Approximate Analytical Solution of Two Coupled Time Fractional Nonlinear Schrödinger Equations

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Abstract In this article, we consider the well known nonlinear Schrödinger equation (NLS) with fractional time derivative and derive its approximate analytical solution using the homotopy analysis method (HAM). We also applied HAM to two coupled time fractional NLS and constructed its approximate solution. The question of convergence of the obtained solution is discussed. The obtained approximate analytical periodic wave solution, solitary wave solution and the effect of time fractional order α are shown graphically.

Keywords Homotopy analysis method · Caputo fractional derivative · Riemann–Liouville fractional integral · Time fractional nonlinear partial differential equations

Mathematics Subject Classification 26A33 · 34A08 · 34K37

Introduction

The study of fractional differential equations has drawn much attention, both from the mathematical and physical point of view, by researchers in nonlinear phenomena in recent years. The primary reason for interest is that the exact description of most of the phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science have been governed by nonlinear equations involving fractional order derivatives [1–6]. Also in reality a physical phenomenon may depend not only the time instant but also the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [1–4]. Nonlinear differential equations exhibiting a wide variety of

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interesting features such as periodic solution, chaos, strange attractors, fractals, etc are not exactly solvable in general and thus deriving their explicit or closed form solutions are of fundamental importance. Considerable number of analytical techniques have been developed to deal with nonlinear differential equations during the past several decades. However, the study of fractional differential equations has been handicapped due to the absence of well defined analytic techniques to deal with them. Recently, we have shown that the Lie transformation group theory provides an useful tool to analyze time fractional nonlinear partial differential equations [7]. We have constructed exact solutions to certain time fractional nonlinear partial differential and differential-difference equations using invariant subspaces [8].

One of the recently developed techniques is the HAM [9–13], which is a combination of the classical perturbation technique and homotopy concept as used in topology, to derive both exact and approximate solutions for linear and nonlinear problems. Liao [9] described the early form of the HAM in his PhD dissertation. Liao [10] introduced a nonzero auxiliary parameter c_0 ¹ in the zeroth order deformation equation to adjust and control the convergence region and rate of homotopy series solution. Liao [11] introduced a nonzero auxiliary function $H(x, t)$ to further generalize the zeroth order deformation equation. One of the advantages of HAM is that it does not need linearization, weak nonlinearity assumptions, discretization, small parameter and linear term in a differential equation. Compared to other numerical and non-numerical techniques, HAM provides a convenient way to control and adjust the convergence region of the solution series. The application of HAM has been demonstrated to a variety of problems arising from science and engineering for all types of boundary and initial conditions governed by nonlinear equations involving both integer and fractional order derivatives [6, 9–21]. Recently, the HAM is extended to solve many nonlinear fractional differential equations (see [6, 17–23] and the references there).

However the application of HAM to derive exact or approximate solution of nonlinear fractional partial differential equations have not been illustrated widely. We would like to mention that the NLS equation and two coupled NLS equations admit rich mathematical structures [24–27]. Thus it is of interest to investigate whether the fractional analog of the above well known equations retain its mathematical structures or not. It is appropriate to mention that Svenkeson et al. [28] have demonstrated a general connection between fractional calculus and the process of subordination. The significance of this connection is that a single fractional trajectory obtained as the solution to a fractional equation of motion can be interpreted as an average over an ensemble of stochastically evolving ordinary trajectories [29]. The subordination implies the existence of two different notions of time [30]. One is the operational time τ , which is the internal time of a single unit, with a unit generating the ordinary dynamics of a non-fractional system. The other notion is experimental time t ; the time as measured by the clock of an external observer.

HAM: Partial Differential Equations

Consider a scalar PDE with two independent variables having the form

$$\mathcal{N}(u(x, t)) = 0, \quad (1)$$

¹ Liao originally used the symbol $\hbar$ to denote the nonzero auxiliary parameter. But it is well known as Planck's constant. To avoid misunderstanding Liao [15] replaced $\hbar$ by c_0 .

where $\mathcal{N}$ is a nonlinear operator. Let $u_0(x, t)$ denote an initial guess of the exact solution $u(x, t)$ satisfying the initial conditions, $c_0 \neq 0$ is an auxiliary parameter. Let $\mathcal{L}$ be an auxiliary linear operator satisfying

$$\mathcal{L}(f(x, t)) = 0, \quad \text{when } f(x, t) = 0. \quad (2)$$

Let $H(x, t)$ be a non-zero auxiliary function. Then using $q \in [0, 1]$ as the embedding parameter, we construct the homotopy

$$\mathcal{H}(\phi, u_0, H, c_0, q) = (1 - q)\mathcal{L}[\phi(x, t; q) - u_0(x, t)] - qc_0 H(x, t)\mathcal{N}(\phi(x, t; q)). \quad (3)$$

The so-called zeroth order deformation equation can be obtained by enforcing the homotopy Eq.(3) to zero, that is,

$$(1 - q)\mathcal{L}[\phi(x, t; q) - u_0(x, t)] = qc_0 H(x, t)\mathcal{N}(\phi(x, t; q)), \quad (4)$$

where $\phi(x, t; q)$ is an unknown function. When $q = 0$, Eq. (4) becomes

$$\mathcal{L}[\phi(x, t; 0) - u_0(x, t)] = 0.$$

Making use of the property Eq. (2) and $u_0(x, t)$ satisfies the initial conditions, we have

$$\phi(x, t; 0) = u_0(x, t),$$

when $q = 1$ since $c_0 \neq 0$, $H \neq 0$, Eq. (4) is equivalent to

$$\mathcal{N}(\phi(x, t; 1)) = 0,$$

which is exactly the same as Eq. (1) provided

$$\phi(x, t; 1) = u(x, t).$$

Thus, it is clear that as q increases from 0 to 1, $\phi(x, t; q)$ varies continuously from the initial guess $u_0(x, t)$ to the exact solution $u(x, t)$ of Eq. (1). In Topology, this is called a *deformation* and $\mathcal{L}[\phi] - \mathcal{L}[u_0]$ and $\mathcal{N}[\phi]$ are said to be homotopic. Our basic assumption is that the solution of Eq. (3) when equated to zero can be expressed as a power series in q . Expanding $\phi(x, t; q)$ as a Taylor series about $q = 0$, we write

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (5)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}. \quad (6)$$

Note that one needs to assume c_0 , $H(x, t)$, $u_0(x, t)$ and $\mathcal{L}$ such that the series Eq. (5) converges at $q = 1$. Then

$$\phi(x, t; 1) = u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (7)$$

represents a solution of Eq. (1) [12]. Define the vector $\mathbf{u}_n(x, t) = (u_0, u_1, \dots, u_n)$. Differentiating Eq. (4) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, the components $u_0(x, t)$, $u_1(x, t)$, $u_2(x, t)$, $\dots$ of Eq. (7) can be determined by solving the so-called m th-order deformation equation

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 H(x, t)R_m(\mathbf{u}_{m-1}(x, t)), \quad (8)$$

where

$$R_m(\mathbf{u}_{\mathbf{m}-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}(\phi(x, t; q))}{\partial q^{m-1}} \right|_{q=0}, \quad (9)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1; \\ 1, & m > 1. \end{cases} \quad (10)$$

Theorem 1 (Convergence Theorem) [10] *If the series solution defined in Eq. (7) is convergent, then it converges to an exact solution of the nonlinear problem Eq. (1).*

It is appropriate to recall that an optimal value of the convergence control parameter c_0 can be found by means of the exact square residual error integrated in the whole region of interest Ω , at the order of approximation M , that is,

$$Res(c_0) = \int_{\Omega} \left[\mathcal{N} \left(\sum_{k=0}^M u_k(x, t) \right) \right]^2 d\Omega. \quad (11)$$

If $Res(c_0)$ in Eq. (11) decreases to zero, then the corresponding series solution Eq. (7) converges faster. So, at the given order of approximation M , the corresponding optimal value of the convergence control parameter c_0 is given by the minimum of $Res(c_0)$, corresponding to a nonlinear algebraic equation of the form

$$\frac{dRes(c_0)}{dc_0} = 0. \quad (12)$$

However, the computation of the exact square residual error defined above needs too much computation time even if the order of approximation is not very high and thus is often useless in practice. To reduce the computational time, Liao [10] has suggested the following:

- Evaluate the expressions of the derivatives of the unknown function $u(x, t)$ with respect to x or t for the fixed values of x and t .
- Plot these unknown quantities such as

$$\left. \frac{\partial u}{\partial t} \right|_{(x_0, t_0)}, \quad \left. \frac{\partial^2 u}{\partial t^2} \right|_{(x_0, t_0)}, \quad \left. \frac{\partial^3 u}{\partial t^3} \right|_{(x_0, t_0)}$$

versus c_0 .

- These curves contain a horizontal line segment the so called c_0 -curve denotes the valid region of c_0 which guarantees the convergence of the solution series. For more details about convergence, see [10, 13, 16].

Before embarking into the details of HAM to time fractional nonlinear PDEs, we would like to recall certain basic definitions and properties of fractional operators which are required for the remaining part of the article.

Definition 1 The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of the function, $f(t) \in L^1([a, b], \mathbb{R}_+)$ denoted by ${}_a J_t^\alpha$, is defined by [1–4]

$${}_a J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t > a, \\ {}_a J_t^0 f(t) = f(t), \quad (13)$$

where Γ is the gamma function.

Definition 2 The Caputo fractional differential operator of order $\alpha > 0$ of the function $f(t) \in L^1([a, b], \mathbb{R}_+)$, denoted by ${}_a D_t^\alpha$, is defined by [1–4]

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} \frac{d^n f(s)}{ds^n} ds, \quad t > a,$$

$${}_a D_t^0 f(t) = f(t), \quad (14)$$

where $n = [\alpha] + 1$, the Caputo derivative of real order introduces a memory formalism as it is an integro-differential operator defined by the convolution of the ordinary derivative with a power law kernel.

For simplicity we denote the operators ${}_0 D_t^\alpha f(t)$ and ${}_0 J_t^\alpha f(t)$ respectively as $D_t^\alpha f(t)$ and $J_t^\alpha f(t)$. Note that the above mentioned operators satisfy the following properties for the suitable functions $f(t)$ and $g(t)$

$$J_t^\alpha (f(t) + g(t)) = J_t^\alpha f(t) + J_t^\alpha g(t), \quad (15)$$

$$J_t^\alpha (D_t^\alpha f(t)) = f(t) - \sum_{r=0}^{n-1} \frac{f^{(r)}(0)}{r!} t^r, \quad n - 1 < \alpha \leq n, \quad (16)$$

$$D_t^\alpha (J_t^\alpha f(t)) = f(t). \quad (17)$$

In this article, the Caputo's definition of fractional differentiation is used.

HAM: Time Fractional Nonlinear Schrödinger Equation

Let us consider a time fractional NLS equation

$$i \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \quad 0 < \alpha \leq 1, \quad i = \sqrt{-1}, \quad (18)$$

where $u(x, t)$ is a complex valued function and the time fractional derivative is in the Caputo sense. Now we look for analytic solution expressed as an infinite series having the form

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t).$$

Let us assume that the initial guess $u_0(x, t)$ and the auxiliary linear operator be

$$u_0(x, t) = u(x, 0) = f(x), \quad (19)$$

$$\mathcal{L}[\phi(x, t; q)] = D_t^\alpha \phi = \frac{\partial^\alpha \phi}{\partial t^\alpha}, \quad (20)$$

From Eq. (18), it is clear that the nonlinear operator is

$$\mathcal{N}(\phi) = \frac{\partial^\alpha \phi}{\partial t^\alpha} - i \left(\frac{\partial^2 \phi}{\partial x^2} + 2|\phi|^2 \phi \right). \quad (21)$$

Substituting Eq. (21) in Eq. (9) along with Eq. (6), we obtain

$$R_m(\mathbf{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} u_j \bar{u}_k u_{m-1-j-k} \right), \quad (22)$$

where the $\bar{u}_k$ is the complex conjugate of u_k . From Eqs. (8) and (20), we write

$$D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 H(x, t) R_m(\mathbf{u}_{m-1}(x, t)). \quad (23)$$

Applying the Riemann–Liouville fractional integral operator J_t^α on both sides, we get

$$J_t^\alpha D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 J_t^\alpha H(x, t) R_m(\mathbf{u}_{m-1}(x, t)). \quad (24)$$

For clarity of presentation we choose $H(x, t) = 1$, and substituting the expression $R_m(\mathbf{u}_{m-1}(x, t))$ given in Eq. (22) into RHS of Eq. (24) we have

$$\begin{aligned} J_t^\alpha D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] \\ = c_0 J_t^\alpha \left[\frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} u_j \bar{u}_k u_{m-1-j-k} \right) \right]. \end{aligned} \quad (25)$$

$$\begin{aligned} J_t^\alpha D_t^\alpha [u_m(x, t)] &= J_t^\alpha D_t^\alpha [\chi_m u_{m-1}(x, t) + c_0 u_{m-1}(x, t)] \\ &\quad - i c_0 J_t^\alpha \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} u_j \bar{u}_k u_{m-1-j-k} \right). \end{aligned} \quad (26)$$

Using the property Eq. (16) along with $u_m(x, 0) = 0$, $m = 1, 2, \dots$ which yields

$$\begin{aligned} u_m(x, t) &= (\chi_m + c_0)(u_{m-1}(x, t) - u_{m-1}(x, 0)) \\ &\quad - i c_0 J_t^\alpha \left[\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} u_j \bar{u}_k u_{m-1-j-k} \right]. \end{aligned} \quad (27)$$

By making use of Eqs. (10) and (27), we get the components of the infinite series solution Eq. (7) as follows (hereafter u_j denotes $u_j(x, t)$)

$$u_0 = f(x), \quad (28)$$

$$u_1 = -i c_0 J_t^\alpha \left[\frac{\partial^2 u_0}{\partial x^2} + 2(u_0)^2 \bar{u}_0 \right], \quad (29)$$

$$u_2 = (1 + c_0)u_1 - i c_0 J_t^\alpha \left[\frac{\partial^2 u_1}{\partial x^2} + 2u_0(\bar{u}_1 u_0 + 2\bar{u}_0 u_1) \right], \quad (30)$$

$$u_3 = (1 + c_0)u_2 - i c_0 J_t^\alpha \left[\frac{\partial^2 u_2}{\partial x^2} + 2(\bar{u}_2 u_0^2 + 2\bar{u}_1 u_0 u_1 + \bar{u}_0(u_1^2 + 2u_0 u_2)) \right], \quad (31)$$

etc.

Periodic Wave Solution

We look for periodic wave solution $u(x, t)$ of Eq. (18) satisfying the initial condition

$$u(x, 0) = a e^{ibx}, \quad (32)$$

where a and b are real constants. Following the procedure outlined in the previous section, we computed the components $u_m(x, t)$ of the infinite series solution Eq. (7) or Eqs. (28)–(31) as follows

$$u_0 = ae^{ibx}, \quad (33)$$

$$u_1 = ae^{ibx} \frac{ic_0 A t^\alpha}{\Gamma(1+\alpha)}, \quad (34)$$

$$\begin{aligned} u_2 &= (1+c_0)u_1 + ae^{ibx} \frac{i^2 c_0^2 A^2 t^{2\alpha}}{\Gamma(1+2\alpha)}, \\ &= ae^{ibx} \left[\frac{c_0(1+c_0) i A t^\alpha}{\Gamma(1+\alpha)} + \frac{i^2 c_0^2 A^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right], \end{aligned} \quad (35)$$

$$\begin{aligned} u_3 &= (1+c_0)u_2 \\ &+ ae^{ibx} \left[\frac{(1+c_0)i^2 c_0^2 A^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{i^3 c_0^3 A^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2a^2 i^3 c_0^3 A^2 t^{3\alpha}}{\Gamma(1+3\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - 2 \right) \right], \\ &= ae^{ibx} \left[\frac{(1+c_0)^2 i c_0 A t^\alpha}{\Gamma(1+\alpha)} + \frac{2(1+c_0)i^2 c_0^2 A^2 t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{i^3 c_0^3 A^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right. \\ &\quad \left. + \frac{2a^2 i^3 c_0^3 A^2 t^{3\alpha}}{\Gamma(1+3\alpha)} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - 2 \right) \right], \end{aligned} \quad (36)$$

where $A = b^2 - 2a^2$. In a similar manner we compute $u_4, u_5, \dots$ and so the infinite series solution Eq. (7) becomes

$$\begin{aligned} u(x, t) &= ae^{ibx} \left[1 + \frac{(c_0^2 + 3c_0 + 3)ic_0 A t^\alpha}{\Gamma(1+\alpha)} + \frac{(2c_0 + 3)i^2 c_0^2 A^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right. \\ &\quad \left. + \left\{ 1 + \frac{2a^2}{A} \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - 2 \right) \right\} \frac{i^3 c_0^3 A^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots \right], \end{aligned} \quad (37)$$

which may not lead to closed form due to presence of the terms like $\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2} - 2 \right)$. If we choose $c_0 = -1$ and $\alpha \rightarrow 1$ then Eqs. (33)–(36) become

$$u_0 = ae^{ibx},$$

$$u_1 = ae^{ibx} (-iA) \frac{t}{1!},$$

$$u_2 = ae^{ibx} (iA)^2 \frac{t^2}{2!},$$

$$u_3 = ae^{ibx} (-iA)^3 \frac{t^3}{3!},$$

and so on. Using Taylor series expansion, the infinite series solution Eqs. (7) or (37) yields the exact periodic wave solution

$$u(x, t) = ae^{i(bx - At)} = ae^{i(bx - (b^2 - 2a^2)t)}. \quad (38)$$

We now investigate the influence of the auxiliary parameter c_0 on the convergence of the solution series Eq. (37) which will be examined by plotting the so called c_0 curve. The c_0 curve for the 8th order approximate solution of Eq. (37) at $x = 0$ and $t = 0.25$ when

$\alpha = 0.9$ is given in Fig. 1a. From the Fig. 1a, it is observed that the valid region for c_0 is $-2 \leq c_0 < 1$. Thus, if we choose the appropriate value for c_0 in this interval at which the approximate solution Eq. (37) converges. Fig. 1b, c represents the absolute value of the approximate and exact periodic wave solutions with $a = 0.50$, $b = 1.0$, $c_0 = -0.1$, $\alpha = 0.9$ and $a = 0.50$, $b = 1.0$, $\alpha = 1.0$ respectively.

Solitary Wave Solution

Next we look for a solitary wave solution $u(x, t)$ of Eq. (18) satisfying the initial condition

$$u(x, 0) = ae^{ibx} \operatorname{sech}(ax). \quad (39)$$

Making use of Eqs. (28)–(31) yields

$$u_0 = ae^{ibx} \operatorname{sech}(ax), \quad (40)$$

$$u_1 = \frac{-ae^{ibx} \operatorname{sech}(ax)(iB + 2abtanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)}, \quad (41)$$

$$u_2 = ae^{ibx} \operatorname{sech}(ax) \left[\frac{-i(1 + c_0)(B - 2iab \tanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right], \quad (42)$$

etc., where $B = a^2 - b^2$. Therefore, from Eq. (7), we have

$$u = ae^{ibx} \operatorname{sech}(ax) \left[1 - \frac{(2 + c_0)(iB + 2abtanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \dots \right]. \quad (43)$$

If we choose $c_0 = -1$ and $\alpha \rightarrow 1$ then Eqs. (40)–(42) become

$$u_0 = ae^{ibx} \operatorname{sech}(ax),$$

$$u_1 = ae^{ibx} \operatorname{sech}(ax)[iB + 2abtanh(ax)] \frac{t}{1!},$$

$$u_2 = -ae^{ibx} \operatorname{sech}(ax) [(a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax)] \frac{t^2}{2!},$$

and so on. Using Taylor series expansion, the infinite series solution Eqs. (7) or (43) yields the exact solitary wave solution

$$u(x, t) = ae^{i(bx+Bt)} \operatorname{sech}(a(x - 2bt)) = ae^{i(bx+(a^2-b^2)t)} \operatorname{sech}(a(x - 2bt)), \quad (44)$$

which is the same solution obtained in [26] by using Hirota bilinearisation method. The c_0 curve for the 6th order approximate solution of Eq. (43) at $x = 0$ and $t = 0.25$ when $\alpha = 0.9$ is given in Fig. 2a. From Fig. 2a, it is observed that the valid region for c_0 is $-1.5 < c_0 < 1$. Thus if we choose the appropriate value for c_0 in this interval at which the approximate solution Eq. (43) converges. Fig. 2b, c represent the absolute value of the approximate and exact solitary wave solutions with $a = 0.50$, $b = 1.0$, $c_0 = -0.1$, $\alpha = 0.9$ and $a = 0.50$, $b = 1.0$, $\alpha = 1.0$ respectively.

Fig. 1 **a** The c_0 curve for the periodic wave solution $|u|$ of Eq. (18) with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **b** The approximate periodic wave solution $|u|$ of Eq. (18) with $a = 0.50$, $b = 1.0$, $c_0 = -0.1$, $\alpha = 0.9$. **c** The exact periodic wave solution $|u|$ of Eq. (18) with $a = 0.50$, $b = 1.0$, $\alpha = 1$

Fig. 2 **a** The c_0 curve for the solitary wave solution $|u|$ of Eq. (18) with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **b** The approximate solitary wave solution $|u|$ of Eq. (18) with $a = 0.50$, $b = 1.0$, $c_0 = -0.1$, $\alpha = 0.9$. **c** The exact solitary wave solution $|u|$ of Eq. (18) with $a = 0.50$, $b = 1.0$, $\alpha = 1$

HAM: Time Fractional Two Coupled Nonlinear Schrödinger Equations

Let us consider the time fractional two coupled NLS equations

$$i \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^2 u}{\partial x^2} + 2(|u|^2 + |v|^2)u = 0, \quad (45)$$

$$i \frac{\partial^\alpha v}{\partial t^\alpha} + \frac{\partial^2 v}{\partial x^2} + 2(|u|^2 + |v|^2)v = 0, \quad 0 < \alpha \leq 1, \quad i = \sqrt{-1}. \quad (46)$$

Let us assume the initial guesses and the auxiliary linear operators be

$$u_0(x, t) = u(x, 0) = f(x), \quad (47)$$

$$v_0(x, t) = v(x, 0) = g(x), \quad (48)$$

$$\mathcal{L}_u[\phi(x, t; q)] = D_t^\alpha \phi = \frac{\partial^\alpha \phi}{\partial t^\alpha}, \quad (49)$$

$$\mathcal{L}_v[\psi(x, t; q)] = D_t^\alpha \psi = \frac{\partial^\alpha \psi}{\partial t^\alpha}, \quad (50)$$

We define the nonlinear operators from Eqs. (45) and (46) as

$$\mathcal{N}_u(\phi, \psi) = \frac{\partial^\alpha \phi}{\partial t^\alpha} - i \left(\frac{\partial^2 \phi}{\partial x^2} + 2(|\phi|^2 + |\psi|^2)\phi \right), \quad (51)$$

$$\mathcal{N}_v(\phi, \psi) = \frac{\partial^\alpha \psi}{\partial t^\alpha} - i \left(\frac{\partial^2 \psi}{\partial x^2} + 2(|\phi|^2 + |\psi|^2)\psi \right). \quad (52)$$

Let c_0^u and c_0^v be the nonzero auxiliary parameters, $H_u(x, t)$ and $H_v(x, t)$ denote the nonzero auxiliary functions. we now construct the so called zeroth order deformation equations

$$(1 - q)\mathcal{L}_u[\phi(x, t; q) - u_0(x, t)] = qc_0^u H_u(x, t)\mathcal{N}_u(\phi(x, t; q), \psi(x, t; q)), \quad (53)$$

$$(1 - q)\mathcal{L}_v[\psi(x, t; q) - v_0(x, t)] = qc_0^v H_v(x, t)\mathcal{N}_v(\phi(x, t; q), \psi(x, t; q)), \quad (54)$$

where $\phi(x, t; q)$ and $\psi(x, t; q)$ are unknown functions. When $q = 0$ along with the property Eqs. (2), (53) and (54) yield

$$\phi(x, t; 0) = u_0(x, t), \quad \psi(x, t; 0) = v_0(x, t). \quad (55)$$

When $q = 1$, we have

$$\phi(x, t; 1) = u(x, t), \quad \psi(x, t; 1) = v(x, t). \quad (56)$$

Thus as q increases from 0 to 1, $\phi(x, t; q)$ and $\psi(x, t; q)$ vary continuously from the initial guesses $u_0(x, t)$, $v_0(x, t)$ to the exact solutions $u(x, t)$, $v(x, t)$ of Eqs. (45), (46) respectively. Expanding $\phi(x, t; q)$ and $\psi(x, t; q)$ as Taylor series with respect to q , one has

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m, \quad (57)$$

$$\psi(x, t; q) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t)q^m, \quad (58)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}, \quad (59)$$

$$v_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \psi(x, t; q)}{\partial q^m} \right|_{q=0}. \quad (60)$$

Assume that $c_0^u, c_0^v, H_u(x, t), H_v(x, t), u_0(x, t), v_0(x, t), \mathcal{L}_u$ and $\mathcal{L}_v$ are properly chosen so that the series Eqs. (57) and (58) are convergent at $q = 1$, using Eq. (56), we obtain

$$u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \quad (61)$$

$$v(x, t) = v_0(x, t) + \sum_{m=1}^{\infty} v_m(x, t). \quad (62)$$

Define the vectors $\mathbf{u}_n(x, t) = (u_0, u_1, \dots, u_n)$, $\mathbf{v}_n(x, t) = (v_0, v_1, \dots, v_n)$. Differentiating Eqs. (53), (54) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called m th-order deformation equations

$$\mathcal{L}_u [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0^u H_u(x, t) R_m^u(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}), \quad (63)$$

$$\mathcal{L}_v [v_m(x, t) - \chi_m v_{m-1}(x, t)] = c_0^v H_v(x, t) R_m^v(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}), \quad (64)$$

where χ_m is defined by Eq. (10),

$$\begin{aligned} R_m^u(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}_u(\phi(x, t; q), \psi(x, t; q))}{\partial q^{m-1}} \right|_{q=0}, \\ &= \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] u_{m-1-j-k} \right), \end{aligned} \quad (65)$$

$$\begin{aligned} R_m^v(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}) &= \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathcal{N}_v(\phi(x, t; q), \psi(x, t; q))}{\partial q^{m-1}} \right|_{q=0}, \\ &= \frac{\partial^\alpha v_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 v_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] v_{m-1-j-k} \right), \end{aligned} \quad (66)$$

where the $\bar{u}_k$ is the complex conjugate of u_k . For simplicity, we choose $c_0^u = c_0^v = c_0$ and $H_u(x, t) = H_v(x, t) = 1$. According to Eqs. (63), (64) and Eqs. (49), (50), one has

$$D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 H_u(x, t) R_m^u(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}), \quad (67)$$

$$D_t^\alpha [v_m(x, t) - \chi_m v_{m-1}(x, t)] = c_0 H_v(x, t) R_m^v(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}). \quad (68)$$

Applying the Riemann–Liouville fractional integral operator J_t^α on both side of the above equations,

$$J_t^\alpha D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 J_t^\alpha R_m^u(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}), \quad (69)$$

$$J_t^\alpha D_t^\alpha [v_m(x, t) - \chi_m v_{m-1}(x, t)] = c_0 J_t^\alpha R_m^v(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}). \quad (70)$$

Substituting Eqs. (65), (66) into Eqs. (69), (70), we have

$$J_t^\alpha D_t^\alpha [u_m(x, t) - \chi_m u_{m-1}(x, t)] = c_0 J_t^\alpha \left[\frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] u_{m-1-j-k} \right) \right],$$

$$J_t^\alpha D_t^\alpha [v_m(x, t) - \chi_m v_{m-1}(x, t)] = c_0 J_t^\alpha \left[\frac{\partial^\alpha v_{m-1}}{\partial t^\alpha} - i \left(\frac{\partial^2 v_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] v_{m-1-j-k} \right) \right].$$

Using the property Eq. (16) along with $u_m(x, 0) = 0$, $v_m(x, 0) = 0$, $m = 1, 2, \dots$ which yield

$$u_m(x, t) = (\chi_m + c_0)(u_{m-1} - u_{m-1}(x, 0)) - i c_0 J_t^\alpha \left[\frac{\partial^2 u_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] u_{m-1-j-k} \right], \quad (71)$$

$$v_m(x, t) = (\chi_m + c_0)(v_{m-1} - v_{m-1}(x, 0)) - i c_0 J_t^\alpha \left[\frac{\partial^2 v_{m-1}}{\partial x^2} + 2 \sum_{j=0}^{m-1} \sum_{k=0}^{m-1-j} [u_j \bar{u}_k + v_j \bar{v}_k] v_{m-1-j-k} \right]. \quad (72)$$

By making use of Eq. (10), the above equations Eqs. (71), (72) yield the components of the infinite series solutions Eqs. (61) and (62) as follows as,

$$u_1 = -i c_0 J_t^\alpha \left[\frac{\partial^2 u_0}{\partial x^2} + 2[u_0^2 \bar{u}_0 + v_0 \bar{v}_0 u_0] \right], \quad (73)$$

$$v_1 = -i c_0 J_t^\alpha \left[\frac{\partial^2 v_0}{\partial x^2} + 2[u_0 \bar{u}_0 v_0 + v_0^2 \bar{v}_0] \right], \quad (74)$$

$$u_2 = (1 + c_0)u_1 - i c_0 J_t^\alpha \left[\frac{\partial^2 u_1}{\partial x^2} + 2[u_0^2 \bar{u}_1 + 2|u_0|^2 u_1 + \bar{v}_1 u_0 v_0 + |v_0|^2 u_1 + \bar{v}_0 u_0 v_1] \right], \quad (75)$$

$$v_2 = (1 + c_0)v_1 - i c_0 J_t^\alpha \left[\frac{\partial^2 v_1}{\partial x^2} + 2[v_0 (\bar{u}_1 u_0 + \bar{u}_0 u_1 + \bar{v}_1 v_0) + (|u_0|^2 + 2|v_0|^2)v_1] \right], \quad (76)$$

$$u_3 = (1 + c_0)u_2 - c_0 i J_t^\alpha \left[\frac{\partial^2 u_2}{\partial x^2} + 2(\bar{u}_2 u_0^2 + 2u_0 |u_1|^2 + 2|u_0|^2 u_2 + 2u_0 v_0 \bar{v}_1 + |v_0|^2 u_2 + u_0 |v_1|^2 + \bar{v}_0 u_1 v_1 + \bar{v}_0 u_0 v_2) \right], \quad (77)$$

$$v_3 = (1 + c_0)v_2 - c_0 i J_t^\alpha \left[\frac{\partial^2 v_2}{\partial x^2} + 2(\bar{u}_2 u_0 v_0 + \bar{u}_0 u_2 v_0 + \bar{v}_2 v_0^2 + \bar{u}_0 u_1 v_1 + 2|v_1|^2 v_0 + \bar{v}_0 v_1^2 + \bar{u}_1(u_1 v_0 + u_0 v_1) + |u_0|^2 v_2 + 2|v_0|^2 v_2) \right], \quad (78)$$

etc.

Periodic Wave Solution

Let

$$u(x, 0) = a_1 e^{ik_1 x} \quad \text{and} \quad v(x, 0) = a_2 e^{ik_2 x},$$

where a_1, a_2, k_1 and k_2 are real constants. Following the procedure outlined in the previous section, we computed the components $u_m(x, t), v_m(x, t)$ of the infinite series solutions Eqs. (61) and (62) as follows

$$u_0 = a_1 e^{ik_1 x}, \quad (79)$$

$$v_0 = a_2 e^{ik_2 x}, \quad (80)$$

$$u_1 = -a_1 e^{ik_1 x} \frac{i C_1 c_0 t^\alpha}{\Gamma(1 + \alpha)}, \quad (81)$$

$$v_1 = -a_2 e^{ik_2 x} \frac{i C_2 c_0 t^\alpha}{\Gamma(1 + \alpha)}, \quad (82)$$

$$u_2 = a_1 e^{ik_1 x} \left[\frac{-i(1 + c_0)C_1 c_0 t^\alpha}{\Gamma(1 + \alpha)} + \frac{(i C_1)^2 c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right], \quad (83)$$

$$v_2 = a_2 e^{ik_2 x} \left[\frac{-i(1 + c_0)C_2 c_0 t^\alpha}{\Gamma(1 + \alpha)} + \frac{(i C_2)^2 c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right], \quad (84)$$

where $C_j = 2(a_1^2 + a_2^2) - k_j^2$, $j = 1, 2$. In a similar manner we compute $u_3, v_3, \dots$ and so the infinite series Eqs. (61) and (62) become

$$u(x, t) = a_1 e^{ik_1 x} \left[1 - \frac{(2 + c_0)i C_1 c_0 t^\alpha}{\Gamma(1 + \alpha)} + \frac{(i C_1)^2 c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \dots \right], \quad (85)$$

$$v(x, t) = a_2 e^{ik_2 x} \left[1 - \frac{(2 + c_0)i C_2 c_0 t^\alpha}{\Gamma(1 + \alpha)} + \frac{(i C_2)^2 c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \dots \right], \quad (86)$$

which may not lead to closed form in general. If we choose $c_0 = -1$ and $\alpha \rightarrow 1$ then Eqs. (79)–(84) become

$$u_0 = a_1 e^{ik_1 x},$$

$$v_0 = a_2 e^{ik_2 x},$$

$$u_1 = a_1 e^{ik_1 x} (i C_1) t,$$

$$v_1 = a_2 e^{ik_2 x} (i C_2) t,$$

$$u_2 = a_1 e^{ik_1 x} (i C_1)^2 \frac{t^2}{2!},$$

$$v_2 = a_2 e^{ik_2 x} (i C_2)^2 \frac{t^2}{2!},$$

and so on. Using Taylor series expansion, the infinite series solutions Eqs. (61) and (62) yield the exact periodic wave solutions

$$u(x, t) = a_1 e^{i(k_1 x + C_1 t)} = a_1 e^{i(k_1 x + (2(a_1^2 + a_2^2) - k_1^2)t)}, \quad (87)$$

$$v(x, t) = a_2 e^{i(k_2 x + C_2 t)} = a_2 e^{i(k_2 x + (2(a_1^2 + a_2^2) - k_2^2)t)}, \quad (88)$$

respectively, which are the same solutions obtained by Tan et. al [25]. The c_0 curves for the 7th order approximate solutions of Eqs. (85) and (86) at $x = 0$ and $t = 0.25$ when $\alpha = 0.9$ are given in Fig. 3a, b. It is observed that the valid region for c_0 is $-1.5 \leq c_0 < 0.5$. Thus if we choose the appropriate value for c_0 in this interval at which the approximate solutions of Eqs. (85) and (86) converges. Fig. 3c, d represent the absolute value of the approximate and exact periodic wave solutions u_{HAM} and u with $a_1 = 0.50$, $k_1 = 1.0$, $c_0 = -0.1$, $\alpha = 0.9$ and $a_1 = 0.50$, $k_1 = 1.0$, $\alpha = 1$ respectively. Fig. 3e, f represent the absolute value of the approximate and exact periodic wave solutions v_{HAM} and v with $a_2 = 1$, $k_2 = 1.5$, $c_0 = -0.1$, $\alpha = 0.9$ and $a_2 = 1$, $k_2 = 1.5$, $\alpha = 1$ respectively.

Solitary Wave Solution

Let

$$u(x, 0) = a\epsilon e^{ibx} \operatorname{sech}(ax) \quad \text{and} \quad v(x, 0) = a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax),$$

where $\epsilon = \frac{1}{\sqrt{1+e^{2c}}}$, a, b, c , and d are real constants. Following the procedure outlined above, we computed the components $u_m(x, t)$, $v_m(x, t)$ of the infinite series solutions Eqs. (61) and (62) as follows

$$u_0 = a\epsilon e^{ibx} \operatorname{sech}(ax), \quad (89)$$

$$v_0 = a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax), \quad (90)$$

$$u_1 = \frac{-a\epsilon e^{ibx} \operatorname{sech}(ax)(iB + 2ab \tanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)}, \quad (91)$$

$$v_1 = \frac{-a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax)(iB + 2ab \tanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)}, \quad (92)$$

$$u_2 = a\epsilon e^{ibx} \operatorname{sech}(ax) \left[\frac{-(1 + c_0)(iB + 2ab \tanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right], \quad (93)$$

$$v_2 = a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax) \left[\frac{-(1 + c_0)(iB + 2ab \tanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} \right], \quad (94)$$

Fig. 3 **a** The c_0 curve for the periodic wave solution u with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **b** The c_0 curve for the periodic wave solution v with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **c** The approximate periodic wave solution $|u|$ with $a_1 = 0.5$, $k_1 = 1$, $c_0 = -0.1$, $\alpha = 0.9$. **d** The exact periodic wave solution $|u|$ with $a_1 = 0.5$, $k_1 = 1$, $\alpha = 1$. **e** The approximate periodic wave solution $|v|$ with $a_2 = 1$, $k_2 = 1.5$, $c_0 = -0.1$, $\alpha = 0.9$. **f** The exact periodic wave solution $|v|$ with $a_2 = 1$, $k_2 = 1.5$, $\alpha = 1$

Fig. 3 continued

Fig. 4 **a** The c_0 curve for the solitary wave solution u with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **b** The c_0 curve for the solitary wave solution v with $x = 0$, $t = 0.25$, $\alpha = 0.9$. **c** The approximate solitary wave solution $|u|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $c_0 = -0.1$, $\alpha = 0.9$. **d** The exact solitary wave solution $|u|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $\alpha = 1$. **e** The approximate solitary wave solution $|v|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $d = 1$, $c_0 = -0.1$, $\alpha = 0.9$. **f** The exact solitary wave solution $|v|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $d = 1$, $\alpha = 1$. **g** Effect of α on the solitary wave solution $|u|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $c_0 = -0.1$, $x = 5$. **h** Effect of α on the solitary wave solution $|v|$ with $a = \frac{1}{\sqrt{2}}$, $b = 1$, $c = 0$, $d = 1$, $c_0 = -0.1$, $x = 5$

Fig. 4 continued

Fig. 4 continued

where $B = a^2 - b^2$ and so on. Therefore, from Eqs. (61) and (62), we have

$$u = a\epsilon e^{ibx} \operatorname{sech}(ax) \left[1 - \frac{(2 + c_0)(iB + 2abtanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabBtanh(ax) - 8a^2b^2tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \dots \right], \quad (95)$$

$$v = a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax) \left[1 - \frac{(2 + c_0)(iB + 2abtanh(ax))c_0 t^\alpha}{\Gamma(1 + \alpha)} - \frac{((a^2 + b^2)^2 - 4iabBtanh(ax) - 8a^2b^2tanh^2(ax))c_0^2 t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \dots \right]. \quad (96)$$

If we choose $c_0 = -1$ and $\alpha \rightarrow 1$ then the equations Eqs. (89)–(94) become

$$u_0 = a\epsilon e^{ibx} \operatorname{sech}(ax),$$

$$v_0 = a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax),$$

$$u_1 = a\epsilon e^{ibx} \operatorname{sech}(ax) [iB + 2abtanh(ax)] \frac{t}{1!},$$

$$\begin{aligned}
v_1 &= a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax) [iB + 2ab \tanh(ax)] \frac{t}{1!}, \\
u_2 &= -a\epsilon e^{ibx} \operatorname{sech}(ax) [(B)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax)] \frac{t^2}{2!}, \\
v_2 &= -a\epsilon e^{c+i(bx+d)} \operatorname{sech}(ax) [(a^2 + b^2)^2 - 4iabB \tanh(ax) - 8a^2b^2 \tanh^2(ax)] \frac{t^2}{2!},
\end{aligned}$$

and so on. Using Taylor series expansion, the infinite series solutions Eqs. (61) and (62) yield the exact solitary wave solutions

$$\begin{aligned}
u(x, t) &= a\epsilon e^{i(bx+Bt)} \operatorname{sech}(a(x - 2bt)) = \frac{ae^{i(bx+(a^2-b^2)t)} \operatorname{sech}(a(x - 2bt))}{\sqrt{1 + e^{2c}}}, \\
v(x, t) &= a\epsilon e^{c+id} e^{i(bx+Bt)} \operatorname{sech}(a(x - 2bt)) = \frac{ae^{c+id} e^{i(bx+(a^2-b^2)t)} \operatorname{sech}(a(x - 2bt))}{\sqrt{1 + e^{2c}}},
\end{aligned}$$

which are the same solutions obtained in [27] by using Hirota bilinearisation method. The c_0 curves for the 6th order approximate solutions of Eqs. (95) and (96) at $x = 0$ and $t = 0.25$ when $\alpha = 0.9$ are given in Fig. 4a, b. It is observed that the valid region for c_0 is $-1.5 < c_0 < 1$. Thus if we choose the appropriate value for c_0 in this interval at which the approximate solutions Eqs. (95) and (96) converges. Fig. 4c, d represent the absolute value of the approximate and exact solitary wave solutions u_{HAM} and u with $a = \frac{1}{\sqrt{2}}$, $b = 1.0$, $c = 0$, $c_0 = -0.1$, $\alpha = 0.9$ and $a = \frac{1}{\sqrt{2}}$, $b = 1.0$, $c = 0$, $\alpha = 1$ respectively. Fig. 4e, f represent the absolute value of the approximate and exact solitary wave solutions v_{HAM} and v with $a = \frac{1}{\sqrt{2}}$, $b = 1.0$, $c = 0$, $d = 1$, $c_0 = -0.1$, $\alpha = 0.9$ and $a = \frac{1}{\sqrt{2}}$, $b = 1.0$, $c = 0$, $d = 1$, $\alpha = 1$ respectively.

Concluding Remarks

In this article, we considered the well known NLS equation and two coupled NLS equations with time fractional derivative and derived its approximate analytical periodic wave and solitary wave solutions using HAM. This method provides us a convenient way to control and adjust the convergence of the series solution. The obtained approximate analytical solutions and the effect of time fractional order α are shown graphically. We would like to mention that the computations were performed by using *MATHEMATICA*. It is well known that memory effects can suppress chaos. Chaos has been found to persist in fractional systems when $\alpha \in (0, 1)$ [31]. In this article, we derived single valued analytic solutions when α is equal to one in the both scalar and two coupled fractional NLS equations. In other words the solution behaviour is regular when $\alpha = 1$. The analysis suggests that the dynamics of the system is strongly sensitive to the values of the order α of the fractional derivative. On the basis of our results we are led to believe that fractional system undergoes a transition from the chaotic dynamics to regular one when $\alpha < 1$ approaches to one.

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