Parameter Uniform Numerical Method for Third Order Singularly Perturbed Turning Point Problems Exhibiting Boundary Layers

N. Geetha¹ · A. Tamilselvan¹ · V. Subburayan²

Published online: 26 May 2015
© Springer India Pvt. Ltd. 2015

Abstract In this paper, a parameter uniform numerical method based on Shishkin mesh is suggested to solve singularly perturbed third order differential equations with a turning point exhibiting boundary layers. An error estimate is derived by using supremum norm. Numerical results are provided to illustrate the theoretical results.

Keywords Singularly perturbed turning point problems · Finite difference scheme · Shishkin mesh

Introduction

A differential equation with a small positive parameter multiplying the highest derivative term subject to boundary conditions belongs to a class of problems known as Singular Perturbation Problems (SPPs). Singular perturbation problems for differential equations occur in many branches of applied mathematics. The presence of small parameter(s) in these problems prevents us from obtaining satisfactory numerical solutions. It is a well-known fact that the solutions of SPPs have a multi-scale character. That is, there are thin layer(s) where the solution varies very rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. Therefore, one has to seek some special methods for the numerical solution of SPPs. Various finite difference schemes have been proposed in the literature to guarantee stability of the schemes for all values of the perturbation parameter.

✉ A. Tamilselvan
mathats@bdu.ac.in

N. Geetha
nhgeetha@gmail.com

V. Subburayan
subburayan123@gmail.com

¹ Bharathidasan University, Tiruchirappalli 620 024, Tamilnadu, India

² SRM University, Kattankulathur, Kanchipuram, Tamilnadu, India

For detailed discussion of analytical and numerical treatment of SPPs one may refer the books O'Malley [12], Doolan [5], Farrel [6], Roos [16] and Miller [10]. Natesan et al. [22–24, 29] suggested some numerical methods which include Asymptotic numerical method, Boundary value technique, Booster method, Shooting method for SPPs. The Non-standard algorithm for linear and nonlinear convection diffusion problem is given in [27, 30]. Adaptively generated grid methods were discussed in [25, 26]. In [28], the author gives a efficient numerical method in which the second order equation is converted into a system of first order equations. In [21], Natividad and Stynes proposed a Richardson extrapolation technique which gives second order accurate.

Singularly Perturbed Turning Point Problems (SPTPPs) arise in various fields of applied mathematics like one dimensional version of stationary convection diffusion problems with a dominant convective term, speed field that changes its sign in the catch basin, geophysics and modeling thermal boundary layers in laminar flow [19].

In general, the analytical and numerical treatments of SPTPPs are more complicated than SPPs without turning points, because the coefficient of the convection term vanishes inside the domain. It is well known that standard discretization methods for solving SPTPPs are sometimes unstable and fail to give accurate results when the perturbation parameter ε is small. In order to overcome this difficulty we apply the classical finite difference scheme on an appropriate piecewise uniform (Shishkin) mesh.

In the past few years, few authors applied some numerical methods for singularly perturbed second order ordinary differential equations with turning points. To cite a few, Abrahamsson [1] derived a priori estimates for the solution and its derivatives of SPPs with a turning point. Wasow [19], O'Malley [12], Watts [20] and Roos et al. [16] studied the qualitative aspects of turning point problems. Farrell [7] and Berger et al. [3] obtained a general sufficient condition for a uniformly convergent scheme for second order turning point problem. Natesan and Ramanujam suggested a computational method for second order SPTPP using classical and Exponentially Fitted Difference Schemes in [13]. Also they applied another technique known as initial value technique in [14] for the same problem in which the numerical solution is obtained by solving suitable initial and terminal value problems. In [11] Kadalbajoo and Patidar applied a numerical method based on cubic spline with nonuniform grid for a second order SPTPP. Natesan et al. [15] proposed a parameter uniform numerical method in which piecewise-uniform mesh is combined with classical finite difference scheme to obtain the numerical solution. For more detail one may refer [18] and the references therein.

Generally the third derivative is the time derivative of acceleration and the third order equations are called Jerk equations ([8, 17]). In [2, 4] asymptotic expansion of solution is given for a third order singularly perturbed turning point problems. Existence of solution for third order semi linear differential equation with turning point is proved in [9]. In ([19], p. 5), Wasow have discussed a modified form of Orr-Sommerfeld equation which is a fourth order singularly perturbed turning point problem.

As mentioned in the abstract, we present a parameter uniform numerical method for singularly perturbed third order differential equations with a turning point exhibiting boundary layers. The present paper is organized as follows. In “Statement of the Problem” section, the problem under study is stated. Method of steps, maximum principle and stability result are discussed in “Maximum Principle and Stability Result” section. Some analytical results are derived in “Analytical Results” section. In “Discrete Problem” section a mesh selection strategy is explained. Further, an upwind finite difference scheme is given and the discrete maximum principle is proved. The error analysis is carried out in “Numerical Solution Estimates” section. Numerical examples are given in “Numerical Results” section which validate our theoretical results.

Statement of the Problem

Motivated by the works of Leung [2], Jian-ping and Zong-chi [4], Mo and Wen [9] and Natesan et al. [15], we consider the following boundary value problem for singularly perturbed third order differential equation with a turning point at $x = 0$:

Find $u \in C^1(\bar{\Omega}) \cap C^3(\Omega)$ such that

$$\begin{cases} \varepsilon u'''(x) + a(x)u''(x) - b(x)u'(x) + c(x)u(x) = f(x), & x \in \Omega = (-1, 1), \\ u(-1) = l_1, \quad u'(-1) = l_2, \quad u'(1) = l_3, \end{cases} \quad (1)$$

$$\begin{cases} |a(x)| \geq \alpha > 0, \quad \text{for } 0 < |x| \leq 1, \quad a(0) = 0, \quad a'(0) < 0, \\ \beta^0 \geq b(x) \geq \beta_0 > 0, \quad \gamma^0 \geq c(x) \geq \gamma_0 > 0, \quad \alpha < \beta_0 - \gamma^0, \\ \text{and } |a'(x)| \geq |a'(0)|/2 \quad \forall x \in \bar{\Omega} = [-1, 1] \end{cases} \quad (2)$$

where a, b, c and f are smooth functions on $\bar{\Omega}$. Using the analysis adopted in [9] we can prove the existence of the solution of the problem (1).

Throughout the paper C, C_1 denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. Let $y : D \rightarrow \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution is the maximum norm $\|w\|_D = \sup_{x \in D} |w(x)|$. In case of vectors $\bar{w} = (w_1, w_2)^T$, we define $|\bar{w}(x)| = (|w_1(x)|, |w_2(x)|)^T$ and $\|\bar{w}\|_D = \max\{\|w_1\|_D, \|w_2\|_D\}$.

Maximum Principle and Stability Result

The above problem (1) is equivalent to the following problem:

Find $\bar{u} = (u_1, u_2)^T$, $u_1 \in C^0(\bar{\Omega}) \cap C^1(\Omega_0)$ and $u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that $\bar{P}\bar{u} = (P_1\bar{u}, P_2\bar{u})^T$ where

$$P_1\bar{u} := u_2(x) - u_1'(x) = 0, \quad x \in \Omega_0 = (-1, 1], \quad (3a)$$

$$P_2\bar{u} := \varepsilon u_2''(x) + a(x)u_2'(x) - b(x)u_2(x) + c(x)u_1(x) = f(x), \quad x \in \Omega, \quad (3b)$$

$$u_1(-1) = l_1, \quad u_2(-1) = l_2, \quad u_2(1) = l_3. \quad (3c)$$

Theorem 1 (Maximum Principle) *Let $\bar{w}(x) = (w_1(x), w_2(x))^T$, $w_1 \in C^0(\bar{\Omega}) \cap C^1(\Omega)$ and $w_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be any function satisfying $P_1\bar{w} \leq 0$, $P_2\bar{w} \leq 0$, $w_1(-1) \geq 0$, $w_2(-1) \geq 0$ and $w_2(1) \geq 0$. Then $\bar{w}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.*

Proof Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ as

$$s_1(x) = 5 + 4x, \quad s_2(x) = 2 + x.$$

Then $\bar{s}(x) > \bar{0}$, for all $x \in \bar{\Omega}$ and $\bar{P}\bar{s}(x) < \bar{0}$, $x \in \Omega$. Further we define

$$\mu = \max \left\{ \max_{x \in \bar{\Omega}} \left(\frac{-w_1(x)}{s_1(x)} \right), \max_{x \in \bar{\Omega}} \left(\frac{-w_2(x)}{s_2(x)} \right) \right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and there exist a point $x_0 \in \Omega$, such that either $\left(\frac{-w_1(x_0)}{s_1(x_0)} \right) = \mu$ or $\left(\frac{-w_2(x_0)}{s_2(x_0)} \right) = \mu$ or both. Also $(\bar{w} + \mu\bar{s})(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Case 1 Assume that $\left(\frac{-w_1(x_0)}{s_1(x_0)}\right) = \mu$. That is $(w_1 + \mu s_1)(x_0) = 0$. Therefore $(w_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$0 > P_1(\bar{w} + \mu \bar{s})(x_0) = (w_2 + \mu s_2)(x_0) - (w_1 + \mu s_1)'(x_0) \geq 0$$

which is a contradiction.

Case 2 Assume that $\left(\frac{-w_2(x_0)}{s_2(x_0)}\right) = \mu$. That is $(w_2 + \mu s_2)(x_0) = 0$. Therefore $(w_2 + \mu s_2)$ attains its minimum at $x = x_0$. Then,

$$0 > P_2(\bar{w} + \mu \bar{s})(x_0) = \varepsilon (w_2 + \mu s_2)''(x_0) + a(x_0) (w_2 + \mu s_2)'(x_0) - b(x_0)(w_2 + \mu s_2)(x_0) + c(x_0) (w_1 + \mu s_1)(x_0) \geq 0.$$

which is a contradiction.

Hence $\bar{w}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$. $\square$

Lemma 1 (Stability Result) Let $\bar{u} = (u_1, u_2)^T$, $u_1 \in C^0(\bar{\Omega}) \cap C^1(\Omega)$, $u_2 \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ be any function. Then for $i = 1, 2$

$$|u_i(x)| \leq C \max \left\{ |u_1(-1)|, |u_2(-1)|, |u_2(1)|, \max_{x \in \bar{\Omega} \cup \{1\}} |P_1 \bar{u}|, \max_{x \in \bar{\Omega}} |P_2 \bar{u}| \right\}, \quad \forall x \in \bar{\Omega}.$$

Proof Define $\bar{\Psi}^\pm = (\Psi_1^\pm, \Psi_2^\pm)$ as

$$\Psi_1^\pm(x) = C(5 + 4x)C_1 \pm u_1(x) \text{ and } \Psi_2^\pm(x) = C(2 + x)C_1 \pm u_2(x), \text{ where}$$

$$C_1 = \max \left\{ |u_1(-1)|, |u_2(-1)|, |u_2(1)|, \max_{x \in \bar{\Omega} \cup \{1\}} |P_1 \bar{u}|, \max_{x \in \bar{\Omega}} |P_2 \bar{u}| \right\}.$$

Note that $\psi_1^\pm(-1) \geq 0$, $\psi_2^\pm(-1) \geq 0$, $\psi_2^\pm(1) \geq 0$ for a proper choice of $C > 0$. It is easy to see that, $P_1(\Psi^\pm(x)) \leq 0$, $\forall x \in \Omega_0$ and $P_2(\Psi^\pm(x)) \leq 0$, $\forall x \in \Omega$. Then by the maximum principle we get the desired result. $\square$

Note: Since the operators P_j , $j = 1, 2$ satisfy the above maximum principle, the solution of the BVP (3a)–(3c) is unique.

Analytical Results

In this section we present some analytical results for the solution and its derivatives. Herein after we shall denote the subdomains of $\bar{\Omega} = [-1, 1]$ as $\Omega_1 = [-1, -\delta]$, $\Omega_2 = [-\delta, \delta]$ and $\Omega_3 = [\delta, 1]$, $0 < \delta \leq 1/2$. The choice of $\delta = 1/2$ can be found in [3].

The following lemma gives estimates for $\bar{u}$ and its derivatives in the intervals Ω_1 and Ω_3 which exclude the turning point $x = 0$.

Lemma 2 Let $\bar{u} = (u_1, u_2)^T$ be the solution of (3a)–(3c) then

$$\begin{aligned} \|u_j^{(k)}(x)\| &\leq \begin{cases} C\varepsilon^{-(k-1)} \max \{\|f\|, \|\bar{u}\|\}, & \text{for } j = 1 \\ C\varepsilon^{-(k)} \max \{\|f\|, \|\bar{u}\|\}, & \text{for } j = 2 \end{cases}, \quad k = 1, 2 \\ \|u_j^{(3)}(x)\| &\leq \begin{cases} C\varepsilon^{-(k-1)} \max \{\|f\|, \|f'\|, \|\bar{u}\|\}, & \text{for } j = 1 \\ C\varepsilon^{-(k)} \max \{\|f\|, \|f'\|, \|\bar{u}\|\}, & \text{for } j = 2 \end{cases} \end{aligned}$$

$\forall x \in \Omega_1 \cup \Omega_3$, where C depends on $\|a\|$, $\|a'\|$, $\|b\|$, $\|b'\|$, $\|c\|$ and $\|c'\|$, $a(x) > 0$ for $x \in \Omega_1$ and $a(x) < 0$ for $x \in \Omega_3$.

Proof Using the technique adopted in [6] the present lemma can be proved in the subdomain Ω_1 . In a similar way one can prove an analogous result in the subdomain Ω_3 .

Using integration by parts we have,

$$\int_{-1}^x a(t)u_2'(t)dt = [a(t)u_2(t)]_{-1}^x - \int_{-1}^x a'(t)u_2(t)dt.$$

Now

$$\begin{aligned} & \left| \int_{-1}^x (a(t)u_2'(t) - b(t)u_2(t) + c(t)u_1(t) - f(t))dt \right| \\ &= \left| \int_{-1}^x a(t)u_2'(t)dt - \int_{-1}^x b(t)u_2(t)dt + \int_{-1}^x c(t)u_1(t)dt - \int_{-1}^x f(t)dt \right| \\ &\leq |a(x)u_2(x)| + |a(-1)u_2(-1)| + \int_{-1}^x |a'(t)u_2(t)|dt \\ &\quad + \int_{-1}^x |b(t)u_2(t)|dt + \int_{-1}^x |c(t)u_1(t)|dt + \int_{-1}^x |f(t)|dt \\ &\leq 2\|f\| + C\|\bar{u}\|, \end{aligned} \quad (4)$$

where C depends on $\|a\|$, $\|a'\|$, $\|b\|$, $\|b'\|$, $\|c\|$ and $\|c'\|$.

By Mean Value Theorem, there exist a point $z \in (-1, -1 + \varepsilon)$ such that

$$\begin{aligned} u_2'(z) &= \frac{u_2(-1 + \varepsilon) - u_2(-1)}{\varepsilon} \\ |\varepsilon u_2'(z)| &= |u_2(-1 + \varepsilon) - u_2(-1)| \\ |\varepsilon u_2'(z)| &\leq 2\|\bar{u}\| \end{aligned} \quad (5)$$

Integrating the differential Eq. (3b) we get

$$\varepsilon \int_{-1}^z u_2''(t)dt = \int_{-1}^z (f(t) - a(t)u_2'(t) + b(t)u_2(t) - c(t)u_1(t))dt. \quad (6)$$

Combining this with (4) and (5) we get $|\varepsilon u_2'(-1)| \leq 2\|f\| + \|\bar{u}\|$. Using Eq. (6) with $z = x$ we see that $|\varepsilon u_2'(x)| \leq \|f\| + C\|\bar{u}\| \quad \forall x \in \Omega_1$, and the required result for $k = 1$ follows.

Then, from the differential Eq. (3b), we have $\varepsilon u_2''(x) = f(x) - a(x)u_2'(x) + b(x)u_2(x) - c(x)u_1(x)$ and $(\varepsilon u_2''(x))' = (f(x) - a(x)u_2'(x) + b(x)u_2(x) - c(x)u_1(x))'$ from which we can obtain successively the required bounds on the second and third derivatives in Ω_1 .

Also we have $u_1' = u_2$, which gives the required bound for u_1 in Ω_1 . $\square$

Let us denote $\beta = b(0)/a'(0)$, β_l and β_s be fixed positive constants such that $\beta_l < 1 < \beta_s$. And also note that $\beta < 0$ always. The following lemma gives estimates for $\bar{u}$ and its derivatives in the interval Ω_2 which includes the turning point $x = 0$.

Lemma 3 Let $\bar{u} = (u_1, u_2)^T$ be the solution of (3a)–(3c), then for $k = 1, 2, 3$

$$\|u_j^{(k)}(x)\| \leq C, \text{ for } j = 1, 2$$

$\forall x \in \Omega_2$, where C depends on $\|a\|, \|a'\|, \|b\|, \|b'\|, \|c\|, \|c'\|, \|f\|, \|f'\|$ and β .

Proof We prove this lemma by adopting the technique as in Berger et al. [3] From the Mean Value Theorem and the assumptions in (2), we have

$$|a(x)| = |a(x) - a(0)| = |x| |a'(\zeta)| \geq |x| |a'(0)|/2 \geq \frac{|x|}{2\beta_s} \beta_0$$

The previous lemma gives the bound for $\bar{u}$ and its derivatives at $x = \pm 1/2$ where C depends on $\|a\|, \|a'\|, \|b\|, \|b'\|, \|c\|$ and $\|c'\|$ if Eqs. (3a)–(3b) is differentiated k times, one finds that the differential equation satisfied by $\bar{z}(x) = (\bar{u})^{(k)}(x)$ is

$$z_2(x) - z_1'(x) = 0 \quad (7)$$

$$\varepsilon z_2''(x) + a(x)z_2'(x) - [b(x) - k((a'(x) - c(x)))z_2(x) + c'(x)z_1(x) = g(x) \quad (8)$$

where g depends on $\bar{u}, \dots, (\bar{u})^{(k-1)}$ and on the k^{th} order derivatives of a, b, c and f . Applying Lemma 1 with $b(x)$ is replaced by $(b - k(a' - c))(x)$, $c(x)$ is replaced by $c'(x)$ using an inductive argument, we obtain the required result. $\square$

To obtain the sharper bounds of solution and its derivatives we decompose the solution $\bar{u}$ into regular and singular components as, $\bar{u} = \bar{v} + \bar{w}$, where $\bar{v} = (v_1, v_2)^T$ and $\bar{w} = (w_1, w_2)^T$. The regular component $\bar{v}$ can be written in the form of $\bar{v} = \bar{v}_0 + \varepsilon \bar{v}_1 + \varepsilon^2 \bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$ and $\bar{v}_2 = (v_{21}, v_{22})^T$ which are defined respectively to be the solutions of the problems:

$$\begin{aligned} \left(\begin{array}{cc} -\frac{d}{dx} & 0 \\ 0 & a(x)\frac{d}{dx} \end{array} \right) \bar{v}_0 + \left(\begin{array}{cc} 0 & 1 \\ c(x) & -b(x) \end{array} \right) \bar{v}_0 = \bar{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad v_{01}(-1) = u_1(-1), \quad v_{02}(-1) = A_1 \\ \left(\begin{array}{cc} -\frac{d}{dx} & 0 \\ 0 & a(x)\frac{d}{dx} \end{array} \right) \bar{v}_1 + \left(\begin{array}{cc} 0 & 1 \\ c(x) & -b(x) \end{array} \right) \bar{v}_1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix} \bar{v}_0, \quad v_{11}(-1) = 0, \quad v_{12}(-1) = 0 \\ \text{and } \bar{P}(\bar{v}_2) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix} \bar{v}_1, \quad v_{21}(-1) = 0, \quad v_{22}(-1) = 0, \\ v_{22}(1) = 0 \end{aligned}$$

Thus the regular component $\bar{v}$ is the solution of

$$\begin{aligned} \bar{P}(\bar{v}) &= \bar{f}, \\ v_1(-1) &= u_1(-1), \quad v_2(-1) = v_{02}(-1) + \varepsilon v_{12}(-1) + \varepsilon^2 v_{22}(-1), \\ v_2(1) &= v_{02}(1) + \varepsilon v_{12}(1) + \varepsilon^2 v_{22}(1). \end{aligned} \quad (9)$$

Then the singular component $\bar{w}$ is the solution of

$$\begin{aligned} \bar{P}(\bar{w}) &= \bar{0}, \\ w_1(-1) &= 0, \quad w_2(-1) = u_1(-1) - v_2(-1), \quad w_2(1) = u_2(1) - v_2(1). \end{aligned} \quad (10)$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{u}$.

Lemma 4 *The smooth component v and singular component w and their derivatives satisfy the bounds for $k=0,1,2,3$*

$$\begin{aligned} ||v_j^{(k)}(x)|| &\leq \begin{cases} C(1 + \varepsilon^{3-k}), & \text{for } j = 1 \\ C(1 + \varepsilon^{2-k}), & \text{for } j = 2 \end{cases} \quad \forall x \in \Omega_1 \cup \Omega_3, \\ |w_j^{(k)}(x)| &\leq \begin{cases} C\varepsilon^{-(k-1)}e^{-\alpha(1+x)/\varepsilon}, & \text{for } j = 1 \\ C\varepsilon^{-k}e^{-\alpha(1+x)/\varepsilon}, & \text{for } j = 2 \end{cases} \quad \forall x \in \Omega_1 \text{ and} \\ |w_j^{(k)}(x)| &\leq \begin{cases} C\varepsilon^{-(k-1)}e^{-\alpha(1-x)/\varepsilon}, & \text{for } j = 1 \\ C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon}, & \text{for } j = 2 \end{cases} \quad \forall x \in \Omega_3. \end{aligned}$$

Proof Following the approach as found in ([6], p. 44), the required bounds on $\bar{v}$, $\bar{w}$ and its derivatives are obtained in the subdomain Ω_1 as follows. In a similar way one can prove an analogous result in the subdomain Ω_3 .

Since $\bar{v}_0(x)$ and $\bar{v}_1(x)$ are independent of ε and $\bar{v}_2(x)$ is the solution of a similar problem to that of (3b) and hence from Lemma 3, for $k = 0, 1, 2$ we have the bounds

$$|\bar{v}_2^{(k)}(x)| \leq \begin{cases} C(1 + \varepsilon^{3-k}), & \text{for } j = 1 \\ C(1 + \varepsilon^{2-k}), & \text{for } j = 2 \end{cases} \quad \forall x \in \Omega_1.$$

For $k = 3$, we have $(\varepsilon v_2''(x))' = (f(x) - a(x)v_2'(x) + b(x)v_2(x) - c(x)v_1(x))'$, which leads to the bound for third derivative of v_2 .

Let us consider the barrier functions

$$\bar{\psi}^\pm(x) = \begin{cases} C \left(\frac{(-\varepsilon/\alpha)e^{-\alpha(1+x)/\varepsilon} - xe^{-\alpha(1-\delta)/\varepsilon + \varepsilon/\alpha}}{1 - e^{-\alpha(1-\delta)/\varepsilon}} \right) \pm w_1(x) \\ C \left(\frac{e^{-\alpha(1+x)/\varepsilon} - e^{-\alpha(1-\delta)/\varepsilon}}{1 - e^{-\alpha(1-\delta)/\varepsilon}} \right) \pm w_2(x) \end{cases} \quad \forall x \in \Omega_1.$$

It is easy to see that $\Psi_1^\pm(-1) \geq 0$, $\Psi_2^\pm(-1) \geq 0$, $\Psi_2^\pm(-\delta) \geq 0$, $P_1\bar{\psi}^\pm(x) \leq 0$ and $P_2\bar{\psi}^\pm(x) \leq 0$. Then by maximum principle, we have $\bar{\psi}^\pm(x) \geq 0$ for all $x \in \Omega_1$, from which the required bound for $\bar{w}$ in Ω_1 is obtained.

Integrating by parts and using pointwise bounds for w_2 given in the above estimate we have,

$$\begin{aligned} \int_x^{-\delta} a(t)w_2'(t)dt &= [a(t)w_2(t)]_x^{-\delta} + \int_x^{-\delta} a'(t)w_2(t)dt \\ &\leq Ce^{-\alpha(1+x)/\varepsilon}, \quad \forall x \in \Omega_1 \end{aligned} \quad (11)$$

By Mean Value Theorem, there exist a point $z \in (-\delta - \varepsilon, -\delta)$ such that

$$\begin{aligned} w_2'(z) &= \frac{w_2(-\delta) - w_2(-\delta - \varepsilon)}{\varepsilon} \\ |w_2'(z)| &= \frac{|w_2(-\delta) - w_2(-\delta - \varepsilon)|}{|\varepsilon|} \\ |w_2'(z)| &\leq C\varepsilon^{-1}e^{-\alpha(1-\delta)/\varepsilon} \end{aligned} \quad (12)$$

Now integrating the differential Eq. (10) defining w_2 we have, as in Lemma 3,

$$\varepsilon w_2'(x) = \varepsilon w_2'(-1) + \int_x^{-\delta} (-a(t)w_2'(t)dt + b(t)w_2(t) - c(t)w_1(t))dt \quad (13)$$

Using (11) and (12) we get,

$$\begin{aligned} |\varepsilon w'_2(-\delta)| &= |-\varepsilon w'_2(z) + \int_x^{-\delta} (-a(t)w'_2(t) + b(t)w_2(t) - c(t)w_1(t))dt| \\ &\leq C e^{-\alpha(1-\delta)/\varepsilon} \end{aligned} \quad (14)$$

Using (11) and (14), it follows that $|\varepsilon w'_2(x)| \leq C e^{-\alpha(1+x)/\varepsilon}$ for all $x \in \Omega_1$, which is the required bound for $k = 1$. Then, from the differential Eq. (10), we have,

$$\begin{aligned} \varepsilon w''_2(x) &= f(x) - a(x)w'_2(x) + b(x)w_2(x) - c(x)w_1(x) \text{ and} \\ (\varepsilon w''_2(x))' &= (f(x) - a(x)w'_2(x) + b(x)w_2(x) - c(x)w_1(x))' \end{aligned}$$

from which we can obtain successively the required bounds on the second and third derivatives of w_2 . Also we have $w'_1 = w_2$, which gives the required bound for w_1 . $\square$

Theorem 2 *The smooth component v and singular component w and their derivatives satisfy the bounds for $k=0,1,2,3$*

$$\begin{aligned} \|v_j^{(k)}(x)\| &\leq \begin{cases} C(1 + \varepsilon^{3-k}), & \text{for } j = 1 \\ C(1 + \varepsilon^{2-k}), & \text{for } j = 2 \end{cases} \quad \text{and} \\ |w_j^{(k)}(x)| &\leq \begin{cases} C\varepsilon^{-(k-1)}(e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), & \text{for } j = 1 \\ C\varepsilon^{-k}(e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), & \text{for } j = 2 \end{cases} \quad \forall x \in \bar{\Omega} \end{aligned}$$

Proof Lemma 3 guarantees that the solution of the SPTPP (3a)–(3c) and its derivatives are smooth in the domain Ω_2 . Hence, the proof is an immediate consequence of the above estimates on $\bar{v}^{(k)}(x)$ and $\bar{w}^{(k)}(x)$. $\square$

Discrete Problem

Mesh Selection Strategy

In this section, the system (3a)–(3c) is discretized using classical finite difference scheme on piecewise uniform meshes (Shishkin mesh). Consider the classical upwind scheme on a piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$, $N \geq 4$ which is constructed by dividing the domain $\bar{\Omega}$ into three subintervals $\Omega_L = [-1, -1 + \tau]$, $\Omega_C = [-1 + \tau, 1 - \tau]$ and $\Omega_R = [1 - \tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$.

The transition parameter τ is chosen to be $\min\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\alpha}\}$.

Finite Difference Method for the Problem (3a)–(3c)

The domain $\bar{\Omega}_\varepsilon^N$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both Ω_L and Ω_R and a uniform mesh with $N/2$ elements in Ω_C . The resulting fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \dots, N$ such that for $x_i \in \bar{\Omega}_\varepsilon^N$,

$$\begin{aligned} P_1^N \bar{U}(x_i) &:= U_2(x_i) - D^- U_1(x_i) = 0, \quad i = 1(1)N \\ P_2^N \bar{U}(x_i) &:= \varepsilon \delta^2 U_2(x_i) + a(x_i) D^* U_2(x_i) - b(x_i) U_2(x_i) \\ &\quad + c(x_i) U_1(x_i) = f(x_i), \quad i = 1(1)N - 1 \\ U_1(x_0) &= u_1(x_0), \quad U_2(x_0) = u_2(x_0), \quad U_2(x_N) = u_2(x_N). \end{aligned} \quad (15)$$

$$\text{where } D^+U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}, \quad D^-U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}}, \quad \delta^2 U_j(x_i) = \frac{D^+U_j(x_i) - D^-U_j(x_i)}{(x_{i+1} - x_{i-1})/2} \text{ and } D^*U_j(x_i) = \begin{cases} D^+U_j(x_i) & \text{if } a(x_i) > 0 \\ D^-U_j(x_i) & \text{if } a(x_i) < 0 \end{cases}.$$

Numerical Solution Estimates

Analogous to the continuous results stated in Theorem 1 and Lemma 1 we prove the following results.

Theorem 3 For any mesh functions $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))^T$, assume that $\Psi_1(x_0) \geq 0$, $\Psi_2(x_0) \geq 0$, $\Psi_2(x_N) \geq 0$, $P_1^N(\bar{\Psi}(x_i)) \leq 0$, $\forall i = 1(1)N$ and $P_2^N(\bar{\Psi}(x_i)) \leq 0$ $\forall i = 1(1)N - 1$. Then $\bar{\Psi}(x_i) \geq \bar{0}$, $\forall x_i \in \bar{\Omega}_\varepsilon^N$.

Proof Define $\bar{s}(x_i) = (s_1(x_i), s_2(x_i))^T$ as

$$s_1(x_i) = 5 + 4x_i \text{ and } s_2(x_i) = 2 + x_i.$$

Then $\bar{s}(x_i) > \bar{0}$, for all $x_i \in \bar{\Omega}_\varepsilon^N$. Further we define

$$\xi = \max \left\{ \max_{x_i \in \bar{\Omega}_\varepsilon^N} \left(\frac{-\Psi_1}{s_1} \right) (x_i), \max_{x_i \in \bar{\Omega}_\varepsilon^N} \left(\frac{-\Psi_2}{s_2} \right) (x_i) \right\}$$

Assume that the theorem is not true. Then $\xi > 0$ and we have $(\bar{\Psi} + \xi \bar{s})(x_i) \geq 0$ for $x_i \in \bar{\Omega}_\varepsilon^N$. For some $i = k$, we may have either $(\Psi_1 + \xi s_1)(x_k) = 0$ (or) $(\Psi_2 + \xi s_2)(x_k) = 0$ or both.

Case 1 $(\Psi_1 + \xi s_1)(x_k) = 0$. Then

$$0 > P_1^N(\bar{\Psi} + \xi \bar{s})(x_k) = (\Psi_2 + \xi s_2)(x_k) - D^-(\Psi_1 + \xi s_1)(x_k) \geq 0,$$

which is a contradiction.

Case 2 $(\Psi_2 + \xi s_2)(x_k) = 0$. Then

$$\begin{aligned} 0 &\geq P_2^N(\bar{\Psi} + \xi \bar{s})(x_k) \\ &= \begin{cases} \varepsilon \delta^2(\Psi_2 + \xi s_2)(x_k) + a(x_k) D^+(\Psi_2 + \xi s_2)(x_k) \\ \quad - b(x_k)(\Psi_2 + \xi s_2)(x_k) + c(x_k)(\Psi_1 + \xi s_1)(x_k) & \text{if } a(x_k) > 0 \\ \varepsilon \delta^2(\Psi_2 + \xi s_2)(x_k) + a(x_k) D^-(\Psi_2 + \xi s_2)(x_k) \\ \quad - b(x_k)(\Psi_2 + \xi s_2)(x_k) + c(x_k)(\Psi_1 + \xi s_1)(x_k) & \text{if } a(x_k) < 0 \end{cases} \\ &> 0. \end{aligned}$$

which is a contradiction.

Hence $\bar{\Psi}(x_i) \geq \bar{0}$, $\forall x_i \in \bar{\Omega}_\varepsilon^N$. $\square$

Lemma 5 Consider the scheme (15). If $\bar{z}(x_i) = (z_1(x_i), z_2(x_i))^T$ is any mesh function then, for all $x_i \in \bar{\Omega}_\varepsilon^N$,

$$|z_j(x_i)| \leq C \max \left\{ |z_1(x_0)|, \max \{|z_2(x_0)|, |z_2(x_N)|\} \right\},$$

$$\max_{1 \leq i \leq N} |P_1^N \bar{z}(x_i)|, \max_{1 \leq i \leq N-1} |P_2^N \bar{z}(x_i)| \Big\}, j = 1, 2.$$

Proof

$$\text{Let } C_1 = C \max \left\{ |z_1(x_0)|, \max \{|z_2(x_0)|, |z_2(x_N)|\}, \right. \\ \left. \max_{1 \leq i \leq N} |P_1^N \bar{z}(x_i)|, \max_{1 \leq i \leq N-1} |P_2^N \bar{z}(x_i)| \right\}.$$

Define the mesh functions $\bar{W}^\pm(x_i)$ as

$$\bar{W}^\pm(x_i) = \begin{pmatrix} C_1(5 + 4x_i) \\ C_1(2 + x_i) \end{pmatrix} \pm \bar{z}(x_i).$$

Then we have $\bar{W}^\pm(x_0) \geq \bar{0}$, $W_2^\pm(x_N) \geq 0$ and $\bar{P}^N \bar{W}^\pm(x_i) \leq \bar{0}$. By Theorem 3 we get the required result. $\square$

The discrete solution $\bar{U}(x_i)$ can be decomposed into the sum $\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$ where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$P_1^N \bar{V}(x_i) = 0, \quad i = 1, 2, \dots, N, \quad P_2^N \bar{V}(x_i) = f(x_i), \quad i = 1, 2, \dots, N-1, \\ \bar{V}(-1) = \bar{v}(-1), \quad V_2(1) = v_2(1). \\ P_1^N \bar{W}(x_i) = 0, \quad i = 1, 2, \dots, N, \quad P_2^N \bar{W}(x_i) = 0, \quad i = 1, 2, \dots, N-1, \\ \bar{W}(-1) = \bar{w}(-1), \quad W_2(1) = w_2(1).$$

The error in the numerical solution can be written in the form

$$(\bar{U} - \bar{u})(x_i) = (\bar{V} - \bar{v})(x_i) + (\bar{W} - \bar{w})(x_i).$$

Lemma 6 At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$, the error of the regular component satisfies the estimate

$$|(\bar{V} - \bar{v})(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$$

Proof This is obtained by using the following standard stability and consistency argument. We consider the local truncation error,

$$\bar{P}(\bar{V} - \bar{v}) = (\bar{P} - \bar{P}^N)\bar{v} = \begin{cases} -(\frac{d}{dx} - D^-)v_1 \\ \varepsilon(\frac{d^2}{dx^2} - \delta^2)v_2 + a(x_i)(\frac{d}{dx} - D^*)v_2. \end{cases} \quad (16)$$

Then by local truncation error estimates and Lemma 4, we obtain

$$|\bar{P}^N(\bar{V} - \bar{v})(x_i)| \leq \begin{pmatrix} -\frac{1}{2}(x_i - x_{i-1})|v_1^{(2)}| \\ \left\{ \begin{array}{l} \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v_2^{(3)}| + \frac{a(x_i)}{2}(x_{i+1} - x_i)|v_2^{(2)}| \quad \text{if } a(x_i) > 0 \\ \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v_2^{(3)}| + \frac{a(x_i)}{2}(x_i - x_{i-1})|v_2^{(2)}| \quad \text{if } a(x_i) < 0 \end{array} \right\} \end{pmatrix} \\ \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$$

Now applying Lemma 5 to the mesh functions $(\bar{V} - \bar{v})(x_i)$ we can easily obtain $|(\bar{V} - \bar{v})(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}$. $\square$

Lemma 7 At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$ the error of the singular component satisfies the estimate

$$|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-1} \ln N \\ CN^{-1} \ln N \end{pmatrix}$$

Proof We consider first the case $\tau = 1/2$ and so $\varepsilon^{-1} \leq C \ln N$ and $h = N^{-1}$. By classical argument and using Lemma 4, we obtain

$$|\bar{P}^N(\bar{W} - \bar{w})(x_i)| \leq \begin{cases} C\varepsilon^{-1}N^{-1} \\ C\varepsilon^{-2}N^{-1}e^{-\alpha(1+x_i)/\varepsilon} \text{ if } a(x_i) > 0 \\ C\varepsilon^{-2}N^{-1}e^{-\alpha(1-x_i)/\varepsilon} \text{ if } a(x_i) < 0 \end{cases} \quad (17)$$

Consider the mesh functions

$$\psi_1^\pm(x_i) = C\varepsilon^{-1}N^{-1}Y_1(x_i) \pm (\bar{W} - \bar{w})(x_i) \text{ and} \quad (18)$$

$$\psi_2^\pm(x_i) = \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha - \gamma)}\varepsilon^{-1}N^{-1}Y_2(x_i) \pm (\bar{W} - \bar{w})(x_i) \quad (19)$$

where γ is any constant satisfying $|a(x)| > \gamma > \alpha$, $Y_1(x_i) = 5 + 4x_i$ and $Y_2(x_i) =$

$$\begin{cases} 1 + (y_{N/2} - 1) \left(\frac{\lambda^{-i} - 1}{\lambda^{-N/2} - 1} \right) & \text{if } i \leq N/2 \\ 1 + (y_{N/2} - 1) \left(\frac{\lambda^i - N - 1}{\lambda^{-N/2} - 1} \right) & \text{if } i \geq N/2 \end{cases}, \text{ where } \lambda = 1 + \gamma h/\varepsilon.$$

Its easy to see that

$$D^+Y_2(x_i) \leq -\gamma/\varepsilon e^{-\gamma(1+x_i)/\varepsilon} \text{ if } i \leq N/2 \quad (20)$$

$$D^-Y_2(x_i) \geq \gamma/\varepsilon e^{-\gamma(1-x_i)/\varepsilon} \text{ if } i \geq N/2 \quad (21)$$

and so Y_2 decreases monotonically with $-1 \leq Y_2(x_i) \leq 0$ and Y_2 increases monotonically with $0 \leq Y_2(x_i) \leq 1$. We then have $\bar{\psi}^\pm(x_0) > 0$, $\psi_2^\pm(x_N) = 0$ and using (17), (20), (21) and the difference equations $(\varepsilon\delta^2 + \gamma D^+)Y_2 = 0$ for $i \leq N/2$, and $(\varepsilon\delta^2 - \gamma D^-)Y_2 = 0$ for $i \geq N/2$, we obtain $\bar{P}^N\bar{\psi}^\pm < 0$.

Then by the discrete maximum principle (Theorem 3) we conclude that $\psi_i^\pm \geq 0$ and so for all $x_i \in \bar{\Omega}_\varepsilon^N$, $|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-1} \ln N \\ CN^{-1} \ln N \end{pmatrix}$.

We now consider the case $\tau = \frac{2\varepsilon}{\alpha} \ln N$.

In this case the mesh is piecewise uniform with the mesh spacing $4\tau/N$ in the subintervals Ω_L , Ω_R and $2\tau/N$ in the subinterval Ω_C . We give separate proofs for coarse and fine mesh subintervals.

The subinterval Ω_C has no boundary layer, both W and w are small, and by the triangle inequality we have

$$|(\bar{W} - \bar{w})(x_i)| \leq |\bar{W}(x_i)| + |\bar{w}(x_i)|. \quad (22)$$

It suffices to bound $\bar{W}(x_i)$ and $\bar{w}(x_i)$ separately.

Now we consider the subinterval $[0, 1 - \tau]$ for our discussion since one can obtain a similar proof for the subinterval $[-1 + \tau, 0]$.

Using Lemma 4 we have

$$|\bar{w}(x_i)| \leq \begin{cases} CN^{-1} \\ CN^{-1} \end{cases}. \quad (23)$$

To obtain a similar bound for $\bar{W}(x_i)$ we introduce the mesh functions $Y_1(x_i) = 5 + 4x_i$ and $Y_2(x_i)$ which is the solution of the constant coefficient finite difference problem

$$\begin{aligned}(\varepsilon\delta^2 + \gamma D^+)Y_2 &= 0 \text{ for } i \leq N/2, \\(\varepsilon\delta^2 - \gamma D^-)Y_2 &= 0 \text{ for } i \geq N/2 \\Y_2(x_0) &= 1 \quad Y_2(x_N) = 1.\end{aligned}$$

From Lemma 7.5 of [10],

$$|\bar{w}(x_i)| \leq |\bar{W}(x_i)| \quad \forall \quad 0 \leq i \leq N.$$

Further the same lemma leads immediately to

$$|\bar{W}(x_i)| \leq \begin{cases} CN^{-1} \\ CN^{-1} \end{cases} \quad \forall \quad N/2 \leq i \leq 3N/4. \quad (24)$$

From the estimates obtained in (23) and (24) we have in the sub interval $[0, 1 - \tau]$

$$|(\bar{W} - \bar{w})(x_i)| \leq |\bar{W}(x_i)| + |\bar{w}(x_i)| \leq \begin{cases} CN^{-1} \\ CN^{-1} \end{cases} \quad \forall \quad N/2 \leq i \leq 3N/4. \quad (25)$$

It remains to prove the results for $x_i \in \Omega_L$ and $x_i \in \Omega_R$. Let $x_i \in \Omega_L$. For $i = 0$ there is nothing to prove. For $x_i \in \Omega_L$ the proof follows the same lines as for the case $\tau = 1/2$ except that we use the discrete maximum principle on Ω_L and the already established bound

$$|\bar{W}(x_{N/4})| \leq \begin{cases} CN^{-1} \\ CN^{-1} \end{cases}.$$

We have in this case,

$$|\bar{P}^N(\bar{W} - \bar{w})(x_i)| \leq \begin{cases} C\tau\varepsilon^{-1}N^{-1} \\ C\tau\varepsilon^{-2}N^{-1}e^{-\alpha(1+x_i)/\varepsilon} \end{cases} \quad (26)$$

For all $i, 0 \leq i \leq N/4$, we introduce the mesh functions

$$\psi_1^\pm(x_i) = C\tau N^{-1}Y_1(x_i) \pm (\bar{W} - \bar{w})(x_i) \text{ and} \quad (27)$$

$$\psi_2^\pm(x_i) = \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha - \gamma)}\tau\varepsilon^{-1}N^{-1}Y_2(x_i) + C'N^{-1} \pm (\bar{W} - \bar{w})(x_i) \quad (28)$$

where $Y_1(x_i) = 5 + 4x_i$, $Y_2(x_i) = 1 + (y_{N/4} - 1)\left(\frac{\lambda^{-i}-1}{\lambda^{-N/4}-1}\right)$ and γ is as before and for all $i, 0 \leq i \leq N/4$.

Its easy to see that

$$D^+Y_2(x_i) \leq -\gamma/\varepsilon e^{-\gamma(1+x_{i+1})/\varepsilon} < 0 \quad (29)$$

and so Y_2 decreases monotonically with $0 \leq i \leq N/4$ we then have $\psi_0^\pm > 0$, $\psi_{N/4}^\pm \geq 0$ and using (26) and (29) and $(\varepsilon\delta^2 + \gamma D^+)Y_2 = 0$, we obtain $\bar{P}^N\psi_i^\pm \leq 0$. Then by the discrete maximum principle we conclude that $\psi_i^\pm \geq 0$, $\forall x_i \in \Omega_L$. Similarly the proof follows for $x_i \in \Omega_R$ and so for all $x_i \in \bar{\Omega}_\varepsilon^N$, $|(\bar{W} - \bar{w})(x_i)| \leq \left(\frac{CN^{-1} \ln N}{CN^{-1} \ln N}\right)$ as required. $\square$

Theorem 4 Let $\bar{u}(x) = (u_1(x), u_2(x))^T$, for all $x \in \bar{\Omega}$ be the solution of (3a)–(3c) and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$, for all $x_i \in \bar{\Omega}_\varepsilon^N$ be the numerical solution of problem (15). Then we have

$$\sup_{0 < \varepsilon \leq 1} \|U_1 - u_1\|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1} \ln N \text{ and } \sup_{0 < \varepsilon \leq 1} \|U_2 - u_2\|_{\tilde{\Omega}_\varepsilon^N} \leq CN^{-1} \ln N$$

Proof Proof follows immediately, if one applies the Lemma 6 and Lemma 7 to $\bar{U} - \bar{u} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$. $\square$

Numerical Results

In this section, two examples are given to illustrate the numerical method discussed in this paper. We use the double mesh principle to estimate the error and compute the rate of convergence in our computed solution.

Define the double mesh differences to be

$$D_{\varepsilon,j}^N = \left\{ \max_{x_i \in \tilde{\Omega}_\varepsilon^N} |U_j^N(x_i) - U_j^{2N}(x_i)| \right\}, \text{ for } j = 1, 2 \text{ and } D_j^N = \max_\varepsilon D_{\varepsilon,j}^N$$

where $U_j^N(x_i)$ and $U_j^{2N}(x_i)$ respectively, denote the numerical solution obtained using N and $2N$ mesh intervals. Further, we calculate the parameter robust order of convergence as

$$p_j^N = \log_2 \left(\frac{D_j^N}{D_j^{2N}} \right), \text{ for } j = 1, 2.$$

The following examples have a turning point at $x = 0$ (Examples 1, 2).

Example 1 Consider the following singularly perturbed turning point problem

$$\begin{aligned} \varepsilon u'''(x) - 5xu''(x) - 4u'(x) + 2u(x) &= -e^x, \quad x \in (-1, 1) \\ u(-1) = 1, \quad u'(-1) = 1, \quad u'(1) &= -2. \end{aligned}$$

The computed maximum pointwise errors D_1^N, D_2^N and the computed rate of convergence p_1^N, p_2^N for the Example 1 are presented in Table 1. In Fig. 1, the graph of the solution components U_1 and U_2 of Example 1 is given. The loglog plot of the maximum pointwise error for the solution components U_1 and U_2 of the Example 1 are given in Fig. 2.

Example 2 Consider the following singularly perturbed turning point problem

$$\begin{aligned} \varepsilon u'''(x) - 5xu''(x) - (x+4)u'(x) + (2+x)u(x) &= -e^x, \quad x \in (-1, 1) \\ u(-1) = 1, \quad u'(-1) = 1, \quad u'(1) &= 1. \end{aligned}$$

Table 1 Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components U_1 and U_2 respectively for Example 1

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	1.0604e-2	4.7925e-3	2.2695e-3	1.0922e-3	4.6046e-4
p_1^N	1.1458	1.0784	1.0551	1.2461	–
D_2^N	4.8789e-2	3.2371e-2	1.9258e-2	1.1560e-2	6.5442e-3
p_2^N	5.9186e-1	7.4925e-1	7.3635e-1	8.2081e-1	–

Fig. 1 Solution graph of Example 1 for $\varepsilon = 2^{-4}$ and $N = 2^{10}$

Fig. 2 Maximum pointwise errors as a function of N and ε for the solution u_1 and u_2 for Example 1

Table 2 Values of D_1^N , p_1^N and D_2^N , p_2^N for the solution components U_1 and U_2 respectively for Example 2

	Number of mesh points N				
	64	128	256	512	1024
D_1^N	5.2573e-3	2.8663e-3	1.6420e-3	9.3378e-4	5.2267e-4
p_1^N	8.7513e-1	8.0372e-1	8.1433e-1	8.3717e-1	—
D_2^N	4.5955e-2	3.1217e-2	1.8770e-2	1.1414e-2	6.5998e-3
p_2^N	5.5789e-1	7.3395e-1	7.1755e-1	7.9035e-1	—

The computed maximum pointwise errors D_1^N , D_2^N and the computed rate of convergence p_1^N , p_2^N for the Example 2 are presented in Table 2. In Fig. 3, the graph of the solution components U_1 and U_2 of Example 2 is given. The loglog plot of the maximum pointwise errors for the solution components U_1 and U_2 of the Example 2 are given in Fig. 4.

Fig. 3 Solution graph of Example 2 for $\varepsilon = 2^{-4}$ and $N = 2^{10}$

Fig. 4 Maximum pointwise errors as a function of N and ε for the solution u_1 and u_2 for Example 2

Acknowledgments The first author wishes to thank Department of Science and Technology, Government of India, for their financial support under the DST - Purse Scheme.

References

1. Abrahamsson, L.R.: A priori estimates for solutions of singular perturbations with a turning point. *Stud. Appl. Math.* **56**, 51–69 (1977)
2. Leung, A.: A third order linear differential equation on the real line with two turning points. *J. Differ. Equ.* **29**, 304–328 (1978)
3. Berger, A.E., Han, H., Kellogg, R.B.: A priori estimates and analysis of a numerical method for a turning point problem. *Math. Comput.* **42**, 465–492 (1984)
4. Jian-ping, C., Zong-chi, L.: Existence of solutions of a singularly perturbed boundary value problem for the third order semi-linear differential equations with a turning point. *Appl. Math. Mech.* **14**(12), 1089–1093 (1993)

5. Doolan, E.P., Miller, J.J.H., Schildres, W.H.A.: Uniform Uumerical Uethods for Uroblems with Unital and Uoundary Uayers. Boole Press, Dublin (1980)
6. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O’Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman and Hall/CRC Press, Boca Raton (2000)
7. Farrell, P.A.: Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem. *SIAM J. Numer. Anal.* **25**(3), 618–643 (1988)
8. Gattlieb, H.P.W.: Question #38. What is the simplest jerk function that gives chaos? *Am. J. Phys.* **64**, 525 (1996)
9. Mo, J., Wen, Z.: A class of boundary value problems for third order differential equation with a turning point. *Appl. Math. Mech.* **31**(8), 1027–1032 (2010)
10. Miller, J.J.H., O’Riordan, E., Shishkin, G.I.: Fitted numerical methods for singularly perturbed problems. Error estimates in the maximum norm for linear problems in one and two dimensions, Revised Edition, world scientific publishing Co., Pvt. Ltd., Singapore (2012)
11. Kadalbajoo, M.K., Patidar, K.C.: Variable mesh spline approximation method for solving singularly perturbed turning point problems having boundary layer(s). *Comput. Math. Appl.* **42**, 1439–1453 (2001)
12. O’Malley, R.E.: Introduction to Singular Perturbations. Academic Press, New York (1974)
13. Natesan, S., Ramanujam, N.: A computational method for solving singularly perturbed turning point problems exhibiting twin boundary layers. *Appl. Math. Comp.* **93**, 259–275 (1998)
14. Natesan, S., Ramanujam, N.: Initial-value technique for singularly perturbed turning point problems exhibiting twin boundary layers. *J. Optim. Theo. Appl.* **99**(1), 37–52 (1998)
15. Natesan, S., Jayakumar, J., Vigo-Aguiar, J.: Parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers. *J. Comp. Appl. Math.* **158**, 121–134 (2003)
16. Roos, H.G., Stynes, M., Tobiska, L.: Numerical Methods for Singularly Perturbed Differential Equations Convection-Diffusion and Flow Problems. Springer, Berlin (2006)
17. Steven, S.H.: Jerk: the time rate of change of acceleration. *Am. J. Phys.* **46**, 1090 (1978)
18. Sharma, K.K., Rai, P., Patidar, K.C.: A review on singularly perturbed differential equations with turning points and interior layers. *Appl. Math. Comput.* **219**, 10575–10609 (2013)
19. Wasow, W.: Linear Turning Point Theory. Springer, New York (1984)
20. Watts, A.M.: A singular perturbation problem with a turning point. *Bull. Aust. Math. Soc.* **5**, 61–73 (1971)
21. Natividad, M.C., Stynes, M.: Richardson extrapolation for a convection-diffusion problem using a shishkin mesh. *Appl. Numer. Math.* **45**, 315–329 (2003)
22. Natesan, S., Ramanujam, N.: A “Booster method” for singular perturbation problems arising in chemical reactor theory by incorporation of asymptotic approximations. *Appl. Math. Comput.* **100**(1), 27–48 (1999)
23. Natesan, S., Ramanujam, N.: ‘Shooting Method’ for the solution of singularly perturbed two-point boundary value problems having less severe boundary layers. *Appl. Math. Comput.* **133**(2–3), 623–641 (2002)
24. Natesan, S., Ramanujam, N.: Improvement of numerical solution of self adjoint singular perturbation problems by incorporation of asymptotic approximations. *Appl. Math. Comput.* **98**(2–3), 119–137 (1999)
25. Mohapatra, J., Natesan, S.: Uniform convergence analysis of finite differences scheme for singularly perturbed delay differential equation on an adaptively generated grid. *Numer. Math.: Theo. Methods Appl.* **3**(1), 1–22 (2010)
26. Mohapatra, J., Natesan, S.: Parameter uniform numerical methods for singularly perturbed mixed boundary value problems using grid equidistribution. *J. Appl. Math. Comput.* **37**, 247–265 (2011)
27. Ramos, H.: A non-standard explicit integration scheme for initial-value problems. *Appl. Math. Comput.* **189**, 710–718 (2007)
28. Vigo-Aguiar, J., Natesan, S.: An efficient numerical method for singular perturbation problems. *J. Comput. Appl. Math.* **192**, 132–141 (2006)
29. Natesan, S., Vigo-Aguiar, J., Ramanujam, N.: A Numerical algorithm for singular perturbation problems exhibiting weak boundary layers. *Comput. Math. Appl.* **45**, 469–479 (2003)
30. Higinio Ramos, J., Vigo-Aguiar, S., Natesan, R., Garca-Rubio, M., Queiruga, A.: Numerical solution of nonlinear singularly perturbed problems on nonuniform meshes by using a non-standard algorithm. *J. Math. Chem.* **48**, 38–54 (2010)