in composite structures is still of significant interests.

In our paper we have analyzed the impact of torsional load on a penny shaped crack in an elastic layer sandwiched between two elastic half-space. The axi-symmetric mixed boundary value problem is reduced to the problem of solving a pair of dual integral equation by using Hankel and Laplace transforms. It is further reduced to Fredholm integral equation of second kind which is solved numerically. After taking numerical inversion of Laplace transform stress intensity factor (SIF) has been calculated at the tip of the crack and presented by means of graphs.

Basic Equations of the Problem

Consider a penny-shaped crack of radius a which lies in a layer of thickness $2b$ with material properties μ_1, ν_1, ρ_1 with reference to the cylindrical polar co-ordinates (r, θ, z) which referred to rectangular co-ordinate system (X, Y, Z) . This layer is bonded between two elastic half space with properties μ_2, ν_2, ρ_2 as shown in Fig. 1.

In this case, the only non-zero component of displacement is the circumferential component u_θ and that the only non-zero component of stress $\sigma_{\theta z}$ is given by the relation

$$\sigma_{\theta z} = \mu \frac{\partial u_\theta}{\partial z} \quad (1)$$

Fig. 1 Geometry of the penny-shaped crack

where μ is the shear modulus of the elastic material and u_θ satisfies the partial differential equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{1}{c_j^2} \frac{\partial^2 u_\theta}{\partial t^2} \quad (j = 1, 2) \quad (2)$$

where $c_j = \left(\frac{\mu_j}{\rho_j}\right)^{\frac{1}{2}}$ is the shear wave velocity and ρ is the density of the material. In Eq. (2) we shall employ subscripts “1” and “2” refers to the layer with the crack and half-space respectively.

To remove the time variable from Eq. (2), we use the Laplace transform pair defined by

$$\bar{f}(p) = \int_0^\infty f(t) e^{-pt} dt \quad (3)$$

$$f(t) = \frac{1}{2\pi i} \int_{Br} \bar{f}(p) e^{pt} dp \quad (4)$$

where Br is the Bromwich path of integration.

Applying Laplace transform in Eq. (2) we have

$$\frac{\partial^2 \bar{u}_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_\theta}{\partial r} - \frac{\bar{u}_\theta}{r^2} + \frac{\partial^2 \bar{u}_\theta}{\partial z^2} = \frac{p^2}{c_j^2} \bar{u}_\theta \quad (5)$$

Formulation and Method of Solution

Apply the torsional load to the penny-shaped crack such that the upper and lower surface will move in opposite directions. It is assumed that the magnitude of this load is τ_0 and since it is applied suddenly from $t = 0$, so we will use the Heaviside unit step function.

The boundary conditions on the plane $z = 0$ for $r \leq a$ and $r > a$ can be written in the following form:

$$\tau_{\theta z}^{(1)}(r, 0, t) = \tau_0 \left(\frac{r}{a} \right) H(t), \quad 0 \leq r \leq a \quad (6)$$

$$u_{\theta}^{(1)}(r, 0, t) = 0, \quad r > a \quad (7)$$

where as the interfaces of the layer $z = \pm b$ are bonded perfectly to the half-spaces, we obtain the following continuity conditions:

$$\tau_{\theta z}^{(1)}(r, b, t) = \tau_{\theta z}^{(2)}(r, b, t) \quad (8)$$

$$u_{\theta}^{(1)}(r, b, t) = u_{\theta}^{(2)}(r, b, t) \quad (9)$$

where $H(t)$ is the Heaviside unit step function.

Taking the Laplace transform of the boundary and continuity conditions (6)–(9), we get

$$\bar{\tau}_{\theta z}^{(1)}(r, 0, p) = \frac{\tau_0 \left(\frac{r}{a} \right)}{p}, \quad 0 \leq r < a \quad (10)$$

$$\bar{u}_{\theta}^{(1)}(r, 0, p) = 0, \quad r > a \quad (11)$$

$$\bar{\tau}_{\theta z}^{(1)}(r, b, p) = \bar{\tau}_{\theta z}^{(2)}(r, b, p) \quad (12)$$

$$\bar{u}_{\theta}^{(1)}(r, b, p) = \bar{u}_{\theta}^{(2)}(r, b, p) \quad (13)$$

To find the solution of Eq. (5), we use the Hankel transform and obtain the Laplace transform of the displacement component for the regions I ($0 < z < b$) and II ($z > b$) in the following form

$$\bar{u}_{\theta}^{(1)}(r, z, p) = \int_0^{\infty} [A_1(s, p) e^{-\gamma_1 z} + A_2(s, p) e^{\gamma_1 z}] J_1(sr) ds \quad (14)$$

$$\text{and} \quad \bar{u}_{\theta}^{(2)}(r, z, p) = \int_0^{\infty} A_3(s, p) e^{-\gamma_2(z-b)} J_1(sr) ds \quad (15)$$

where

$$\begin{aligned} \gamma_1 &= (s^2 + k_1^2)^{\frac{1}{2}}, & \gamma_2 &= (s^2 + k_2^2)^{\frac{1}{2}} \\ k_1 &= p/c_1, & k_2 &= p/c_2 \\ c_1 &= \sqrt{\frac{\mu_1}{\rho_1}}, & c_2 &= \sqrt{\frac{\mu_2}{\rho_2}} \end{aligned} \quad (16)$$

and $J_1(\cdot)$ is the bessel function of the first kind. Moreover, since the composite geometry is symmetrical so we consider only the solution in the upper half-space, $z \geq 0$.

In Eqs. (14) and (15), A_1, A_2, A_3 are the constants which are to be determined later on. With the help of Eqs. (14) and (15), the expression for $\bar{\tau}_{\theta z}$ is

$$\bar{\tau}_{\theta z}^{(1)} = -\mu_1 \int_0^{\infty} \gamma_1 \{A_1(s, p) e^{-\gamma_1 z} - A_2(s, p) e^{\gamma_1 z}\} J_1(sr) ds \quad (17)$$

$$\text{and} \quad \bar{\tau}_{\theta z}^{(2)} = -\mu_2 \int_0^{\infty} \gamma_2 A_3(s, p) e^{-\gamma_2(z-b)} J_1(sr) ds \quad (18)$$

Using the above two expressions, the continuity conditions (12) and (13) yield

$$\int_0^{\infty} [A_1(s, p) e^{-\gamma_1 b} + A_2(s, p) e^{\gamma_1 b}] J_1(sr) ds = \int_0^{\infty} A_3(s, p) J_1(sr) ds \quad (19)$$

$$\text{and } \mu_1 \int_0^\infty \gamma_1 \left\{ A_1(s, p) e^{-\gamma_1 b} - A_2(s, p) e^{\gamma_1 b} \right\} J_1(sr) ds = \mu_2 \int_0^\infty \gamma_2 A_3(s, p) J_1(sr) ds \quad (20)$$

Inverting (19) and (20) by means of Hankel inversion formula, we obtain the pair of equations

$$A_1(s, p) e^{-\gamma_1 b} + A_2(s, p) e^{\gamma_1 b} = A_3(s, p) \quad (21)$$

$$\gamma_1 A_1(s, p) e^{-\gamma_1 b} - \gamma_1 A_2(s, p) e^{\gamma_1 b} = G \gamma_2 A_3(s, p) \quad (22)$$

where $G = \mu_2/\mu_1$.

Solving the pair of Eqs. (21) and (22) for $A_3(s, p)$ and $A_2(s, p)$ in terms of $A_1(s, p)$, we have

$$A_3(s, p) = \frac{2\gamma_1}{\gamma_1 + G \gamma_2} e^{-\gamma_1 b} A_1(s, p) \quad (23)$$

$$\text{and } A_2(s, p) = \left(\frac{\gamma_1 - G \gamma_2}{\gamma_1 + G \gamma_2} \right) e^{-2\gamma_1 b} A_1(s, p) \quad (24)$$

Applying (14) and (17) in the boundary conditions (11) and (10) we obtain the dual integral equations in terms of the unknown function $B_1(s, p)$.

$$\int_0^\infty B_1(s, p) J_1(sr) ds = 0, \quad r > a \quad (25)$$

$$\int_0^\infty s P_1(s, p) B_1(s, p) J_1(sr) ds = -\frac{\tau_0 \left(\frac{r}{a} \right)}{p \mu_1}, \quad 0 \leq r < a \quad (26)$$

where

$$B_1(s, p) = -A_1(s, p) \left[\frac{\gamma_1 (1 + e^{-2\gamma_1 b}) - G \gamma_2 (e^{-2\gamma_1 b} - 1)}{\gamma_1 + G \gamma_2} \right] \quad (27)$$

$$P_1(s, p) = \frac{\gamma_1}{s} \left[\frac{1 + \left(\frac{\gamma_1 + G \gamma_2}{G \gamma_2 - \gamma_1} \right) e^{2\gamma_1 b}}{1 - \left(\frac{\gamma_1 + G \gamma_2}{G \gamma_2 - \gamma_1} \right) e^{2\gamma_1 b}} \right] \quad (28)$$

To solve the dual integral equations (25) and (26), the form of $B_1(s, p)$ that satisfies Eq. (25) can be found to be

$$B_1(s, p) = \frac{4 \tau_0 a^{\frac{5}{2}}}{3 \mu_1 p (2\pi)^{\frac{1}{2}}} \sqrt{s} \int_0^1 \sqrt{\xi} \Phi_3^*(\xi, p) J_{\frac{3}{2}}(sa\xi) d\xi \quad (29)$$

where $\Phi_3^*(\xi, p)$ in Eq. (29), is unknown function to be determined later from the Fredholm integral equation of the second kind.

Using the formula

$$\int_0^\infty \sqrt{s_1} J_1(R s_1) J_{\frac{3}{2}}(s_1 \xi) ds_1 = \begin{cases} 0, & 0 < \xi < R \\ \sqrt{\frac{2}{\pi}} \frac{R}{\xi^{\frac{3}{2}} \sqrt{\xi^2 - R^2}}, & 0 < R < \xi \end{cases} \quad (30)$$

the expression (25) for $R > 1$ is automatically satisfied with $s_1 = sa$, $R = \frac{r}{a}$.

Applying the formula

$$J_{\frac{3}{2}}(s_1 \xi) = -\frac{\sqrt{\xi}}{s_1} \frac{d}{d\xi} \left\{ \xi^{-\frac{1}{2}} J_{\frac{1}{2}}(s_1 \xi) \right\}$$

and introducing the function

$$\Phi^*(\xi, p) = \xi^{-\frac{1}{2}} \frac{d}{d\xi} \{\xi \Phi_3^*(\xi, p)\} \quad (31)$$

we obtain

$$B_1\left(\frac{s_1}{a}, p\right) = \frac{4 \tau_0 a^2}{3 \mu_1 p (2\pi s_1)^{\frac{1}{2}}} \left[\int_0^1 \Phi^*(\xi, p) J_{\frac{1}{2}}(s_1 \xi) d\xi - \Phi_3^*(1, p) J_{\frac{1}{2}}(s_1) \right] \quad (32)$$

After some algebraic manipulation, the expression for $P_1\left(\frac{s_1}{a}, p\right)$ approaches to -1 for large values of s_1 , that is

$$1 + P_1\left(\frac{s_1}{a}, p\right) \longrightarrow 0 \quad \text{as } s_1 \longrightarrow \infty.$$

So, the Eq. (26) in terms of dimensionless quantities, can now be written as

$$\begin{aligned} & \int_0^\infty s_1 B_1\left(\frac{s_1}{a}, p\right) J_1(R s_1) ds_1 \\ &= \frac{\tau_0 a^2 R}{p \mu_1} + \int_0^\infty s_1 \left[1 + P_1\left(\frac{s_1}{a}, p\right) \right] B_1\left(\frac{s_1}{a}, p\right) J_1(R s_1) ds_1, \quad (R < 1) \end{aligned}$$

With the help of Eqs. (29) and (32) and using the result

$$\int_0^\infty t^{\frac{1}{2}} J_1(at) J_{\frac{1}{2}}(bt) dt = \begin{cases} 0, & 0 < a < b \\ \sqrt{\frac{2}{\pi}} \frac{\sqrt{b}}{a \sqrt{a^2 - b^2}}, & 0 < b < a \end{cases}$$

the above equation becomes

$$\begin{aligned} \frac{4 \tau_0 a^2}{3 \mu_1 p (2\pi)^{\frac{1}{2}}} \sqrt{\frac{2}{\pi}} \int_0^R \frac{\sqrt{\xi}}{R \sqrt{R^2 - \xi^2}} \Phi^*(\xi, p) d\xi &= \frac{\tau_0 R a^2}{p \mu_1} + \frac{4 \tau_0 a^2}{3 \mu_1 p (2\pi)^{\frac{1}{2}}} \int_0^1 \sqrt{\xi} \Phi_3^*(\xi, p) d\xi \\ &\quad \times \int_0^\infty \left[1 + P_1\left(\frac{s_1}{a}, p\right) \right] s_1^{\frac{3}{2}} J_1(R s_1) J_{\frac{3}{2}}(\xi s_1) ds_1 \\ \text{or, } \int_0^R \frac{\sqrt{\xi}}{\sqrt{R^2 - \xi^2}} \Phi^*(\xi, p) d\xi &= \frac{3\pi R^2}{4} + \sqrt{\frac{\pi}{2}} R \int_0^1 \sqrt{\xi} \Phi_3^*(\xi, p) d\xi \\ &\quad \times \int_0^\infty s_1^{\frac{3}{2}} M\left(\frac{s_1}{a}, p\right) J_1(R s_1) J_{\frac{3}{2}}(\xi s_1) ds_1 = F(R) \end{aligned} \quad (33)$$

where $M\left(\frac{s_1}{a}, p\right) = 1 + P_1\left(\frac{s_1}{a}, p\right)$

and $F(R) = \frac{3\pi R^2}{4} + \sqrt{\frac{\pi}{2}} R \int_0^1 \sqrt{\xi} \Phi_3^*(\xi, p) d\xi \int_0^\infty s_1^{\frac{3}{2}} M\left(\frac{s_1}{a}, p\right) J_1(R s_1) J_{\frac{3}{2}}(\xi s_1) ds_1.$

To solve the Eq. (33), we adopt Abel's integral formula by setting

$$\sqrt{\xi} \Phi^*(\xi, p) = \frac{2}{\pi} \frac{d}{d\xi} \int_0^\xi \frac{R F(R)}{\sqrt{\xi^2 - R^2}} dR$$

and with the help of expression (31) the given Eq. (33) becomes

$$\begin{aligned} \xi \Phi_3^*(\xi, p) &= \frac{2}{\pi} \int_0^\xi \frac{R}{\sqrt{\xi^2 - R^2}} \left[\frac{3\pi R^2}{4} + \sqrt{\frac{\pi}{2}} R \int_0^1 \sqrt{u} \Phi_3^*(u, p) du \right. \\ &\quad \left. \times \int_0^\infty s_1^{\frac{3}{2}} M\left(\frac{s_1}{a}, p\right) J_1(R s_1) J_{\frac{3}{2}}(u s_1) ds_1 \right] dR \end{aligned} \quad (34)$$

Introducing Hankel transform, we obtain from (34) the following Fredholm integral equation of the second kind

$$\Phi_3^*(\xi, p) + \int_0^1 \Phi_3^*(u, p) L_3(\xi, u, p) du = \xi^2 \quad (35)$$

where

$$L_3(\xi, u, p) = -\sqrt{\xi u} \int_0^\infty s_1 \left[1 + P_1\left(\frac{s_1}{a}, p\right) \right] J_{\frac{3}{2}}(s_1 u) J_{\frac{3}{2}}(s_1 \xi) ds_1 \quad (36)$$

Stress Intensity Factor

In order to determine stress intensity factor (SIF) $K_1(t)$ from its Laplace transform $K_1^*(p)$, the stress components in the matrix layer are expanded in terms of the local co-ordinates r_1 and θ_1 for small values of r_1 . The local co-ordinates (r_1, θ_1) are related to (r, θ) as the following

$$\left. \begin{aligned} r &= a + r_1 \cos \theta_1 \\ z &= r_1 \sin \theta_1 \end{aligned} \right\} \quad (37)$$

where $x = r \cos \theta$, $y = r \sin \theta$.

From Eq. (17), we obtain

$$\bar{\tau}_{\theta z}^{(1)}(r, 0, p) = -\mu_1 \int_0^\infty s P_1(s, p) B_1(s, p) J_1(sr) ds, \quad (r > a) \quad (38)$$

Since $P_1(s, p) \rightarrow -1$ as $s \rightarrow \infty$, the Eq. (38) is given by

$$\begin{aligned} \bar{\tau}_{\theta z}^{(1)}(r, 0, p) &= \frac{4\tau_0}{3\pi p} \left[\int_0^1 \frac{\sqrt{\xi}}{R\sqrt{R^2 - \xi^2}} \phi^*(\xi, p) d\xi - \phi_3^*(1, p) \frac{1}{R\sqrt{R^2 - 1}} \right] \\ &= -\frac{4\tau_0}{3\pi p} \frac{\phi_3^*(1, p)}{(R\sqrt{R^2 - 1})} + O(1) \quad (R > 1) \end{aligned} \quad (39)$$

To find the singularity it is important to evaluate the term $|\bar{\tau}_{\theta z}^{(1)}(r, 0, p)|$. So,

$$\begin{aligned} |\bar{\tau}_{\theta z}^{(1)}(r, 0, p)| &= \frac{4\tau_0}{3\pi p} \frac{\phi_3^*(1, p)}{(R\sqrt{R^2 - 1})} \quad (R > 1) \\ &= \frac{4\tau_0}{3\pi p} \frac{\phi_3^*(1, p)}{\left(\frac{r}{a}\sqrt{\frac{r}{a} - 1}\sqrt{\frac{r}{a} + 1}\right)} \quad (R > 1) \\ &= \frac{4\tau_0}{3\pi p} \frac{\sqrt{a} \phi_3^*(1, p)}{\left(\frac{r}{a}\sqrt{r_1}\sqrt{\frac{r}{a} + 1}\right)} \quad (r > a) \end{aligned} \quad (40)$$

where $r_1 = r - a$.

Fig. 2 Stress intensity factor ($K_1(t)/\tau_0$) against time t ($a = 1.5$, $b = 3.0$)

The Laplace transform of stress-intensity factor is defined by

$$K_1^*(p) = \lim_{r \rightarrow a} [|\bar{\tau}_{\theta z}^{(1)}(r, 0, p)| \sqrt{r-a}] \quad (41)$$

Using Eq. (40), the Eq. (41) yields

$$K_1^*(p) = \frac{2\sqrt{2a}}{3\pi} \tau_0 \frac{\phi_3^*(1, p)}{p} \quad (42)$$

Taking Laplace inverse transform, the stress intensity factor $K_1(t)$ is given by

$$K_1(t) = \frac{2\sqrt{2a}}{3\pi} \tau_0 \frac{1}{2\pi i} \int_{Br} \frac{\phi_3^*(1, p)}{p} e^{pt} dp \quad (43)$$

where Br is the Bromwich path of integration.

Numerical Results and Discussion

To calculate stress intensity factor $K_1(t)$, it is necessary to obtain the value of $\phi_3^*(1, p)$ and for which the integral equation (35) has been solved by the method of Fox and Goodwin [27] for different values of radius of crack a and the layer thickness b . For the Laplace inversion, we use Zakian Algorithm [28, 29] in Eq. (43) (Appendix). After solving the integral Eq. (35) and then Eq. (43) for the engineering elastic constants of Aluminum alloy ($\rho_1 = 2.7$, $\mu_1 = 28$) as layer 1 and Brass ($\rho_2 = 8.4$, $\mu_2 = 39$) as layer 2, the stress intensity factor has been calculated at the tip of the crack and the value of $K_1(t)/\tau_0$ has been plotted against the time t for different values of a and b .

In Figs. 2 and 3, $K_1(t)/\tau_0$ has been plotted against t for different values of crack radius a (1.5, 2.0) and for the layer thickness $b = 3.0$. In all cases it is observed that SIF increases first and attains maximum value near about $t = 0.5$ and then decreases to minimum value near $t = 1.0$ and then shows wave like nature and finally decreases as t increases.

Fig. 3 Stress intensity factor ($K_I(t)/\tau_0$) against time t ($a = 2.0$, $b = 3.0$)

Conclusions

The primary motivation of this problem is to arrest the propagation of crack when load is increased. This is calculated by stress intensity factor (SIF). From Figs. 2 and 3, it is observed that SIF initially increases but after a certain increase it decreases showing wave like nature. This ensures the arrest of crack propagation or expansion of crack.

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Appendix

Numerical inversion of the Laplace transform:

Let the Laplace transform $F(p)$ of $f(t)$ be given by

$$F(p) = \int_0^{\infty} e^{-pt} f(t) dt, \quad p \geq 0 \quad (p = \text{transform parameter})$$

For the Laplace inversion we use here Zakian's Algorithm.

Zakian's Algorithm:

This algorithm is one of a class of algorithms in which $f(t)$ is computed as a sum of weighted evaluations of $F(p)$.

$$f(t) = \sum_{i=1}^N K_i F(p_i)$$

where the values of K_i , p_i and N are dictated by a particular method. The development of Zakian's algorithm is given in Rice et al. [30]. A significant feature of the derivation is the specification that the time function can be related to a finite series of exponential functions

Table 1 Set of five constants for α_i and K_i for the Zakian's Method

i	α_i	K_i
1	$12.83767675 + i \ 1.666063445$	$-36902.08210 + i \ 196990.4257$
2	$12.22613209 + i \ 5.012718792$	$61277.02524 - i \ 95408.62551$
3	$10.93430308 + i \ 8.409673116$	$-28916.56288 + i \ 18169.18531$
4	$8.776434715 + i \ 11.92185389$	$4655.361138 - i \ 1.901528642$
5	$5.225453361 + i \ 15.72952905$	$-118.7414011 - i \ 141.3036911$

$$\sum_{i=1}^N K_i e^{\alpha_i t}.$$

This significance of this specification is that Zakian's Algorithm is very accurate for over-damped and slightly underdamped systems. But it is not accurate for systems with prolonged oscillations.

Given $F(p)$ and a value of time t , the following equation implements Zakian's Algorithm and allows us to calculate the numerical value of $f(t)$.

$$f(t) = \frac{2}{t} \sum_{i=1}^5 REAL\left(K_i F\left(\frac{\alpha_i}{t}\right)\right)$$

Table 1 gives the set of five complex constants for α_i and for K_i as in [11].

Zakian's Algorithm is simple to implement and computes quickly. But note that the initial value, $f(t)$ at $t = 0$, cannot be computed. Also, when there are oscillatory systems, $f(t)$ becomes inaccurate after approximately the second cycle.

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