Application of Sumudu Transform in Generalized Fractional Reaction–Diffusion Equation

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Abstract In this paper we investigate the solution of a generalized nonlinear fractional reaction diffusion equation by the application of Sumudu Transform.

Keywords Generalized fractional derivatives · Fractional diffusion equation · Generalized Mittag-Leffler function · Sumudu transform and Fourier transform

Mathematics Subject Classification Primary 26A33 · Secondary 33E12

Introduction

Reaction–diffusion systems (RDS) have been extensively used in the study of self-organization phenomena in physics, biology, chemistry, ecology, etc. see Smoller [1], Grindrod [2], Gilding and Kersner [3], and Wilhelmsson and Lazzaro [4]. These systems show that diffusion can produce the spontaneous formation spatiotemporal patterns. For details, refer to the work of Nicolis and Pri-Gogine [5]. A general model for reaction diffusion systems is discussed by Henry and Wearne [6, 7]. For the some applications of the Laplace decomposition method for solving linear and nonlinear fractional diffusion–wave equations, see [5, 8–10].

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A similar approach was recently used by Manne et al. [11] to investigate effects on the propagation of nonlinear wave fronts. The simplest reaction–diffusion models can be described by an equation

$$\frac{\partial N}{\partial t} = D \frac{\partial^2 N}{\partial x^2} + \gamma F(N) \quad (1)$$

where D is the diffusion coefficient and $F(N)$ is a nonlinear function representing reaction kinetics. For $F(N) = \gamma N(1 - N)$, Eq. (1) reduces to Fisher Kolmogorov equation and for $F(N) = \gamma N(1 - N^2)$, it reduces to the real Ginsburg-Landau equation. A generalization of (1) has been considered by Manne et al. [11] in the form

$$\frac{\partial^2 N}{\partial t^2} + a \frac{\partial N}{\partial t} = v^2 \frac{\partial^2 N}{\partial x^2} + \xi^2 N(x, t) \quad (2)$$

where ξ indicates the strength of the nonlinearity of the system. Saxena et al. [12] generalized above equation using time-fractional derivatives. In this work, we present a straight forward method for the systematic derivation of the solution of nonlinear reaction–diffusion equations that is connected with nonlinear waves, which is also more general than the Eq. (2).

Definitions and Preliminaries

Definition 2.1 Over the set of functions,

$$A = \left\{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_i}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\} \quad (3)$$

the Sumudu Transform is

$$G(u) = S[f(t)] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \quad (4)$$

For further detail and properties of this transform, see [13–18]. We will establish the following results which are directly applicable in the analysis of reaction diffusion systems.

Definition 2.2 A generalization of the Mittag-Leffler function, see [19].

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \quad (5)$$

where $\Re(\alpha) > 0$; $z \in \mathbb{C}$.

A generalization of Mittag-Leffler function $E_\alpha(z)$ of (5) is defined and studied by Wiman [20], as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (6)$$

where $\alpha, \beta \in \mathbb{R}$ ($\alpha > 0$; $\beta > 0$); $z \in \mathbb{C}$

A generalization of Mittag-Leffler function $E_{\alpha,\beta}(z)$ of (6) is introduced by Prabhakar [21], as follows:

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \quad (7)$$

where $\alpha, \beta \in \mathbb{C}$; $R(\alpha) > 0$; $R(\beta) > 0$; $z \in \mathbb{C}$, γ_k denotes the familiar Pochhammer symbol or the shifted factorial.

Recently, a remarkably large family of Generalized Riemann–Liouville fractional derivatives of order α ($0 < \alpha < 1$) and type β ($0 \leq \beta \leq 1$) was introduced as follows [22, 23].

Definition 2.3 The right-sided fractional derivative $\left(D_{a+}^{\alpha, \beta}\right)$ and the left sided fractional derivative $\left(D_{a-}^{\alpha, \beta}\right)$ of order α ($0 < \alpha < 1$) and type β ($0 \leq \beta \leq 1$) with respect to x are defined by

$$\left(D_{a\pm}^{\alpha, \beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \frac{d}{dx} \left(I_{a\pm}^{(1-\beta)(1-\alpha)} f\right)\right)(x) \quad (8)$$

whenever the right hand side of (8) exist. This generalization yields the classical Riemann–Liouville fractional derivative operator when $\beta = 0$ moreover for $\beta = 1$, it gives the fractional derivative operator introduced by Liouville [24], which is often attributed to Caputo now-a-days and which should more appropriately be referred to as the Liouville–Caputo fractional derivative, see [25].

The fractional derivative operator $\left(D_{a\pm}^{\alpha, \beta}\right)$ can be rewritten in the following form :

$$\left(D_{a\pm}^{\alpha, \beta}\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \left(D_{a\pm}^{\alpha+\beta-\alpha\beta} f\right)\right)(x). \quad (9)$$

Definition 2.4 Caputo [25] introduced fractional derivative in the following form

$$\begin{aligned} {}_0 D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad m-1 < \alpha < m, \quad R(\alpha) > 0 \\ &= \frac{d^m f}{dt^m}, \quad \text{if } \alpha = m. \end{aligned} \quad (10)$$

Definition 2.5 The Weyl fractional operator defined by

$$-_\infty D_x^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_{-\infty}^t \frac{f(u)}{(t-u)^{\mu-n+1}} du, \quad (11)$$

where $n = [\mu]$ is an integral part of $\mu > 0$.

Its Fourier transform is given by Metzler and Klafter [26].

$$F\{-_\infty D_t^\mu f(t)\} = (ik)^\mu f^*(k) \quad (12)$$

where $f^*(k)$ denotes the Fourier Transform of the function $f(x)$. Further modification of result (12) is given Metzler and Klafter [27].

$$F\{-_\infty D_x^\mu f(x)\} = -|k|^\mu f^*(k).$$

Main Results

We will establish the following results which are directly applicable in the analysis of reaction diffusion systems.

Lemma 3.1 In the complex plane C , for any $R\{\alpha\} > 0$, $R\{\beta\} > 0$ and $\omega \in C$

$$S^{-1} \left[u^{\gamma-1} (1 - \omega u^\beta)^{-\delta} \right] = t^{\gamma-1} E_{\beta, \gamma}^\delta (\omega t^\beta). \quad (13)$$

We can prove the result in another way

$$S \left[t^{\gamma-1} E_{\beta, \gamma}^\delta (\omega t^\beta) \right] = \int_0^\infty e^{-t} (ut)^{\gamma-1} E_{\beta, \gamma}^\delta (\omega (ut)^\beta) dt. \quad (14)$$

By using Eq. (7)

$$u^{\gamma-1} \sum_{n=0}^\infty \frac{(\delta)_n (\omega u^\beta)^n}{n!} = u^{\gamma-1} (1 - \omega u^\beta)^{-\delta}. \quad (15)$$

By applying inverse Sumudu Transform, we get our required result.

Lemma 3.2 The complex plane C , for any $R\{\alpha\} > 0$, $R\{\beta\} > 0$

$$S^{-1} \left[\frac{1}{u(u^{-\alpha} + au^{-\beta} + b)} \right] = \sum_{r=0}^\infty (-b)^r t^{\alpha(r+1)-1} E_{\alpha-\beta, \alpha(r+1)}^{r+1} [-at^{\alpha-\beta}]. \quad (16)$$

Lemma 3.3 The complex plane C , for any $R\{\alpha\} > 0$, $R\{\beta\} > 0$

$$S^{-1} \left[\frac{u^{-1+\alpha_1(1-\alpha)}}{u^{-\alpha} + au^{-\beta} + b} \right] = \sum_{r=0}^\infty (-b)^r t^{(r+1-\alpha_1)\alpha+\alpha_1-1} E_{\alpha-\beta, (r+1-\alpha_1)\alpha+\alpha_1}^{r+1} (-at^{\alpha-\beta}). \quad (17)$$

Lemma 3.4 The complex plane C , for any $R\{\alpha\} > 0$, $R\{\beta\} > 0$

$$S^{-1} \left[\frac{au^{-1\alpha_1+(1-\beta)}}{u^{-\alpha} + au^{-\beta} + b} \right] = \sum_{r=0}^\infty a \cdot (-b)^r t^{+\alpha_1(1-\beta)+(r+1)\alpha-1} E_{\alpha-\beta, \alpha_1(1-\beta)+(r+1)\alpha}^{r+1} (-at^{\alpha-\beta}). \quad (18)$$

Lemma 3.5 The Sumudu transform of the generalized fractional derivative is given by

$$S \left[{}_0D_t^{\alpha, \beta} f(x) \right] = S \left(\pm I_{a^\pm}^{\beta(1-\alpha)} \frac{d}{dx} \left(I_{a^\pm}^{(1-\beta)(1-\alpha)} f \right) \right) (x). \quad (19)$$

By using Laplace Sumudu duality, see [14–17, 28], we get the following result

$$u^{-\alpha} S[f(x)] u - \sum_{k=0}^{n-1} u^{k-n+\beta(n-\alpha)} \frac{d^k}{dx^k} \left(I_0^{(1-\beta)(n-\alpha)} f \right) (0+).$$

Theorem 3.1 Consider the fractional reaction diffusion equation

$${}_0D_t^{\alpha, \alpha_1} N(x, t) + a_0 D_t^{\beta, \alpha_1} N(x, t) = v^2 {}_{-\infty}D_x^\gamma N(x, t) + \xi^2 N(x, t) + \psi(x, t) \\ 0 < \alpha < 1, 0 < \beta < 1, 0 < \alpha_1 \leq 1 \quad (20)$$

with initial conditions

$$I_t^{(1-\alpha_1)(1-\alpha)} N(x, 0) = \phi_1(x) \\ I_t^{(1-\alpha_1)(1-\beta)} N(x, 0) = \phi_2(x)$$

where v is a diffusion coefficient, ψ is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula:

$$\begin{aligned} N(x, t) = & \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{(r+1-\alpha_1)\alpha+\alpha_1-1} \phi_1^*(k) \exp(-ikx) \\ & \times E_{\alpha-\beta, (r+1-\alpha_1)\alpha+\alpha_1}^{r+1}(-at^{\alpha-\beta}) dk \\ & - \sum_{r=0}^{\infty} a \frac{(-b)^r}{\sqrt{2\pi}} \times \int_{-\infty}^{\infty} t^{\alpha_1(1-\beta)+(r+1)\alpha-1} \phi_2^*(k) \exp(-ikx) \\ & \times E_{\alpha-\beta, \alpha_1(1-\beta)+(r+1)\alpha}^{r+1}(at^{\alpha-\beta}) dk \\ & + \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_0^t \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k, t-\tau) \exp(-ikx) E_{\alpha-\beta, \alpha(r+1)}^{r+1} \\ & (-a\tau^{\alpha-\beta}) dk d\tau \end{aligned} \quad (21)$$

where $\alpha > \beta$ and $E_{\beta, \gamma}^{\delta}(z)$ is the generalized Mittag-Leffler function, defined in (6) and $b = v^2 |k|^{\gamma} - \xi^2$.

Proof Applying the Sumudu Transform with respect to the time variable t , we find,

$$\begin{aligned} u^{-\alpha} \overline{N}(x, u) - u^{-1+\alpha_1(1-\alpha)} \left[I_0^{(1-\alpha_1)(1-\alpha)} N(x, 0) \right] + au^{-\beta} \overline{N}(x, u) \\ - au^{-1+\alpha_1(1-\beta)} \left[I_t^{(1-\alpha_1)(1-\beta)} N(x, 0) \right] \\ = v^2 {}_{-\infty} D_x^{\gamma} \overline{N}(x, u) + \xi^2 \overline{N}(x, u) + \overline{\psi}(x, u) \end{aligned} \quad (22)$$

Using initial conditions, Eq. (22) can be written as

$$\begin{aligned} u^{-\alpha} \overline{N}(x, u) - u^{-1+\alpha_1(1-\alpha)} \phi_1(x) + au^{-\beta} \overline{N}(x, u) - au^{-1+\alpha_1(1-\beta)} \phi_2(x) \\ = v^2 {}_{-\infty} D_x^{\gamma} \overline{N}(x, u) + \xi^2 \overline{N}(x, u) + \overline{\psi}(x, u) \end{aligned} \quad (23)$$

Taking Fourier transform of above Eq. (23) we obtain

$$\begin{aligned} u^{-\alpha} \overline{N}^*(k, u) - u^{-1+\alpha_1(1-\alpha)} \phi_1^*(x) + au^{-\beta} \overline{N}^*(k, u) - au^{-1+\alpha_1(1-\beta)} \phi_2^*(x) \\ = v^2 |K|^{\gamma} \overline{N}^*(k, u) + \xi^2 \overline{N}^*(k, u) + \overline{\psi}^*(k, u) \end{aligned} \quad (24)$$

Solving for $\overline{N}^*(k, u)$

$$\overline{N}^*(k, u) = \frac{u^{-1+\alpha_1(1-\alpha)} \phi_1^*(k)}{u^{-\alpha} + au^{-\beta} + b} - \frac{u^{-1+\alpha_1(1-\beta)} \phi_2^*(k)}{u^{-\alpha} + au^{-\beta} + b} + \frac{\overline{\psi}^*(k, u)}{u^{-\alpha} + au^{-\beta} + b} \quad (25)$$

where $b = v^2 |k|^{\gamma} - \xi^2$, Inverting the Sumudu transform with the help of Eqs. (18) and (21).

$$\begin{aligned}
N^*(k, t) = & \sum_{r=0}^{\infty} (-b)^r t^{(r+1-\alpha_1)\alpha+\alpha_1-1} E_{\alpha-\beta, (r+1-\alpha_1)\alpha+\alpha_1}^{r+1} (-at^{\alpha-\beta}) \phi_1^*(k) \\
& - \sum_{r=0}^{\infty} a \cdot (-b)^r t^{\alpha_1(1-\beta)+(r+1)\alpha-1} E_{\alpha-\beta, \alpha_1(1-\beta)+(r+1)\alpha}^{r+1} (-at^{\alpha-\beta}) \phi_2^*(k) \\
& + \sum_{r=0}^{\infty} (-b)^r \int_0^t \psi^*(k, t-\tau) \tau^{\alpha(r+1)-1} E_{\alpha-\beta, \alpha(r+1)}^{r+1} (-a\tau^{\alpha-\beta}) d\tau \quad (26)
\end{aligned}$$

on using the convolution theorem of Sumudu transform (see [14, 16, 17]). Now by applying inverse Fourier transform we get the required result of reaction diffusion equation in terms of generalized Mittag-Leffler function.

$$\begin{aligned}
N(x, t) = & \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{(r+1-\alpha_1)\alpha+\alpha_1-1} \phi_1^*(k) \exp(-ikx) \\
& \times E_{\alpha-\beta, (r+1-\alpha_1)\alpha+\alpha_1}^{r+1} (-at^{\alpha-\beta}) dk \\
& - \sum_{r=0}^{\infty} a \frac{(-b)^r}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} t^{\alpha_1(1-\beta)+(r+1)\alpha-1} \phi_2^*(k) \exp(-ikx) \\
& \times E_{\alpha-\beta, \alpha_1(1-\beta)+(r+1)\alpha}^{r+1} (-at^{\alpha-\beta}) dk \\
& + \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_0^t \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k, t-\tau) \exp(-ikx) E_{\alpha-\beta, \alpha(r+1)}^{r+1} \\
& (-a\tau^{\alpha-\beta}) dk d\tau \quad (27)
\end{aligned}$$

In particular, if we set $\alpha_1 = 1$ we get fractional diffusion reaction equation in following form studied by Gupta and Sharma [10].

$$\begin{aligned}
{}_0 D_t^\alpha N(x, t) + a_0 D_t^{\beta_1} N(x, t) = & \nu^2 {}_{-\infty} D_x^\gamma N(x, t) + \xi^2 N(x, t) + \psi(x, t) \\
0 \leq \alpha \leq 1, & \quad 0 \leq \beta \leq 1 \quad (28)
\end{aligned}$$

with initial conditions

$$N(x, 0) = f(x), \quad \text{for } x \in R$$

where ν is a diffusion coefficient, ψ is constant which describes the nonlinearity in the system, and is a nonlinear function for reaction kinetics then there holds the following formula.

$$\begin{aligned}
N(x, t) = & \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{(r+1-\alpha_1)\alpha+\alpha_1} f^*(k) \exp(-ikx) \times E_{\alpha-\beta, 1+\alpha r}^r (-at^{\alpha-\beta}) dk \\
& + \sum_{r=0}^{\infty} \frac{(-b)^r}{\sqrt{2\pi}} \int_0^t \tau^{\alpha(r+1)-1} \int_{-\infty}^{\infty} \psi^*(k, t-\tau) \exp(-ikx) E_{\alpha-\beta, \alpha(r+1)}^{r+1} \\
& (-a\tau^{\alpha-\beta}) dk d\tau. \quad (29)
\end{aligned}$$

□

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