On Hydromagnetic Flow of an Oldroyd-B Fluid Between Two Oscillating Plates

Arun Kumar Ghosh¹ · Sanjib Kumar Datta² ·
Pulakesh Sen³

Published online: 22 May 2015
© Springer India Pvt. Ltd. 2015

Abstract An initial value investigation is made of the motion of an incompressible, viscoelastic, electrically conducting Oldroyd-B fluid bounded by two infinite rigid non-conducting plates. The flow is generated impulsively from rest in the fluid due to rectilinear oscillations of given frequencies superimposed on the plates in their own planes in presence of an external magnetic field acting transversely to the plates. The operational method is used to derive exact solutions for the fluid velocity and the shear stress on the walls. The quantitative evaluation of the results is considered when two plates oscillate in phase but with different frequencies. The results are shown graphically for different time periods of oscillations of the plates which represent the cases: (i) the lower plate oscillates with a time period less than the upper, (ii) both the plates oscillate with the same time period and (iii) the lower plate oscillates with a time period greater than the upper. It is seen that the effect of fluid elasticity on the flow depends on the advancing and retarding motion of the plates. On the other hand, the magnetic field damps the fluid motion for all values of the time period of oscillations of the plates. The drag on the plates, for small and large time, are shown graphically when the time period of oscillation of the lower plate is small. For small values of time, the drag on the lower plate is negative with the amplitude decreasing continuously till the oscillations occur from moderately large values of time. Similar phenomenon also occurs for the drag which is positive at the upper plate for small values of time. In all cases, the amount of drag on the plates increases with the increase of the elasticity of the fluid and the magnetic field. Some particular results for the fluid velocity are derived as special cases of the present solution.

✉ Arun Kumar Ghosh
akg10143@yahoo.co.in

Sanjib Kumar Datta
sanjib_kr_datta@yahoo.co.in

Pulakesh Sen
psendnc2011@gmail.com

¹ Jadavpur University, Kolkata 700032, India

² Department of Mathematics, University of Kalyani, Nadia, Kalyani 741213, India

³ Department of Mathematics, D. N. C. College, Murshidabad, Aurangabad 742201, W.B, India

Keywords Hydromagnetic oscillatory flow · Oldroyd-B fluid

Mathematics Subject Classification 76 T 05

Introduction

In recent years considerable attention has been given to the study of flow problems associated with Oldroyd-B fluids due to their increasing applications in various branches of science and engineering. The investigation in this direction was carried out by Ghosh and Mitra [1] who considered Stokes' problems for linear viscoelastic fluids using Oldroyd-B fluid model. Ray et al. [2] discussed the low Reynolds number stability of MHD plane poiseuille flow of an Oldroyd-B fluid. Hayat et al. [3–7] constructed solutions of a number of problems concerning the flow of an Oldroyd-B fluid in various geometrical configurations which are (i) Some simple flows of an Oldroyd-B fluid, (ii) Exact solutions of flow problems of Oldroyd-B fluid, (iii) Hydromagnetic Couette flow of an Oldroyd-B fluid in a rotating system and (iv) Hall effect on flows of an Oldroyd-B through porous medium for cylindrical geometries and others. Asghar et al. [8] presented the solution of a problem concerning Hall effects on unsteady hydromagnetic flows of an Oldroyd-B fluid. Khan et al. [9–11] found (i) Exact solution for MHD flow of a generalized Oldroyd-B fluid with modified Darcy's law, (ii) Influence of Hall current on the flows of a generalized Oldroyd-B fluid in a porous space and (iii) New exact solutions of magnetohydrodynamic flows of an Oldroyd-B fluid. Fetecau et al. [12, 13] analyzed the problem of unsteady flow of an Oldroyd-B fluid induced by the impulsive motion of a plate between two side walls perpendicular to the plate in one paper while in the other they described the Rayleigh- Stokes problem for an edge in a generalized Oldroyd-B fluid. Tong et al. [14] obtained exact solutions for the unsteady rotational flow of a non-Newtonian fluid in an annular pipe. Most recently, Ghosh et al. [15–17] solved some problems associated with the flow of an Oldroyd-B fluid which are (i) Hydromagnetic flow of an Oldroyd-B fluid near a pulsating plate, (ii) Hydromagnetic channel flow of an Oldroyd-B fluid induced by tooth pulses and (iii) Hydromagnetic rotating flow of an Oldroyd-B fluid near an oscillating plate. In the present paper, we intend to construct solution of an unsteady hydromagnetic flow of an Oldroyd-B fluid between two oscillating plates with a view to its applications in the analysis of viscoelastic boundary layers, spreading of protective coating on electronic and optical devices and in the detection of magnetic effect on electrically conducting flows of biological and geophysical fluids.

The problem is concerned with the unsteady hydromagnetic flow of an incompressible, electrically conducting viscoelastic Oldroyd-B fluid bounded by two infinite rigid non-conducting plates. The motion is supposed to generate impulsively from rest in the fluid due to rectilinear oscillations of given frequencies superimposed on the plates in their respective planes in presence of an external magnetic field acting transversely to the plates. The operational method is used to obtain exact solutions for the fluid velocity and the shear stress exerted by the fluid on the plates. The results corresponding to fluid velocity of some particular cases are also derived from the general solution of the present problem. The quantitative analysis is taken up to examine the influence of fluid elasticity, the magnetic field and the time-periods of oscillations of the plates on the fluid velocity and on the wall shear stress.

Basic Equations

The constitutive equations for an Oldroyd-B fluid flow problem [3,4] are

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} \quad (1)$$

$$\mathbf{S} + \lambda_1 \frac{D\mathbf{S}}{Dt} = \mu \left[\mathbf{1} + \lambda_2 \frac{D}{Dt} \right] \mathbf{A}_1, \quad (2)$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the fluid pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{S}$ is the extra stress tensor, and μ is the viscosity coefficient, λ_1 is the relaxation time, λ_2 is the retardation time (assumed constants).

The tensor $\mathbf{A}_1$ is defined as

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T \quad (3)$$

In a cartesian system, $\frac{D}{Dt}$ (upper convected time derivative) operating on any tensor $\mathbf{B}_1$ is

$$\frac{D\mathbf{B}_1}{Dt} = \frac{\partial \mathbf{B}_1}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{B}_1 - (\nabla \mathbf{V}) \mathbf{B}_1 - \mathbf{B}_1 (\nabla \mathbf{V})^T \quad (4)$$

It is to be mentioned here that this model includes the viscous fluid as a particular case for $\lambda_1 = \lambda_2$, the Maxwell fluid when $\lambda_2 = 0$ and an Oldroyd-B fluid when $0 < \lambda_2 < \lambda_1 < 1$.

The stress equations of motion for an incompressible electrically conducting Oldroyd-B fluid in presence of an external magnetic field are

$$\nabla \mathbf{V} = \mathbf{0} \quad (5)$$

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B} \quad (6)$$

$$\nabla \mathbf{B} = \mathbf{0}, \quad (7a)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (7b)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (8a)$$

$$\mathbf{J} = \sigma [\mathbf{E} + \mathbf{V} \times \mathbf{B}], \quad (8b)$$

where $\mathbf{V}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is the fluid velocity, ρ is the fluid density, $\mathbf{J}$ is the current density, $\mathbf{B}$ is the magnetic flux density, $\mathbf{E}$ is the electric field, μ_0 is the magnetic permeability (assumed constant), and σ is the electrical conductivity (assumed finite).

It is to be noted here that the electromagnetic effect entering in Eqs. (6–8b) are approximations under the assumption of the small magnetic Reynolds number. More precisely, the above equations hold good if the condition $V \ll \sigma L$, in which V is the characteristic velocity and L is the characteristic scale of motion, is satisfied. It is also noteworthy that the inequality given above holds in many cases of interest.

Mathematical Formulation of the Problem

In this problem, we consider the motion of a viscoelastic electrically conducting Oldroyd-B fluid between two oscillating plates at a distance h apart where the x -axis is taken along the lower plate in the direction of flow with the origin at the lower plate and y -axis perpendicular to it. The unsteady motion is generated in the fluid due to simultaneous action of rectilinear

Fig. 1 Geometry of the flow configuration

oscillations of different frequencies and amplitudes subjected on the plates. An external magnetic field of uniform strength B_0 is applied on the fluid system in a direction parallel to y -axis. We assume that no external electric field is acting on the fluid and the magnetic Reynolds number is very small. This implies that the current in the fluid is mainly due to the induced electric field and the applied magnetic field remains essentially unaltered by the electric current flowing in the fluid. We also assume that the induced magnetic field produced by the motion of the fluid is negligible compared to the applied magnetic field so that the Lorentz force term in Eq. (6) becomes $-\sigma B_0^2 \mathbf{V}$. Since the motion is a plain one and the plates are infinitely long, we assume that all the physical variables are independent of x and z . The geometry of the flow configuration is shown in the Fig. 1.

Now, from the equation of continuity (5) and from the physical conditions of the problem, we take

$$\mathbf{V} = \{\mathbf{u}(\mathbf{y}, t), 0, 0\} \quad (9a)$$

and

$$\mathbf{S} = \mathbf{S}(\mathbf{y}, t) \quad (9b)$$

The equations of motion in (6) then reduces to

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial S_{xy}}{\partial y} - \sigma B_0^2 u \quad (10a)$$

$$\frac{\partial p}{\partial y} = \frac{\partial S_{yy}}{\partial y} \quad (10b)$$

and

$$\frac{\partial p}{\partial z} = 0 \quad (10c)$$

It follows from (2), (9a) and (9b) that

$$S_{xx} + \lambda_1 \left[\frac{\partial S_{xx}}{\partial t} - 2 S_{xy} \frac{\partial u}{\partial y} \right] = -2 \mu \lambda_2 \left(\frac{\partial u}{\partial y} \right)^2 \quad (11a)$$

$$S_{xy} + \lambda_1 \left[\frac{\partial S_{xy}}{\partial t} - S_{yy} \frac{\partial u}{\partial y} \right] = \mu \left(\frac{\partial u}{\partial y} \right) + \lambda_2 \mu \left(\frac{\partial^2 u}{\partial y \partial t} \right) \quad (11b)$$

$$S_{yy} + \lambda_1 \frac{\partial S_{yy}}{\partial t} = 0 \quad (11c)$$

The Eq. (11c) gives

$$S_{yy} = A(y) e^{-t/\lambda_1}, \quad (12)$$

where $A(y)$ is an arbitrary function of y . But S_{yy} is known to be zero for $t < 0$. This implies that $A(y)$ must be zero. Hence S_{yy} is zero always. Consequently, from (10a) and (11b) and in the absence of the pressure gradient along x direction, we get

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial t} = \nu \left(1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t} \right) u \quad (13)$$

subject to no-slip boundary conditions at the plates given by

$$u = ae^{i\omega_1 t} + be^{-i\omega_1 t} \quad \text{at } y = 0, t > 0 \quad (14a)$$

and

$$u = ce^{i\omega_2 t} + de^{-i\omega_2 t} \quad \text{at } y = h, t > 0 \quad (14b)$$

where (a,b,c,d) are complex constants so that u becomes real on the plates with frequencies of oscillations (ω_1 , and ω_2).

The initial conditions of the problem are

$$u = 0, \frac{\partial u}{\partial t} = 0 \quad \text{at } t \leq 0 \quad \text{for all } y \in (0, h) \quad (14c)$$

Now, on introducing the dimensionless quantities given by

$$\bar{y} = \frac{y}{\sqrt{\nu \lambda_1}}, \quad \bar{t} = \frac{t}{\lambda_1}, \quad D = \frac{h}{\sqrt{\nu \lambda_1}}, \quad \bar{u} = \frac{u}{U_0}, \quad k = \frac{\lambda_2}{\lambda_1} (\leq 1), \quad \text{and } n = \frac{\sigma \lambda_1 B_0^2}{\rho} \quad (15)$$

and the non-dimensional flow parameters are

$$(a', b', c', d') = U_0 (a, b, c, d) \quad \text{and} \quad (\sigma_1, \sigma_2) = \lambda_1 (\omega_1, \omega_2) \quad (16)$$

On dropping the primes from the above, the problem now reduces to solving the equation

$$\left(1 + \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial t} = \left(1 + k \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} - n \left(1 + \frac{\partial}{\partial t} \right) u \quad (17)$$

subject to boundary and initial conditions:

$$u = ae^{i\sigma_1 t} + be^{-i\sigma_1 t} \quad \text{at } y = 0, t > 0 \quad (18a)$$

$$u = ce^{i\sigma_2 t} + de^{-i\sigma_2 t} \quad \text{at } y = D, t > 0 \quad (18b)$$

and

$$u = 0, \frac{\partial u}{\partial t} = 0 \quad \text{at } t \leq 0 \quad \text{for all } y \in (0, D) \quad (18c)$$

Solution of the Problem

In order to solve the boundary value problem, we introduce the Laplace Transform $\bar{u}(y, s)$ of $u(y, t)$ defined in Myint-U and Debnath [18].

Applying the Laplace Transform in 18a, b with the initial conditions, we obtain the transformed equation with s as the Laplace Transform variable as

$$\frac{d^2 \bar{u}}{dy^2} - p^2 \bar{u} = 0, \quad (19)$$

where $p^2 = \frac{(n+s)(1+s)}{1+ks}$ with the boundary conditions

$$\bar{u} = \frac{a}{s - i\sigma_1} + \frac{b}{s + i\sigma_1} \quad \text{at } y = 0 \quad (20a)$$

and

$$\bar{u} = \frac{c}{s - i\sigma_2} + \frac{d}{s + i\sigma_2} \quad \text{at } y = D \quad (20b)$$

where s is the Laplace transform variable.

Hence, The transformed solution for the fluid velocity $\bar{u}(y, s)$ becomes

$$\begin{aligned} \bar{u}(y, s) = & \left(\frac{a}{s - i\sigma_1} + \frac{b}{s + i\sigma_1} \right) \frac{\sinh p(D - y)}{\sinh pD} \\ & + \left(\frac{c}{s - i\sigma_2} + \frac{d}{s + i\sigma_2} \right) \frac{\sinh py}{\sinh pD} \end{aligned} \quad (21)$$

We now apply the inverse Laplace Transform to obtain the exact solution for the fluid velocity in the form

$$\begin{aligned} u(y, t) = & \frac{1}{2} e^{i\sigma_1 t} \frac{\sinh(\alpha + i\beta)(D - y)}{\sinh(\alpha + i\beta)D} + \frac{1}{2} e^{-i\sigma_1 t} \frac{\sinh(\alpha - i\beta)(D - y)}{\sinh(\alpha - i\beta)D} \\ & + \frac{1}{2} e^{i\sigma_2 t} \frac{\sinh(\gamma + i\delta)y}{\sinh(\gamma + i\delta)D} + \frac{1}{2} e^{-i\sigma_2 t} \frac{\sinh(\gamma - i\delta)y}{\sinh(\gamma - i\delta)D} \\ & - \frac{2\pi k}{D^2} \sum_{m=0}^{\infty} m(-1)^m \frac{\sin m\pi(D - y)}{D} (\Delta_1 e^{s_1 t} + \Delta_2 e^{s_2 t}) \\ & - \frac{2\pi k}{D^2} \sum_{m=0}^{\infty} m(-1)^m \frac{\sin m\pi y}{D} (\Delta_3 e^{s_1 t} + \Delta_4 e^{s_2 t}), \end{aligned} \quad (22)$$

where

$$(\alpha, \beta) = \frac{1}{\sqrt{2}} \left(\sqrt{A^2 + B^2} \pm A \right)^{\frac{1}{2}}, \quad (\gamma, \delta) = \frac{1}{\sqrt{2}} \left(\sqrt{C^2 + D^2} \pm C \right)^{\frac{1}{2}} \quad (23)$$

$$A = \frac{n + \sigma_1^2 (nk + k - 1)}{1 + k^2 \sigma_1^2}, \quad (24a)$$

$$B = \frac{\sigma_1 \{n(1 - k) + 1 + k\sigma_1^2\}}{1 + k^2 \sigma_1^2} \quad (24b)$$

$$C = \frac{n + \sigma_2^2 (nk + k - 1)}{1 + k^2 \sigma_2^2}, \quad (24c)$$

$$D = \frac{\sigma_2 \{n(1-k) + 1 + k\sigma_2^2\}}{1 + k^2\sigma_2^2} \quad (24d)$$

$$(s_1, s_2) = \frac{1}{2} \left\{ -(1 + n + km^2\pi^2) \pm \sqrt{(1 + n + km^2\pi^2)^2 - 4(n + m^2\pi^2)} \right\} \quad (25)$$

$$\Delta_1 = \left[\frac{a}{s_1 - i\sigma_1} + \frac{b}{s_1 + i\sigma_1} \right] \left[1 - \frac{(1-k)(1-nk)}{(1+ks_1)^2} \right]^{-1}, \quad (26a)$$

$$\Delta_2 = \left[\frac{a}{s_2 - i\sigma_1} + \frac{b}{s_2 + i\sigma_1} \right] \left[1 - \frac{(1-k)(1-nk)}{(1+ks_2)^2} \right]^{-1} \quad (26b)$$

$$\Delta_3 = \left[\frac{c}{s_1 - i\sigma_2} + \frac{d}{s_1 + i\sigma_2} \right] \left[1 - \frac{(1-k)(1-nk)}{(1+ks_1)^2} \right]^{-1}, \quad (26c)$$

$$\Delta_4 = \left[\frac{c}{s_2 - i\sigma_2} + \frac{d}{s_2 + i\sigma_2} \right] \left[1 - \frac{(1-k)(1-nk)}{(1+ks_2)^2} \right]^{-1} \quad (26d)$$

The real form of (22) yields

$$\begin{aligned} u(y, t) = & a[(A_1 \cos \sigma_1 t - B_1 \sin \sigma_1 t) + i(B_1 \cos \sigma_1 t + A_1 \sin \sigma_1 t)] \\ & + b[(A_1 \cos \sigma_1 t - B_1 \sin \sigma_1 t) - i(B_1 \cos \sigma_1 t + A_1 \sin \sigma_1 t)] \\ & + c[(A_2 \cos \sigma_2 t - B_2 \sin \sigma_2 t) + i(B_2 \cos \sigma_2 t + A_2 \sin \sigma_2 t)] \\ & + d[(A_2 \cos \sigma_2 t - B_2 \sin \sigma_2 t) - i(B_2 \cos \sigma_2 t + A_2 \sin \sigma_2 t)] \\ & - \frac{2\pi k}{D^2} \sum_{m=1}^{\infty} m(-1)^m \frac{\sin m\pi(D-y)}{D} (\Delta_1 e^{s_1 t} + \Delta_2 e^{s_2 t}) \\ & - \frac{2\pi k}{D^2} \sum_{m=1}^{\infty} m(-1)^m \frac{\sin m\pi y}{D} (\Delta_3 e^{s_1 t} + \Delta_4 e^{s_2 t}), \end{aligned} \quad (27)$$

where

$$A_1 = \frac{1}{k_3} \{k_1 \cos(D-y)\beta \cdot \sinh(D-y)\alpha + k_2 \sin(D-y)\beta \cdot \cosh(D-y)\alpha\} \quad (28)$$

$$B_1 = \frac{1}{k_3} \{k_1 \sin(D-y)\beta \cdot \cosh(D-y)\alpha - k_2 \cos(D-y)\beta \cdot \sinh(D-y)\alpha\} \quad (29)$$

$$A_2 = \frac{1}{k'_3} \{k'_1 \cos Dy\delta \cdot \sinh Dy\gamma + k'_2 \sin Dy\delta \cdot \cosh Dy\gamma\} \quad (30)$$

$$B_2 = \frac{1}{k'_3} \{k'_1 \sin Dy\delta \cdot \cosh Dy\gamma - k'_2 \cos Dy\delta \cdot \sinh Dy\gamma\} \quad (31)$$

$$k_1 = \sinh \alpha \cdot \cos \beta, \quad (32a)$$

$$k_2 = \cosh \alpha \cdot \sin \beta, \quad (32b)$$

$$k_3 = k_1^2 + k_2^2 \quad (32c)$$

$$k'_1 = \sinh \gamma \cdot \cos \delta, \quad (33a)$$

$$k'_2 = \cosh \gamma \cdot \sin \delta, \quad (33b)$$

$$k'_3 = k_1'^2 + k_2'^2. \quad (33c)$$

The result (22) describe the fluid velocity between two oscillating plates in the general case. Many particular results follow easily from the above solutions. For example, if

$D = 1$, $a = b = 0$, $c = d = \frac{V}{2}$ and $\omega_1 = \omega_2 = \omega$, (22) provides the result similar to Mitra and Bhattacharyya [19]. On the other hand, when $D = 1$, $\omega_1 = \omega_2 = 0$, the result analogous to that of Mitra and Bhattacharyya [20] will emerge. Moreover, when $D = 1$, $(a, b) = (u_0/2i, -u_0/2i)$, $(c, d) = (u_1/2i, -u_1/2i)$ and $\sigma_1 = \sigma_2 = \sigma$, the resulting solutions provide an extension of the work of Stanisic et al. [21] for the case of an Oldroyd-B fluid.

Some Special Cases

(i). The Steady-state solutions:

Proceeding to the limit $t \rightarrow \infty$ in (22), we find that the transient effects die out for all frequencies σ_r ($r = 1, 2$) and the ultimate steady-state is attained and the fluid velocity assumes the form

$$u(y, t) = a e^{i\sigma_1 t} \frac{\sinh(\alpha + i\beta)(d - y)}{\sinh(\alpha + i\beta)} + b e^{i\sigma_1 t} \frac{\sinh(\alpha - i\beta)(d - y)}{\sinh(\alpha - i\beta)} \\ + c e^{i\sigma_2 t} \frac{\sinh(\gamma + i\delta)y}{\sinh(\gamma + i\delta)} + d e^{-i\sigma_2 t} \frac{\sinh(\gamma - i\delta)y}{\sinh(\gamma - i\delta)} \quad (34)$$

The result (34) represents Stokes-Hartmann boundary layers in an Oldroyd-B fluid formed in combination on the plates. The thickness of these layers in non-dimensional form are of the order $\frac{\sqrt{2}}{(\sqrt{A^2+B^2+A})^{1/2}}$ on $y = 0$ and $\frac{\sqrt{2}}{(\sqrt{C^2+D^2+C})^{1/2}}$ on $y = D$.

Obviously, the thickness of the layers decreases with the decrease in fluid elasticity and the magnetic field. Consequently, the attainment of the steady-state is much quicker in an electrically conducting Oldroyd-B fluid compared to the corresponding hydromagnetic situation.

Further, the thickness of the layers formed on the respective plates becomes identical when $\sigma_2 = \sigma_2 = 0$ (non-oscillatory case) and when $\sigma_2 = \sigma_2 = \sigma$ (oscillating with same frequency).

In the limit $h \rightarrow \infty$, the steady-state solution for the non-dimensional fluid velocity corresponding to the single oscillating plate is given by

$$u(y, t) = a e^{i\sigma_1 t - (\alpha + i\beta)y} + b e^{-i\sigma_1 t - (\alpha - i\beta)y}. \quad (35)$$

When $a = b = 1/2$,

$$u(y, t) = e^{-\alpha y} \left[\frac{e^{i(\sigma_1 t - \beta y)} + e^{-i(\sigma_1 t - \beta y)}}{2} \right], \text{ i.e., } u(y, t) = e^{-\alpha y} \cos[\sigma_1 t - \beta y], \quad (36)$$

where

$$\alpha = \left[\frac{\sqrt{n^2 + \sigma_1^2} + n}{2} \right]^{1/2}, \quad \text{and} \quad \beta = \left[\frac{\sqrt{n^2 + \sigma_1^2} - n}{2} \right]^{1/2}. \quad (37)$$

This is the Stokes' solution corresponding to the hydromagnetic flow of an Oldroyd-B fluid near an oscillating plate given in [1].

Further, when $k = 1$ (viscous case), (36) provides the fluid velocity in dimensional form as,

$$u(y, t) = e^{-\alpha_1 \frac{y}{\sqrt{2\nu}}} \cos[\omega_1 t - y\sqrt{\omega_1/2\nu}]. \quad (38)$$

(ii) Hydromagnetic Flow (with different frequency):

In the limit $k \rightarrow 1$, the fluid velocity (22) provides the result corresponding to hydromagnetic flow of clean viscous fluid between two oscillating plates as

$$\begin{aligned} u(y, t) = & a[(A_3 \cos \sigma_1 t - B_3 \sin \sigma_1 t) + i(B_3 \cos \sigma_1 t + A_3 \sin \sigma_1 t)] \\ & + b[(A_3 \cos \sigma_1 t - B_3 \sin \sigma_1 t) - i(B_3 \cos \sigma_1 t + A_3 \sin \sigma_1 t)] \\ & + c[(A_4 \cos \sigma_2 t - B_4 \sin \sigma_2 t) + i(B_4 \cos \sigma_2 t + A_4 \sin \sigma_2 t)] \\ & + d[(A_4 \cos \sigma_2 t - B_4 \sin \sigma_2 t) - i(B_4 \cos \sigma_2 t + A_4 \sin \sigma_2 t)] \\ & - \frac{2\pi}{D} \sum_{m=1}^{\infty} m(-1)^m \left[\Delta_{11} \frac{\sin m\pi(D-y)}{D} + \Delta_{12} \frac{\sin m\pi y}{D} \right] e^{-(n+m^2\pi^2)t} \end{aligned} \quad (39)$$

where

$$A_3 = \frac{1}{l_3} \{l_1 \cos(D-y)\beta_1 \cdot \sinh(D-y)\alpha_1 + l_2 \sin(D-y)\beta_1 \cdot \cosh(D-y)\alpha_1\} \quad (40)$$

$$B_3 = \frac{1}{l_3} \{l_1 \sin(D-y)\beta_1 \cdot \cosh(D-y)\alpha_1 - l_2 \cos(D-y)\beta_1 \cdot \sinh(D-y)\alpha_1\} \quad (41)$$

$$A_4 = \frac{1}{l'_3} \{l'_1 \cos Dy\delta_1 \cdot \sinh Dy\gamma_1 + l'_2 \sin Dy\delta_1 \cdot \cosh Dy\gamma_1\} \quad (42)$$

$$B_4 = \frac{1}{l'_3} \{l'_1 \sin Dy\delta_1 \cdot \cosh Dy\gamma_1 - l'_2 \cos Dy\delta_1 \cdot \sinh Dy\gamma_1\} \quad (43)$$

$$l_1 = \sinh \alpha_1 \cdot \cos \beta_1, \quad (44a)$$

$$l_2 = \cosh \alpha_1 \cdot \sin \beta_1, \quad (44b)$$

$$l_3 = l_1^2 + l_2^2 \quad (44c)$$

$$l'_1 = \sinh \gamma_1 \cdot \cos \delta_1, \quad (45a)$$

$$l'_2 = \cosh \gamma_1 \cdot \sin \delta_1, \quad (45b)$$

$$l'_3 = l'^2_1 + l'^2_2 \quad (45c)$$

$$(\alpha_1, \beta_1) = \sqrt{\frac{1+\sigma_1^2}{2}} \left(\sqrt{(n^2 + \sigma_1^2)} \pm n \right)^{\frac{1}{2}}, (\gamma_1, \delta_1) = \sqrt{\frac{1+\sigma_2^2}{2}} \left(\sqrt{(n^2 + \sigma_2^2)} \pm n \right)^{\frac{1}{2}} \quad (46)$$

$$\Delta_{11} = \left[\frac{a}{s_1 - i\sigma_1} + \frac{b}{s_1 + i\sigma_1} \right], \quad (47a)$$

$$\Delta_{12} = \left[\frac{c}{s_1 - i\sigma_2} + \frac{d}{s_1 + i\sigma_2} \right] \quad (47b)$$

$$\text{and } s_1 = -(n + m^2\pi^2) \quad (48)$$

(iii) **Hydromagnetic Flow (with equal frequency):**

If the plates oscillate with the same frequency, i.e. $\sigma_1 = \sigma_2 = \sigma$, then by putting $D = 1$ and $a = b = c = d = 1/2$, we get a simpler form of Eqs. (39–48) as follows:

$$u(y, t) = (A_3 \cos \sigma t - B_3 \sin \sigma t) - 2\pi \sum_{m=1}^{\infty} m(-1)^m [\sin m\pi(1-y) + \sin \pi y] \frac{n + m^2\pi^2}{(n + m^2\pi^2)^2 + \sigma^2} e^{-(n+m^2\pi^2)t} \quad (49)$$

where

$$A_3 = \frac{1}{l_3} \{l_1 \cos(1-y)\beta_1 \cdot \sinh(1-y)\alpha_1 + l_2 \sin(1-y)\beta_1 \cdot \cosh(1-y)\alpha_1\} \quad (50)$$

$$B_3 = \frac{1}{l_3} \{l_1 \sin(1-y)\beta_1 \cdot \cosh(1-y)\alpha_1 - l_2 \cos(1-y)\beta_1 \cdot \sinh(1-y)\alpha_1\} \quad (51)$$

$$(\alpha_1, \beta_1) = \sqrt{\frac{1+\sigma^2}{2}} \left(\sqrt{(n^2 + \sigma^2)} \pm n \right)^{\frac{1}{2}}, \quad (52)$$

(iv) **Non-oscillatory Hydromagnetic Flow:**

In the non-oscillatory situation $\sigma_1 = \sigma_2 = 0$. Accordingly, the fluid velocity between the plates, in this case, as obtained from (22), is given by

$$u(y, t) = (a+b) \frac{\sinh \sqrt{n}(1-y)}{\sinh \sqrt{n}} + (c+d) \frac{\sinh \sqrt{ny}}{\sinh \sqrt{n}} - 2\pi \sum_{m=1}^{\infty} m(-1)^m \sin m\pi(1-y) (F_1 e^{s_1 t} + F_2 e^{s_2 t}) - 2\pi \sum_{m=1}^{\infty} m(-1)^m \sin m\pi y (F_3 e^{s_1 t} + F_4 e^{s_2 t}) \quad (53)$$

where

$$F_1 = \frac{a+b}{s_1} \left[1 - \frac{(1-k)(1-nk)}{(1+ks_1)^2} \right]^{-1}, \quad (54a)$$

$$F_2 = \frac{a+b}{s_2} \left[1 - \frac{(1-k)(1-nk)}{(1+ks_2)^2} \right]^{-1} \quad (54b)$$

$$F_3 = \frac{c+d}{s_1} \left[1 - \frac{(1-k)(1-nk)}{(1+ks_1)^2} \right]^{-1}, \quad (54c)$$

$$F_4 = \frac{c+d}{s_2} \left[1 - \frac{(1-k)(1-nk)}{(1+ks_2)^2} \right]^{-1} \quad (54d)$$

and s_1, s_2 are given by (25).

The first two terms of (53) represent the steady Hartman layer flow similar to that of classical fluid while the unsteady part contains both the effects of elasticity and magnetic field. Hence, in the non-oscillatory case, the ultimate flow becomes identical with that of the classical fluid flow.

The Wall Shear Stresses

(i) The shear stress on the lower plate ($y = 0$), in dimensionless form, is given by

$$\begin{aligned} \tau_0 = & -\frac{1}{2} \frac{1 + ik\sigma_1}{1 + i\sigma_1} \left(e^{i\sigma_1 t} - e^{-t} \right) (\alpha + i\beta) \frac{\cosh(\alpha + i\beta)D}{\sinh(\alpha + i\beta)D} \\ & -\frac{1}{2} \frac{1 - ik\sigma_1}{1 - i\sigma_1} \left(e^{-i\sigma_1 t} - e^{-t} \right) (\alpha - i\beta) \frac{\cosh(\alpha - i\beta)D}{\sinh(\alpha - i\beta)D} \\ & +\frac{1}{2} \frac{1 + ik\sigma_2}{1 + i\sigma_2} \left(e^{i\sigma_2 t} - e^{-t} \right) (\gamma + i\delta) \frac{1}{\sinh(\gamma + i\delta)D} \\ & +\frac{1}{2} \frac{1 - ik\sigma_2}{1 - i\sigma_2} \left(e^{-i\sigma_2 t} - e^{-t} \right) (\gamma - i\delta) \frac{1}{\sinh(\gamma - i\delta)D} \\ & +\frac{2\pi^2 k}{D^3} \sum_{m=0}^{\infty} m^2 \left[\frac{1 + ks_1}{1 + s_1} (e^{s_1 t} - e^{-t}) \Delta_1 + \frac{1 + ks_2}{1 + s_2} (e^{s_2 t} - e^{-t}) \Delta_2 \right] \\ & -\frac{2\pi^2 k}{D^3} \sum_{m=0}^{\infty} m^2 (-1)^m \left[\frac{1 + ks_1}{1 + s_1} (e^{s_1 t} - e^{-t}) \Delta_3 + \frac{1 + ks_2}{1 + s_2} (e^{s_2 t} - e^{-t}) \Delta_4 \right] \end{aligned} \quad (55)$$

where $\alpha, \beta, \gamma, \delta, \sigma_1, \sigma_2, \Delta_1, \Delta_2, \Delta_3, \Delta_4$ are given by (23) to (26) respectively, which in real form is

$$\begin{aligned} \tau_0 = & L_5 \cos \sigma_1 t + M_5 \sin \sigma_1 t + L_6 \cos \sigma_2 t + M_6 \sin \sigma_2 t + Ne^{-t} \\ & -2\pi^2 \sum_{m=1}^{\infty} m^2 \left[\Delta_1 (1 + ks_1) \frac{e^{s_1 t} - e^{-t}}{1 + s_1} + \Delta_2 (1 + ks_2) \frac{e^{s_2 t} - e^{-t}}{1 + s_2} \right] \\ & +2\pi^2 \sum_{m=1}^{\infty} m^2 (-1)^m \left[\Delta_3 (1 + ks_1) \frac{e^{s_1 t} - e^{-t}}{1 + s_1} + \Delta_4 (1 + ks_2) \frac{e^{s_2 t} - e^{-t}}{1 + s_2} \right] \end{aligned} \quad (56)$$

where

$$L_5 = L_3 - \sigma_1 M_3, \quad (57a)$$

$$M_5 = M_3 + \sigma_1 L_3, \quad (57b)$$

$$L_6 = L_4 - \sigma_2 M_4, \quad (57c)$$

$$M_6 = M_4 + \sigma_2 L_4 \quad (57d)$$

$$L_3 = k\sigma_1 M_1 - L_1, \quad (58a)$$

$$M_3 = k\sigma_1 L_1 + M_1, \quad (58b)$$

$$L_4 = k\sigma_2 M_2 - L_2, \quad (58c)$$

$$M_4 = k\sigma_2 L_2 + M_2 \quad (58d)$$

$$L_1 = \frac{1}{k_3} \{ (k_1 \beta - k_2 \alpha) \sin \beta \cdot \sinh \alpha - (k_1 \alpha + k_2 \beta) \cos \beta \cdot \cosh \alpha \} \quad (59a)$$

$$M_1 = \frac{1}{k_3} \{ (k_2 \alpha - k_1 \beta) \cos \beta \cdot \cosh \alpha - (k_1 \alpha + k_2 \beta) \sin \beta \cdot \sinh \alpha \} \quad (59b)$$

$$L_2 = \frac{1}{k'_3} [k'_1 \gamma + k'_2 \delta], \quad (59c)$$

Fig. 2 Fluid velocity profiles for different values of k when $n = 0.5$, $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

Fig. 3 Fluid velocity profiles for different values of k when $n = 0.5$, $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

$$M_2 = \frac{1}{k'_3} [k'_1 \delta - k'_2 \gamma] \quad (59d)$$

and k_1 , k_2 , k_3 , k'_1 , k'_2 , k'_3 are given by the Eqs. (32) and (33) respectively.

Using (22) the shear stress τ_D , at the upper oscillating plate ($y = D$) in dimensionless form, can similarly be written down.

Fig. 4 Fluid velocity profiles for different values of k when $n = 0.5$, $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

Fig. 5 Fluid velocity profiles for different values of n when $k = 0.4$, $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

(ii) Stress on the upper plate (at $y = D$):

The shear stress on the lower plate ($y = D$), in dimensionless form becomes

$$\tau_D = -\frac{1}{2} \frac{1 + ik\sigma_1}{1 + i\sigma_1} \left(e^{i\sigma_1 t} - e^{-t} \right) (\alpha + i\beta) \frac{1}{\sinh(\alpha + i\beta)D} - \frac{1}{2} \frac{1 - ik\sigma_1}{1 - i\sigma_1} \left(e^{-i\sigma_1 t} - e^{-t} \right) (\alpha - i\beta) \frac{1}{\sinh(\alpha - i\beta)D}$$

Fig. 6 Fluid velocity profiles for different values of n when $k = 0.4$, $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

Fig. 7 Fluid velocity profiles for different values of n when $k = 0.4$, $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

$$\begin{aligned}
 & + \frac{1}{2} \frac{1 + ik\sigma_2}{1 + i\sigma_2} (e^{i\sigma_2 t} - e^{-t}) (\gamma + i\delta) \frac{\cosh(\gamma + i\delta)D}{\sinh(\gamma + i\delta)D} \\
 & + \frac{1}{2} \frac{1 - ik\sigma_2}{1 - i\sigma_2} (e^{-i\sigma_2 t} - e^{-t}) (\gamma - i\delta) \frac{\cosh(\gamma - i\delta)D}{\sinh(\gamma - i\delta)D} \\
 & + \frac{2\pi^2 k}{D^3} \sum_{m=0}^{\infty} m^2 (-1)^m \left[\frac{1 + ks_1}{1 + s_1} (e^{s_1 t} - e^{-t}) \Delta_1 + \frac{1 + ks_2}{1 + s_2} (e^{s_2 t} - e^{-t}) \Delta_2 \right] \\
 & - \frac{2\pi^2 k}{D^3} \sum_{m=0}^{\infty} m^2 \left[\frac{1 + ks_1}{1 + s_1} (e^{s_1 t} - e^{-t}) \Delta_3 + \frac{1 + ks_2}{1 + s_2} (e^{s_2 t} - e^{-t}) \Delta_4 \right] \quad (60)
 \end{aligned}$$

Fig. 8 Small time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

Fig. 9 Large time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

which in real form becomes

$$\begin{aligned} \tau_D = & L_7 \cos \sigma_1 t + M_7 \sin \sigma_1 t + L_8 \cos \sigma_2 t + M_8 \sin \sigma_2 t + N_1 e^{-t} \\ & - 2\pi^2 \sum_{m=1}^{\infty} m^2 (-1)^m \left[\Delta_1 (1 + ks_1) \frac{e^{s_1 t} - e^t}{1 + s_1} + \Delta_2 (1 + ks_2) \frac{e^{s_2 t} - e^t}{1 + s_2} \right] \\ & + 2\pi^2 \sum_{m=1}^{\infty} m^2 \left[\Delta_3 (1 + ks_1) \frac{e^{s_1 t} - e^t}{1 + s_1} + \Delta_4 (1 + ks_2) \frac{e^{s_2 t} - e^t}{1 + s_2} \right] \end{aligned} \quad (61)$$

Fig. 10 Small time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

Fig. 11 Large time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

where

$$L_7 = L_{13} - \sigma_1 M_{13}, \quad (62a)$$

$$M_7 = M_{13} + \sigma_1 L_{13}, \quad (62b)$$

$$L_8 = L_{14} - \sigma_2 M_{14}, \quad (62c)$$

$$M_8 = M_{14} + \sigma_2 L_{14} \quad (62d)$$

Fig. 12 Small time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

Fig. 13 Large time shear stress profiles at the lower plate ($y = 0$) when $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

$$L_{13} = k\sigma_1 M_{11} - L_{11}, \quad (63a)$$

$$M_{13} = k\sigma_1 L_{11} + M_{11} \quad (63b)$$

$$L_{14} = k\sigma_2 M_{12} - L_{12}, \quad (63c)$$

$$M_{14} = k\sigma_2 L_{12} + M_{12} \quad (63d)$$

$$L_{11} = -\frac{1}{k_3} \{ (k_1 \alpha + k_2 \beta), \quad (64a)$$

$$M_{11} = \frac{1}{k_3} \{ (k_2 \alpha - k_1 \beta) \quad (64b)$$

Fig. 14 Small time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

$$L_{12} = \frac{1}{k'_3} [(k'_2\gamma - k'_1\delta) \sin \delta \sinh \gamma + (k'_1\gamma + k'_2\delta) \cos \delta \cosh \gamma] \quad (65a)$$

$$M_{12} = \frac{1}{k'_3} [(k'_1\delta - k'_2\gamma) \cos \delta \cosh \gamma + (k'_1\gamma + k'_2\delta) \sin \delta \sinh \gamma] \quad (65b)$$

$$N_1 = \frac{\sigma_1 M_{13} - L_{13}}{1 + \sigma_1^2} + \frac{\sigma_2 M_{14} - L_{14}}{1 + \sigma_2^2} \quad (66)$$

Numerical Results and Discussions

To proceed with the numerical evaluation of the general results (22), (56) and (61) in some particular cases, we first suppose that the time period of one complete oscillation of the lower plate is T_1 while that of the upper plate is T_2 . Consequently, we have $\sigma_1 = \frac{2\pi}{T_1}$ and $\sigma_2 = \frac{2\pi}{T_2}$ with $\sigma = \frac{T_1}{T_2}$.

We now discuss the relevant results corresponding to three different types of movements of the plates: $\sigma < 1$, $\sigma = 1$ and $\sigma > 1$. In order that the plates oscillate in phase with each other in all the three cases, we put $a = b = c = d = \frac{1}{2}$ in the general results. The modified form of the general results for the fluid velocities and the wall shear stresses on the plates are then evaluated numerically for various values of the flow parameters k , n , T_1 , σ and also for different values of y and t . The velocity profiles are represented graphically for values of y from 0 to 1 when $t = \frac{T_1}{8}, \frac{2T_1}{8}, \frac{3T_1}{8}, \frac{4T_1}{8}, \frac{5T_1}{8}, \frac{6T_1}{8}, \frac{7T_1}{8}$ and T_1 . In Figs. 2, 3 and 4, the velocity profiles for the conducting Oldroyd-B fluid are shown for different values of the elastic parameter k , when $\sigma = 0.5$, 1.0 and 1.5 respectively while the other parameters n and T_1 are kept to the values 0.5 and 2 in each case. It is observed from Figs. 2, 3 and 4 that when the time period of oscillations of the lower plate is small, i.e. $T_1 = 2.0$ (for large frequency), the fluid velocity increases with the increase of the elastic parameter k when the

Fig. 15 Large time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 0.5$ ($T_1 < T_2$)

Fig. 16 Small time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

motion of the plates are advancing and decreases with the same when the motion of the plates are receding. Such a phenomena occurs due to the fact that the viscoelastic fluids neither grow nor decay as fast as clean viscous fluids. It is to be noted here that $k = 1$ represents clean viscous fluid and the fluid will be more and more viscoelastic for smaller and smaller values of k .

We further notice from the figures mentioned above that the effect of elasticity is more pronounced near the plate having a time period larger than the other, provided the time periods

Fig. 17 Large time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 1.0$ ($T_1 = T_2$)

Fig. 18 Small time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

of oscillations of both the plates are small and different. In Figs. 5, 6 and 7 the velocity profiles for the conducting viscoelastic fluids are shown for different values of the magnetic field in each of three cases of σ when T_1 is 2.0. It is clear from the figures that the magnetic field has a strong damping effect on the fluid velocity irrespective of the values of σ . Finally, Figs. 8, 10 and 12 represent the small time variation of shear stress values on the lower plate corresponding to three different values of σ when the time period of oscillation of the lower plate is 2.0 while those for the upper plate are shown in Figs. 14, 16 and 18 respectively. It

Fig. 19 Large time shear stress profiles at the upper plate ($y = 1$) when $T_1 = 2.0$ and $\sigma = 1.5$ ($T_1 > T_2$)

is seen that the drag values on both the plates when the small time is considered, are large with the increase of the elasticity k and insignificant with increase of magnetic field n . On the other hand, the large time oscillations of drag values on both the plates corresponding to three different values of σ when the time period of oscillation of the lower plate is 2.0 are presented in Figs. 9, 11, 13 and 15, 17, 19 respectively. In these cases, both the elasticity k and the magnetic field n have increasing effects on the amplitude of drag values on the plates.

Conclusion

Hydromagnetic channel flow of an electrically conducting viscoelastic Oldroyd-B fluid between two infinite rigid non-conducting parallel plates when the plates start impulsively from rest with different time period (frequency), amplitude and phase has been discussed in this paper. The operational method is used to obtain the exact solutions for the fluid velocity and the shear stresses on the walls. The quantitative evaluation of the results is considered when two plates oscillate in phase but with different frequency. The results are shown graphically for different time periods of oscillations of the plates which represent the cases: (i) the lower plate oscillates with a time period less than the upper, (ii) both the plates oscillate with the same time period and (iii) the lower plate oscillates with a time period greater than the upper. It is seen that, in general, the fluid motion is amplified and damped with the increase of elasticity k when the motion of the plates are advancing and receding respectively. Moreover, the effect of elasticity is found more pronounced near the plate having a time period larger than the other, provided the time periods of oscillations of both the plates are small and different. The magnetic field produces a damping effect on the flow irrespective of the motions of the plates. We further observe that the shear stress is negative on the lower plate and positive on the upper plate with their oscillations starting after a moderately large values of time. In all the cases, the magnitudes of wall shear stresses increase with the increase of both the elastic parameter k and the magnetic field n .

The problem although idealized bears a good resemblance with the fluid motion contained between the two layers of the earth, particularly at the time of earthquake, when both the layers execute rectilinear oscillations with different frequencies in their own planes in presence of an external magnetic field acting transversely on them. Moreover, such a problem is usually encountered in polymer and process industries, the movement of biological fluids and for the detection of the magnetic effect on the blood flow in cardiovascular system. Finally, the initial value approach to the problems of the above nature appears to be richer in contents and utility compared to the investigations made by earlier researchers.

References

1. Ghosh, A.K., Mitra, S.: On Stokes' problems for linear viscoelastic fluids. *Bull. de L' Acad. Pol. des Sci.* **25**(8), 55 (1977)
2. Roy, R.N., Samad, A., Chaudhury, T.K.: Low Reynolds number stability of MHD plane Poiseuille flow of an Oldroyd fluid. *Int. J. Math. Math. Sci.* **23**(9), 617–625 (2000)
3. Hayat, T., Siddiqui, A.M., Asghar, S.: Some simple flows of an Oldroyd-B fluid. *Int. J. Eng. Sci.* **39**, 135–147 (2001)
4. Hayat, T., Hutter, K., Asghar, S., Siddiqui, A.M.: MHD flows of an Oldroyd-B fluid. *Math. Comput. Model.* **36**, 987–995 (2002)
5. Hayat, T., Khan, M., Ayub, M.: Exact solutions of flow problems of an Oldroyd-B fluid. *Appl. Math. Comput.* **151**, 105–119 (2004)
6. Hayat, T., Nadeem, S., Asghar, S.: Hydromagnetic couette flow of an Oldroyd-B fluid in a rotating system. *Int. J. Eng. Sci.* **42**, 65–78 (2004)
7. Hayat, T., Husain, M., Khan, M.: Hall effect on flows of an Oldroyd-B fluid through porous medium for cylindrical geometries. *Comput. Math. Appl.* **52**, 269–282 (2006)
8. Asghar, S., Parveen, S., Hanif, S., Siddiqui, A.M., Hayat, T.: Hall effects on the unsteady hydromagnetic flows of an Oldroyd-B fluid. *Int. J. Eng. Sci.* **41**, 609–619 (2003)
9. Khan, M., Hayat, T., Asghar, S.: Exact solution for MHD flow of a generalized Oldroyd-B fluid with modified Darcy's law. *Int. J. Eng. Sci.* **44**, 333–339 (2006)
10. Khan, M., Maqbool, K., Hayat, T.: Influence of Hall current on the flows of a generalized Oldroyd-B fluid in a porous space. *Acta Mech.* **184**, 1–13 (2006)
11. Khan, M., Mahmood, S., Fetecau, C.: New exact solution for magnetohydrodynamic flows of an Oldroyd-B fluid. *ZAMP* **60**(6), 1206–1219 (2009). doi:[10.1007/s00033-008-8029-7](https://doi.org/10.1007/s00033-008-8029-7)
12. Fetecau, C., Hayat, T., Khan, M., Fetecau, C.: Unsteady flow of an Oldroyd-B fluid induced by the impulsive motion of a plate between two side walls perpendicular to the plate. *Acta Mech.* **198**, 21–33 (2008)
13. Fetecau, C., Jamil, M., Constantin, F., Vieru, D.: The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid. *ZAMP* **60**(5), 921–933 (2009)
14. Tong, D., Liu, Y.: Exact solutions for the unsteady rotational flow of non-Newtonian fluid in an annular pipe. *Int. J. Eng. Sci.* **43**, 281–289 (2005)
15. Ghosh, A.K., Sana, P.: On hydromagnetic flow of an Oldroyd-B fluid near a pulsating plate. *Acts Astronaut.* **64**, 272–280 (2009)
16. Ghosh, A.K., Sana, P.: On hydromagnetic channel flow of an Oldroyd-B fluid induced by tooth pulses. *Magnetohydrodynamics* **44**(3), 325–340 (2008)
17. Ghosh, A.K., Sana, P.: On hydromagnetic rotating flow of an Oldroyd-B fluid near an oscillating plate. *ZAMP* doi:[10.1007/s00033-009-8060-3](https://doi.org/10.1007/s00033-009-8060-3) (2009)
18. Myint-U, T., Debnath, L.: *Partial Differential Equations for Scientists and Engineers*, 3rd edn. North Holland, New York (1987)
19. Mitra, P., Bhattacharyya, P.: On the hydromagnetic flow of a dusty fluid between two parallel plates, one being stationary and the other oscillating. *J. Phys. Soc. Jpn.* **50**(3), 995–1001 (1981)
20. Mitra, P., Bhattacharyya, P.: Unsteady hydromagnetic laminar flow of a conducting dusty fluid between two parallel plates started impulsively from rest. *Acta Mech.* **39**, 171–182 (1981)
21. Stanisic, M.M., Fetz, B.H., Mickelsen Jr, H.P., Czumak, F.M.: On the flow of a hydromagnetic fluid between two oscillating flat plates. *J. Aerosp. Sci.* **29**, 116–117 (1962)