The Approximate Solution of High-Order Nonlinear Ordinary Differential Equations by Improved Collocation Method with Terms of Shifted Chebyshev Polynomials

Yalçın Öztürk¹ · Mustafa Gülsu²

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Abstract In this paper, we present a direct computational method for solving the higher-order nonlinear differential equations by using collocation method. This method transforms the nonlinear differential equation into the system of nonlinear algebraic equations with unknown shifted Chebyshev coefficients, via Chebyshev–Gauss collocation points. The solution of this system yields the Chebyshev coefficients of the solution function. The method is valid for both initial-value and boundary-value problems. Several examples are presented to illustrate the accuracy and effectiveness of the method by the approximate solutions of very important equations of applied mathematics such as Lane–Emden equation, Riccati equation, Van der Pol equation. The approximate solutions can be very easily calculated using computer program Maple 13.

Keywords Nonlinear differential equation · Shifted Chebyshev polynomial · Collocation method · Lane–Emden equations · Riccati equations · Van der Pol equation

Introduction

Non-linear differential equations are indispensable tools for modelling many physical phenomenon such as chemical reactions, spring–mass systems, and beam bending. These equations are also useful in ecology and economics. The most popular of this type equations are Riccati, Emden–Fowler, Duffing, Van der Pol, Rayleigh and Yermakov’s equations. Most nonlinear differential equations do not have exact solutions, so approximation and numerical techniques must be used. Recently, several numerical methods to solve nonlinear differen-

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✉ Yalçın Öztürk
yozturk@mu.edu.tr

Mustafa Gülsu
mgulsu@mu.edu.tr

¹ Ula Ali Koçman Vocational School, Mugla Sıtkı Koçman University, Mugla, Turkey

² Department of Mathematics, Faculty of Science, Mugla Sıtkı Koçman University, Mugla, Turkey

tial equations have been given such as Adomian decomposition method [1,2], linearization method [3], a sixth-degree B-spline method [4], the decomposition method [5]. Moreover, a Homotopy perturbation method (HPM) [6] has been proposed to solve a quadratic Riccati equation. A quasilinearization approach to the nonlinear problems in physics has been discussed in [7]. Chebyshev collocation matrix method [8] has been presented the numerical solution of nonlinear differential equations. The method in [8] transforms the nonlinear differential equation into the matrix equation, which corresponds to a system of nonlinear algebraic equations with unknown Chebyshev coefficients, via Chebyshev collocation points.

In this paper, it is concerned with the application of shifted Chebyshev collocation method to obtain the numerical solution of nonlinear differential equation of the form:

$$\sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) y^s(x) y^{(k)}(x) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) y^{(s)}(x) y^{(k)}(x) = f(x) \quad (1)$$

with conditions

$$\sum_{k=0}^{m-1} \left(a_{ik} y^{(k)}(a) + b_{ik} y^{(k)}(b) + c_{ik} y^{(k)}(c) \right) = \alpha_i, \quad i = 0, 1, \dots, m-1. \quad (2)$$

where $y^{(0)}(x) = y(x)$ and $y(x) \in C^m[0, L]$ is an unknown function. $Q_{k,s}(x)$, $P_{k,s}(x)$ and $f(x)$ known functions which are defined on $[0, L]$.

We seek the approximate solution of Eq. (1) with Eq. (2) as the truncated shifted Chebyshev series defined by

$$y_N(x) = \sum_{r=0}^N a_r T_{L,r}^*(x), \quad x \in [0, L] \quad (3)$$

where N is any positive integer and $N > m$ and $T_{L,r}^*(x)$, $r = 0, 1, \dots, N$ denote the shifted Chebyshev polynomials which can be obtained with the aid of the following recurrence formula [9,10]:

$$T_{L,r+1}^*(x) = 2 \left(\frac{2x}{L} - 1 \right) T_{L,r}^*(x) - T_{L,r-1}^*(x), \quad r = 1, 2, \dots$$

where $T_{L,0}^*(x) = 1$, $T_{L,1}^*(x) = \frac{2x}{L} - 1$. The analytic form of the shifted Chebyshev polynomials $T_{L,r}^*(x)$ of degree r is given by

$$T_{L,r}^*(x) = r \sum_{p=0}^r (-1)^{r-p} \frac{(r+p-1)! 2^{2p}}{(r-p)!(2p)! L^p} x^p \quad (4)$$

where $T_{L,r}^*(0) = (-1)^r$ and $T_{L,r}^*(L) = 1$. The orthogonality condition is

$$\int_0^L T_{L,j}^*(x) T_{L,i}^*(x) w_L(x) dx = h_k \delta_{ji}$$

where $w_L(x) = (Lx - x^2)^{-1/2}$ and $h_i = b_i \pi / 2$, $b_0 = 2$, $b_i = 1$, $k \geq 1$. By Eq. (4), we have the k -th derivatives of $T_{L,r}^*(x)$

$$(T_{L,r}^*)^{(k)}(x) = T_{L,r}^{*,k}(x) = r \sum_{p=m}^r (-1)^{r-p} p(p-1) \dots (p-k+1) \frac{(r+p-1)! 2^{2p}}{(r-p)!(2p)! L^k} x^{p-k} \quad (5)$$

where $p \geq m-1$.

Solution Method

Using Eq. (3), we have the k -th derivatives of approximate solution $y_N(x)$

$$y_N^{(k)}(x) = \sum_{r=0}^N a_r (T_{L,r}^*)^{(k)}(x) = \sum_{r=k}^N a_r T_{L,r}^{*,k}(x) \quad (6)$$

From (1), (3) and (6), we have

$$\begin{aligned} & \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) \left(\sum_{r=0}^N a_r T_{L,r}^*(x) \right)^s \left(\sum_{r=k}^N a_r T_{L,r}^{*,k}(x) \right) \\ & + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) \left(\sum_{r=s}^N a_r T_{L,r}^{*,s}(x) \right) \left(\sum_{r=k}^N a_r T_{L,r}^{*,k}(x) \right) = f(x) \end{aligned} \quad (7)$$

We now collocate Eq. (7) at $N - m + 1$ points x_p , $p = 0, 1, \dots, N - m$, we obtain following equations

$$\begin{aligned} & \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x_q) \left(\sum_{r=0}^N a_r T_{L,r}^*(x_q) \right)^s \left(\sum_{r=k}^N a_r T_{L,r}^{*,k}(x_q) \right) \\ & + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x_q) \left(\sum_{r=s}^N a_r T_{L,r}^{*,s}(x_q) \right) \left(\sum_{r=k}^N a_r T_{L,r}^{*,k}(x_q) \right) = f(x_q) \end{aligned} \quad (8)$$

where x_p are roots of $T_{L,m}^*(x)$. Also, by substituting Eq. (6) in the conditions Eq. (2) we obtain k -times equations as follows:

$$\begin{aligned} & \sum_{k=0}^{m-1} \left(a_{ik} \sum_{r=k}^N a_r T_{L,r}^{*,k}(a) + b_{ik} \sum_{r=0}^N a_r T_{L,r}^{*,k}(b) + c_{ik} \sum_{r=0}^N a_r T_{L,r}^{*,k}(c) \right) \\ & = \alpha_i, i = 0, 1, \dots, m-1. \end{aligned} \quad (9)$$

Equation 8, together with k -times equations of conditions Eq. (9) give $(N + 1)$ non-linear algebraic equations. Solving these nonlinear algebraic systems, we obtain the unknown shifted Chebyshev coefficients a_r , $r = 0, 1, \dots, N$. Consequently $y_N(x)$ to approximate $y(x)$ given in Eq. (1) can be calculated.

Error Analysis

In this section, we present convergence analysis of the mention method. We assume that $y(x)$ is a sufficiently smooth function on $[0,1]$ and $I_N(x)$ is the interpolating polynomial to y at x_i , where x_i , $i = 0, 1, \dots, n$ are the Chebyshev-Gauss grid points, then we have [9, 10]

$$y(x) - I_N(x) = \frac{y^{(N+1)}(\lambda)}{(N+1)!} \prod_{i=0}^N (x - x_i), \lambda \in [0, 1] \quad (10)$$

Therefore, we have [29–32]

$$|y(x) - I_N(x)| \leq \frac{1}{2^{2N+1}} \left\| y^{(N+1)}(x) \right\|_{\infty}. \quad (11)$$

Definition [9] The least-square norm is defined by

$$\|y\|_2 = \left(\int_a^b w(x) |y(x)|^2 dx \right)^{1/2}$$

where $w(x)$ is a non-negative weight function.

Theorem Suppose that the known functions in Eq. (1) are real $(N + 1)$ -times continuously differential functions on the $[0, 1]$ and

$$y_N(x) = \sum_{r=0}^N a_r T_{L,r}^*(x)$$

are the shifted Chebyshev polynomials expansion of the exact solution.

Let

$$\overline{y}_N(x) = \sum_{r=0}^N \overline{a}_r T_{L,r}^*(x)$$

be the approximate solution obtained by proposed method, then there exist real number α such that

$$\|y(x) - y_N(x)\|_2 \leq \alpha \frac{1}{2^{N+1}} \|y^{(N+1)}(x)\|_\infty + \sqrt{\frac{3\pi}{8}} \|A - \overline{A}\|_2 \quad (12)$$

where

$$A = [a_0 \ a_1 \ \cdots \ a_N] \text{ and } \overline{A} = [\overline{a}_0 \ \overline{a}_1 \ \cdots \ \overline{a}_N].$$

Proof Let $y_N(x)$ is real-valued polynomials of degree $\leq N$ and $y_N(x)$ is the best approximation of $y(x)$. We can write

$$\|y(x) - y_N(x)\|_2 \leq \|y(x) - \overline{y}_N(x)\|_2 + \|\overline{y}_N(x) - y_N(x)\|_2$$

Using Eq. (11), we obtain

$$\begin{aligned} \|y(x) - y_N(x)\|_2 &= \left(\int_0^1 |y(x) - y_N(x)|^2 dx \right)^{1/2} \\ &\leq \left(\int_0^1 \left[\frac{1}{2^{N+1}(N+1)!} \|y^{(N+1)}(x)\|_\infty \right]^2 dx \right)^{1/2} \\ &= \sqrt{L} \frac{1}{2^{2N+1}(N+1)!} \left(\|y^{(N+1)}(x)\|_\infty \right)^{N+1} \end{aligned}$$

and we have

$$\|y(x) - \overline{y}_N(x)\|_2 = \left(\int_0^1 \left[\sum_{r=0}^N (a_r - \overline{a}_r) T_r^*(x) \right]^2 dx \right)^{1/2}$$

$$\begin{aligned} &\leq \left(\int_0^1 \left[\sum_{r=0}^N (a_r - \overline{a_r})^2 \right] \left[\sum_{r=0}^N |T_r^*(x)|^2 \right] dx \right)^{1/2} \\ &= \left[\sum_{r=0}^N (a_r - \overline{a_r})^2 \right]^{1/2} \left(\sum_{r=0}^N \int_0^1 |T_r^*(x)|^2 dx \right)^{1/2} = \sqrt{\frac{3\pi}{8}} \|A - \overline{A}\|. \end{aligned}$$

Moreover, we can easily check the accuracy of the method. Since the truncated Chebyshev series (3) is an approximate solutions of Eq. (1), when the function $y_j^N(x)$, $j = 0, 1, \dots, m$ and its first derivatives are substituted in Eq. (1) the resulting equation must be satisfied approximately [32]; that is, for $x_i \in [0, 1]$, $i = 0, 1, 2, \dots$

$$\left| \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x_i) y^s(x_i) y^{(k)}(x_i) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x_i) y^{(s)}(x_i) y^{(k)}(x_i) - f(x_i) \right| \cong 0$$

On the other hand, the error can be estimated by the function [10]

$$E_N(x) = \sum_{k=0}^m \sum_{s=0}^n Q_{k,s}(x) y^s(x) y^{(k)}(x) + \sum_{k=1}^m \sum_{s=1}^m P_{k,s}(x) y^{(s)}(x) y^{(k)}(x) - f(x).$$

□

Numerical Results

We show the efficiency of the present method using the following examples. In Tables, $N_e = |y(x) - y_N(x)|$ is absolute error for selected points.

Example 1 Consider the following Riccati differential equation [6, 11]:

$$y'(x) - 2y(x) + y^2(x) = 1$$

with initial conditions

$$y(0) = 0.$$

Exact solution of this problem is

$$y(x) = 1 + \sqrt{2} \tanh \left(\sqrt{2}x + \frac{1}{2} \log \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right).$$

Let us suppose that $y(x)$ is approximated by a truncated Chebyshev series in the form

$$y_6(x) = \sum_{i=0}^6 a_i T_{1,r}^*(x), \quad x \in [0, 1] \quad (13)$$

Using Eq. (8), we have

$$\sum_{r=1}^6 a_r T_{1,r}^{*,1}(x_q) - 2 \sum_{r=0}^6 a_r T_{1,r}^*(x_q) + \left(\sum_{r=0}^6 a_r T_{1,r}^*(x_q) \right)^2 = 1 \quad (14)$$

with $q = 0, 1, 2, \dots, 5$, where x_q are roots of the shifted Chebyshev polynomials $T_{1,6}^*(x)$

$$x_0 = \frac{1}{2} - \frac{1}{2} \cos\left(\frac{\pi}{14}\right), \quad x_1 = \frac{1}{2} - \frac{1}{2} \cos\left(\frac{3\pi}{14}\right), \quad x_2 = \frac{1}{2} - \frac{1}{2} \cos\left(\frac{5\pi}{14}\right),$$

$$x_3 = \frac{1}{2}, \quad x_4 = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{5\pi}{14}\right), \quad x_5 = \frac{1}{2} + \frac{1}{2} \cos\left(\frac{3\pi}{14}\right)$$

and

$$y(0) = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 = 0 \quad (15)$$

Thus, we obtain seven nonlinear algebraic equations with the seven unknown by Eqs. 14–15. Now solving equations, we obtain the coefficients:

$$a_0 = 0.803725, \quad a_1 = 0.873802, \quad a_2 = 0.044314, \quad a_3 = -0.019645,$$

$$a_4 = -0.0037358, \quad a_5 = 0.001331, \quad a_6 = 0.000291.$$

This coefficients are substituting into Eq. (13), we get the approximate solution for $N = 6$

$$y_6(x) = 0.999923x + 1.003816x^2 + 0.287515x^3 - 0.0879134x^4$$

$$- 1.110271x^5 + 0.597349x^6$$

This problem was solved by HPM [6] and Adomian decomposition method (ADM) [11]. In Table 1, we give the comparison of the absolute errors HPM with 9 terms, ADM with 6 terms and Present method for $N = 6$. Moreover, the numerical results and the absolute errors are compared in Figs. 1 and 2 respectively. Numerical results show that the Present method is more accurate than the others methods.

Example 2 Consider a fifth order nonlinear differential equation [8]:

$$y^{(5)}(x) - e^{-x}y^2(x) = 0$$

Table 1 Numerical result for Example 1

x	Exact solution	PM		ADM		HPM	
		N = 6	N _e = 6	Numerical solution	Absolute errors	Numerical solution	Absolute errors
0.0	0.000000	0.000000	0.000E–0	0.000000	0.000E–0	0.000000	0.000E–0
0.1	0.110295	0.110298	0.350E–5	0.1102678	0.271E–4	0.110295	0.340E–7
0.2	0.241976	0.241979	0.291E–5	0.241642	0.334E–3	0.241972	0.445E–5
0.3	0.395104	0.395108	0.395E–5	0.393924	0.118E–2	0.395027	0.775E–4
0.4	0.567812	0.567807	0.421E–5	0.565631	0.218E–2	0.567223	0.588E–3
0.5	0.756014	0.755998	0.161E–4	0.754166	0.184E–2	0.753190	0.282E–2
0.6	0.953566	0.953572	0.676E–5	0.955968	0.240E–2	0.943490	0.100E–0
0.7	1.159489	1.153000	0.515E–5	1.166675	0.137E–1	1.123821	0.291E–1
0.8	1.346363	1.346357	0.590E–5	1.381290	0.349E–1	1.274579	0.717E–1
0.9	1.526911	1.526793	0.118E–3	1.594332	0.674E–1	1.371096	0.155E–0
1.0	1.689498	1.690420	0.921E–E–3	1.799999	0.110E–0	1.384920	0.304E–0

Fig. 1 Comparison of numerical solutions for ADM, HPM and present method

Fig. 2 Comparison of absolute errors for numerical methods ADM, HPM and PM

subject to the boundary conditions $y(0) = y'(0) = y''(0) = 1$, $y(1) = y'(1) = e$ and the exact solution is $y(x) = e^x$. Taking $N = 6, 10$, we obtain the approximate solutions

$$y_6(x) = 1 + x + 0.5x^2 + 0.166726x^3 + 0.041533x^4 + 0.008163x^5 + 0.001558x^6$$

$$y_{10}(x) = 1 + x + 0.5x^2 + 0.166666x^3 + 0.041666x^4 + 0.083333x^5 + 0.00138892x^6$$

$$+ 0.198269E - 3x^7 + 0.250859E - 4x^8 + 0.246432E - 5x^9 + 0.421894E - 6x^{10}$$

Comparison of the absolute errors obtained by Chebyshev collocation matrix method (CCMM) and Present method are given in Table 2.

Example 3 Let us consider following the Lane–Emden equation

$$y''(x) + \frac{\alpha}{x}y'(x) + g(x, y) = h(x) \quad (16)$$

with initial conditions

$$y(0) = \alpha_0, y'(0) = \alpha_1 \quad (17)$$

Table 2 Numerical result for Example 2

x	PM		CCMM	
	$N_e = 6$	$N_e = 10$	$N_e = 6$	$N_e = 10$
0.0	0.10E–13	0.00E–00	0.00E–0	0.20E–15
0.1	0.45E–7	0.30E–13	0.80E–7	0.20E–12
0.2	0.23E–6	0.90E–13	0.40E–6	0.10E–11
0.3	0.41E–6	0.35E–12	0.10E–5	0.20E–11
0.4	0.24E–6	0.12E–11	0.20E–5	0.50E–11
0.5	0.49E–6	0.10E–11	0.30E–5	0.70E–11
0.6	0.16E–5	0.67E–12	0.40E–5	0.90E–11
0.7	0.27E–5	0.18E–11	0.40E–5	0.10E–10
0.8	0.27E–5	0.83E–12	0.30E–5	0.10E–10
0.9	0.13E–5	0.26E–12	0.10E–5	0.60E–11
1.0	0.10E–13	0.00E–00	0.20E–1	0.30E–15

A discussion of the formulation of these models and the physical structure of the solutions can be found in the literature. For example, for $g(x, y) = y^n(x)$, $h(x) = 0$, Eq. (16) is the standart Lane–Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics and for $g(x, y) = e^y$, $h(x) = 0$, Eq. (16) is the isothermal gas sphere equation, where the temperature remains constant [12–20].

Case I Let us consider $\alpha = 2$, $g(x, y) = y(x)$, $h(x) = 6 + 12x + x^2 + x^3$ and $y(0) = 0$, $y'(0) = 0$ in Eq. (16). We applied the given method in this paper, we have

$$\sum_{r=1}^3 a_r T_{1,r}^{*,2}(x_q) - \frac{2}{x} \sum_{r=0}^3 a_r T_{1,r}^{*,1}(x_q) + \sum_{r=0}^3 a_r T_{1,r}^*(x_q) = 6 + 12x_p + x_p^2 + x_p^3 \quad (18)$$

and substituting roots of $T_{1,2}^*(x)$ into Eq. (18), we obtain

$$a_0 + 58.846787a_1 - 190.351251a_2 + 275.138438a_3 = 6.808635$$

$$a_0 + 8a_1 + 15a_2 - 24a_3 = 12.375$$

with conditions

$$a_0 - a_1 + a_2 - a_3 = 0$$

$$2a_1 - 8a_2 + 18a_3 = 0.$$

Therefore, it is obtained that 4 linear algebraic equations with 4 unknown. Solving above linear equations, we obtain the coefficients

$$a_0 = 0.6875 \quad a_1 = 0.96875 \quad a_2 = 0.3125 \quad a_3 = 0.03125$$

Then so approximate solution is obtained for $N = 3$

$$y_3(x) = x^3 + x^2$$

which is the exact solution.

Case II Taking $g(x, y) = y^3(x)$, $h(x) = 0$ and $y(0) = 1$, $y'(0) = 0$. The solution to this initial value problem is well known to be $y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}$. In Table 3, we give the absolute errors for various N and these results are plotted in Fig. 3.

Table 3 Error values of Case II in Example 3

x	$N_e = 6$	$N_e = 7$	$N_e = 8$	$N_e = 9$
0.2	0.433E-6	0.172E-8	0.551E-8	0.904E-9
0.4	0.146E-5	0.183E-7	0.844E-8	0.189E-8
0.6	0.444E-6	0.956E-7	0.970E-8	0.264E-8
0.8	0.119E-5	0.173E-6	0.167E-7	0.800E-9
1.0	0.623E-5	0.499E-6	0.588E-7	0.117E-7

Fig. 3 Comparison of absolute errors of Case II in Example 3**Table 4** Numerical results for CaseIII in Example 3

x	Exact solution	PM (N = 9)		HCM	
		Approximate values	Absolute errors	Approximate values	Absolute errors
0.00	0.0000000000	0.0000000000	0.000E-0	0.0000000000	0.000E-0
0.01	-0.0001999900	-0.0001999905	0.598E-9	-0.0001970587	0.293E-5
0.10	-0.0199006617	-0.0199005895	0.721E-7	-0.0198967225	0.394E-5
0.50	-0.4462871026	-0.4462872851	0.182E-6	-0.4462840851	0.302E-5
1.00	-1.3862943611	-1.3862955089	0.114E-5	-1.3862934297	0.931E-6

Case III In this case, taking $g(x, y) = 4(2e^y + e^{y/2})$, $h(x) = 0$, $\alpha = 2$ and $y(0) = 0$, $y'(0) = 0$. In this case Eq. (16) has the exact solution $y(x) = -2\ln(1 + x^2)$. This type of equation has been solved by [16] with Hermite collocation method (HCM). We apply our method to solve Eq. (16) with Case III. Then, Table 4 shows the comparison of exact solution and HCM obtained by the new method proposed in this paper. In order to compare the present method with exact solution, the resulting graph of Case III is shown in Fig. 4 for various N .

Example 4 Let us consider the Van der Pol equation [8, 21, 22]

$$y''(x) - 0.05(1 - y^2(x))y'(x) + y(x) = 0 \quad (19)$$

Fig. 4 Comparison of absolute errors of Case III in Example 3

Table 5 Comparison of absolute errors for some numerical methods for Example 4

x	CCMM	Runge–Kutta method	Gear method	Present method
0.0	0	0	0	0
0.1	0.0500416561	0.0500416616	0.0500416561	0.0500416560
0.2	0.0998321663	0.0998321792	0.0998321668	0.0998321667
0.3	0.1488697052	0.1488697072	0.1488697063	0.1488697063
0.4	0.1966564148	0.1966564313	0.1966564167	0.1966564165
0.5	0.2427036829	0.2427036961	0.2427036728	0.2427036725
0.6	0.2865373683	0.2865373842	0.2865373709	0.2865373708
0.7	0.3277031549	0.3277032041	0.3277031577	0.3277031578
0.8	0.3657715085	0.3657715349	0.3657715117	0.3657715113
0.9	0.4003425800	0.4003426519	0.4003425836	0.4003425833
1.0	0.4310507066	0.4310507546	0.4310507104	0.4310507094

with conditions $y(0) = 0$, $y'(0) = 0.5$. For comparison the solutions at the selected points, Eq. (19) is solved by CCMM [8], Runge Kutta method [21] and Gear single-step extrapolation method [22]. Numerical results are given in Table 5 and numerical results are plotted in Fig. 5. From Fig. 5 and Table 5, We can see that the numerical results are the same line.

Example 5 Consider the nonlinear differential equation

$$y'''(x) + y''(x) + y'(x)y(x) = -e^{-2x}$$

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1$$

with exact solution $y(x) = e^{-x}$. In Table 6, we compare the maximum absolute errors and L_2 errors for various N . These results are the similar values.

Fig. 5 Comparison of numerical methods for Example 4**Table 6** Numerical result for Example 5

Present method	Maximum absolute errors	L_2 errors
$N = 5$	0.119252×10^{-4}	0.704723×10^{-5}
$N = 8$	0.445095×10^{-6}	0.158113×10^{-6}
$N = 10$	0.524856×10^{-12}	0.424679×10^{-12}

Table 7 Numerical result for Example 6

Present method	Maximum absolute errors	L_2 errors
$N = 6$	0.516389×10^{-4}	0.238789×10^{-4}
$N = 8$	0.695539×10^{-7}	0.269757×10^{-7}
$N = 10$	0.665694×10^{-10}	0.866025×10^{-9}
$N = 12$	0.425280×10^{-13}	0.417133×10^{-12}

Example 6 Consider the following nonlinear differential equation with initial conditions:

$$\begin{aligned}
 &y^{(4)}(x) + y''' - y'' + (1 - y^2(x))y' = x^3 \cos^2(x) \sin(x) \\
 &\quad - x^2 \cos^3(x) + 2x \cos(x) - 2 \cos(x) + 6 \sin(x) \\
 &y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = -3
 \end{aligned}$$

The exact solution of this problem $y = x \cos(x)$. To obtained numerical results maximum absolute errors and L_2 errors are compare in Table 7. In Fig. 6, we display the absolute errors for various N .

Fig. 6 Comparison of absolute error for Example 6

Conclusion

In this article, we present the effective numerical method for solving nonlinear differential equations. The present method provides a reliable technique which gives highly accurate results and requires less work than traditional techniques and existing numerical methods.

In Examples, we get numerical solution of the important nonlinear differential equations such as Riccati equations, Emden–Fowler equations, Van der Pol equations. An valuable feature of this method is that the analytical solution is obtained as demonstrated in Example 3 when the exact solution is polynomial. Moreover, Tables and Figures show that this method gives better approximate solutions than the other methods.

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