A Novel (G'/G) -Expansion Method and its Application to the $(3 + 1)$ -Dimensional Burger's Equations

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Abstract In this article, a novel (G'/G) -expansion method is used to look for the traveling wave solutions of nonlinear evolution equations. We construct abundant traveling wave solutions involving parameters to the $(3 + 1)$ -dimensional integrable Burger's equations by means of the suggested method. The performance of the method is reliable, useful and gives more new general exact solutions than the existing methods. The novel (G'/G) -expansion method provides more general forms of solutions.

Keywords The novel (G'/G) -expansion method · $(3 + 1)$ -Dimensional Burger's equations · Solitary wave solutions · Exact solution · Auxiliary equation

Introduction

It is well known that nonlinear evolution equations (NLEEs) are widely used to describe complex phenomena in various fields of science, especially in physics, plasma physics, fluid physics, quantum field theory, biophysics, chemical kinematics, electricity, propagation of shallow water waves, high-energy physics, condensed matter physics, quantum mechanics, optical fibers and so on. The analytical solutions of such equations are of fundamental importance to reveal the inner mechanism of the phenomena. In mathematics and physics, a soliton is a self-reinforcing solitary wave, a wave packet that upholds its profile while it travels at constant speed. In the past years, many powerful and direct methods have been developed to find special solutions, such as, improved F-expansion method [1], Weierstrass elliptic function method [2], Jacobi elliptic function expansion method [3] and tanh-function method [4–8], inverse scattering transform method [9], Hirota's bilinear method [10], Backlund transform method [11], Exp-function method [12–16], Painleve expansion method [17], extended tanh-method [18–20], homogeneous balance method [21, 22] and so on.

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Lately, Wang et al. [23] introduced a direct method, called (G'/G) -expansion method and demonstrated that it is a powerful method for seeking analytic solutions of NLEEs. For additional references see the articles [24–29]. In order to establish the efficiency and persistence of (G'/G) -expansion method and to extend the range of applicability, further research has been carried out by several researchers. For instance, Zhang et al. [30] made a generalization of (G'/G) -expansion method for the evolution equations with variable coefficients. Zhang et al. [31] also presented an improved (G'/G) -expansion method to seek more general traveling wave solutions. Zayed [32] presented a new approach of (G'/G) -expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0$, e_2, e_1, e_0 are arbitrary constants, and obtained new exact solutions. Zayed [33] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = A + BG^2(\xi)$, where A and B are arbitrary constants.

In this article, we use a novel (G'/G) -expansion method introduced by Alam et al. [34] to solve the NLEEs in mathematical physics and engineering. To illustrate the originality, consistency and advantages of the method, the $(3 + 1)$ -dimensional Burgers equations are solved and abundant new families of exact solutions are found.

The Novel (G'/G) -Expansion Method

Suppose the nonlinear evolution equation is of the form

$$P(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \dots) = 0, \quad (1)$$

where P is a polynomial in $u(x, t)$ and its partial derivatives wherein the highest order partial derivatives and the nonlinear terms are concerned. The main steps of the method are as follows:

Step 1 Combining the real variables x, y, z and t by a compound variable ξ , we suppose that

$$u(x, y, z, t) = u(\xi), \quad \xi = x + y + z \pm Vt, \quad (2)$$

where V is the speed of the traveling wave. Equation (2) transforms Eq. (1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$Q(u, u', u'', u''', \dots) = 0, \quad (3)$$

where Q is a function of $u(\xi)$ and its derivatives in which prime stands for derivative with respect to ξ .

Step 2 Assume the solution of Eq. (3) can be expressed as:

$$u(\xi) = \sum_{i=-m}^m \alpha_i (k + \Phi(\xi))^i, \quad (4)$$

where

$$\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}. \quad (5)$$

Herein α_{-m} or α_m may be zero, but both of them can not be zero simultaneously. α_i ($i = 0, \pm 1, \pm 2, \dots, \pm m$) and k are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ordinary differential equation:

$$GG'' = AGG' + BG^2 + C(G')^2, \quad (6)$$

where prime denotes the derivative with respect ξ ; A , B , and C are real constants.

The Cole–Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = \frac{G'(\xi)}{G(\xi)}$ reduces the Eq. (6) into the Riccati equation:

$$\Phi'(\xi) = B + A\Phi(\xi) + (C - 1)\Phi^2(\xi) \quad (7)$$

Eq. (7) has individual twenty five solutions (see Zhu [35] for details).

Step 3 The value of the positive integer m can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order come out in Eq. (3). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D\left[\frac{d^p u(\xi)}{d\xi^p}\right] = n + p, \quad D\left[u^p \left(\frac{d^q u(\xi)}{d\xi^q}\right)^s\right] = np + s(n + q).$$

Step 4 Substitute Eq. (4) including Eqs. (5) and (6) into Eq. (3), we obtain polynomials in $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^i$ and $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^{-i}$, ($i = 0, 1, 2, \dots, m$) Collect each coefficient of the resulted polynomials to zero, yields an over-determined set of algebraic equations for α_i ($i = 0, \pm 1, \pm 2, \dots, \pm m$), k and V .

Step 5 Suppose the value of the constants can be obtained by solving the algebraic equations obtained in Step 4. The values of the constants together with the solutions of Eq. (6) yield abundant exact traveling wave solutions of the nonlinear evolution Eq. (1).

Application

In this section, we will employ the novel (G'/G) -expansion method to obtain some new and more general exact traveling wave solutions of the celebrated $(3 + 1)$ -dimensional Burger's equations.

Let us consider the $(3 + 1)$ -dimensional Burgers equations,

$$\begin{aligned} u_t &= auu_x + bvu_x + cwu_x + u_{xx} + u_{yy} + u_{zz}, \\ u_x &= v_y, \\ u_z &= w_y. \end{aligned} \quad (8)$$

where a , b and c are nonzero constants. The inverse strong symmetries and strong symmetries of a $(3 + 1)$ -dimensional Burgers equation was given explicitly in [36]. In [37], the multilinear variable separation approach was extended to study Eq. (8). In [38], a new prior variable separation approach was developed to determine a new general solution for Eq. (8). In [39], Bäcklund transformation method combined with Hirota's bilinear method and tanh-coth method are used to determine two distinct structures of regular soliton solutions, singular soliton solutions and new single soliton solutions.

We will study the traveling wave solutions to system (8). Substituting $u = u(\xi)$, $\xi = x + y + z - Vt$ into system (8), we obtain

$$\begin{aligned} -Vu &= auu' + bvu' + cwu' + 3u'', \\ u' &= v', \\ u' &= w'. \end{aligned} \quad (9)$$

Integrating the last two equations gives $u = v = w$, where the constants of integration are considered zeros. The first equation in the system after integrating once becomes:

$$Vu + \frac{a+b+c}{2}u^2 + 3u' + C_1 = 0. \quad (9a)$$

where C_1 is an integration constant. Inserting (4) and (6) and considering the homogeneous balance between u' and u^2 in Eq. (9), we obtain $2m = m + 1$. i.e. $m = 1$. Therefore, the solution formula (4) becomes

$$u(\xi) = \alpha_{-1}(k + \mu(\xi))^{-1} + \alpha_0 + \alpha_1(k + \mu(\xi)). \quad (10)$$

Substituting Eq. (10) into Eq. (9), the left hand side transforms into polynomials in $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^i$, ($i = 0, 1, 2, \dots, m$) and $\left(k + \frac{G'(\xi)}{G(\xi)}\right)^{-i}$, ($i = 0, 1, 2, \dots, N$). Equating the coefficients of similar power of these polynomials to zero, we obtain an over-determine set of algebraic equations (for simplicity we leave out to display the equations) for $\alpha_0, \alpha_1, \alpha_{-1}, k, C_1$ and V . Solving the over-determined set of algebraic equations by making using the symbolic computation software, such as Maple 13, we obtain

Set 1

$$\begin{aligned} \alpha_1 &= \frac{6(1-C)}{a+b+c}, \alpha_0 = \alpha_0, \alpha_{-1} = 0, k = k, V = -3A + 6k(C-1) - (a+b+c)\alpha_0, \\ C_1 &= \frac{1}{2(a+b+c)}(36BC - 72k^2C + 36kA - 36B - 12\alpha_0bkC - 12\alpha_0akC \\ &\quad - 12\alpha_0ckC + 36k^2 + 2ca\alpha_0^2 + 2bc\alpha_0^2 + 6Aa\alpha_0 + 6Ab\alpha_0 + 6Ac\alpha_0 + 2ab\alpha_0^2 \\ &\quad - 36ACk + 12bk\alpha_0 + 12ak\alpha_0 + 12ck\alpha_0 + a^2\alpha_0^2 + b^2\alpha_0^2 + 36k^2C^2 + c^2\alpha_0^2), \end{aligned} \quad (11)$$

where α_0, k, A, B and C are arbitrary constants.

Set 2

$$\begin{aligned} \alpha_{-1} &= \frac{6(B+k^2C-k^2-kA)}{a+b+c}, \alpha_0 = \alpha_0, \alpha_1 = 0, k = k, \\ V &= 3A + 6k - 6kC - (a+b+c)\alpha_0, \\ C_1 &= \frac{1}{2(a+b+c)}(36BC - 72k^2C + 36kA - 36B + 12\alpha_0bkC + 12\alpha_0akC \\ &\quad + 12\alpha_0ckC + 36k^2 + 2ca\alpha_0^2 + 2bc\alpha_0^2 + 2ab\alpha_0^2 - 6Aa\alpha_0 - 6Ab\alpha_0 - 6Ac\alpha_0 \\ &\quad - 36ACk - 12ak\alpha_0 - 12bk\alpha_0 - 12ck\alpha_0 + a^2\alpha_0^2 + b^2\alpha_0^2 + c^2\alpha_0^2 + 36k^2C^2), \end{aligned} \quad (12)$$

where α_0, k, A, B and C are arbitrary constants.

Set 3

$$\begin{aligned} \alpha_1 &= \frac{6(1-C)}{a+b+c}, \alpha_{-1} = -\frac{3(4BC-A^2-4B)}{2(a+b+c)(1-C)}, \alpha_0 = \alpha_0, k = -\frac{A}{2(1-C)}, \\ V &= -(a+b+c)\alpha_0, \\ C_1 &= \frac{1}{2(a+b+c)}(144BC - 144B + 2ab\alpha_0^2 + 2bc\alpha_0^2 + 2ca\alpha_0^2 + a^2\alpha_0^2 + b^2\alpha_0^2 \\ &\quad + c^2\alpha_0^2 - 36A^2), \end{aligned} \quad (13)$$

where α_0, A, B and C are arbitrary constants.

Substituting Eqs. (11)–(13) into solution formula (10), we obtain

$$u_1(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times (k + (G'/G)). \quad (14)$$

$$u_2(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a+b+c} \times (k + (G'/G))^{-1}. \quad (15)$$

$$u_3(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left(-\frac{A}{2(1-C)} + (G'/G) \right) - \frac{3(4BC - A^2 - 4B)}{2(a+b+c)(1-C)} \\ \times \left(-\frac{A}{2(1-C)} + (G'/G) \right)^{-1}. \quad (16)$$

where $\xi = x + y + z - Vt$; α_0 , k , A , B and C are arbitrary constants.

Substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (14) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_1^1(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \tanh \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right) \right\}, \quad (17)$$

$$u_1^2(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \coth \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right) \right\}, \quad (18)$$

$$u_1^3(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \\ \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \left(\tanh(\sqrt{\Delta} \xi) \pm i \operatorname{sech}(\sqrt{\Delta} \xi) \right) \right) \right\}, \quad (19)$$

$$u_1^4(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \\ \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \left(\coth(\sqrt{\Delta} \xi) \pm \operatorname{csch}(\sqrt{\Delta} \xi) \right) \right) \right\}, \quad (20)$$

$$u_1^5(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \\ \times \left\{ k - \frac{1}{4(C-1)} \left(2A + \sqrt{\Delta} \left(\tanh \left(\frac{1}{4} \sqrt{\Delta} \xi \right) + \coth \left(\frac{1}{4} \sqrt{\Delta} \xi \right) \right) \right) \right\}, \quad (21)$$

$$u_1^6(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \\ \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \frac{\pm \sqrt{\Delta(F^2 + H^2)} - F \sqrt{\Delta} \cosh(\sqrt{\Delta} \xi)}{F \sinh(\sqrt{\Delta} \xi) + H} \right) \right\}, \quad (22)$$

$$u_1^7(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \\ \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \frac{\pm \sqrt{\Delta(F^2 + H^2)} + F \sqrt{\Delta} \cosh(\sqrt{\Delta} \xi)}{F \sinh(\sqrt{\Delta} \xi) + H} \right) \right\}, \quad (23)$$

where F and H are real constants.

$$u_1^8(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}{\sqrt{\Delta} \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) - A \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)} \right\}, \quad (24)$$

$$u_1^9(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}{\sqrt{\Delta} \cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) - A \sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)} \right\}, \quad (25)$$

$$u_1^{10}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \cosh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \sinh(\sqrt{\Delta}\xi) - A \cosh(\sqrt{\Delta}\xi) \pm i\sqrt{\Delta}} \right\}, \quad (26)$$

$$u_1^{11}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \sinh(\sqrt{\Delta}\xi)}{\sqrt{\Delta} \cosh(\sqrt{\Delta}\xi) - A \sinh(\sqrt{\Delta}\xi) \pm \sqrt{\Delta}} \right\}. \quad (27)$$

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_1^{12}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left(k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} \tan\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) \right) \right), \quad (28)$$

$$u_1^{13}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{-\Delta} \cot\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) \right) \right\}, \quad (29)$$

$$u_1^{14}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \sqrt{-\Delta} \left(\tan(\sqrt{-\Delta}\xi) \pm \sec(\sqrt{-\Delta}\xi) \right) \right) \right\}, \quad (30)$$

$$u_1^{15}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{1}{2(C-1)} \left(A + \sqrt{-\Delta} \left(\cot(\sqrt{-\Delta}\xi) \pm \csc(\sqrt{-\Delta}\xi) \right) \right) \right\}, \quad (31)$$

$$u_1^{16}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{1}{4(C-1)} \left(-2A + \sqrt{-\Delta} \left(\tan\left(\frac{1}{4}\sqrt{-\Delta}\xi\right) - \cot\left(\frac{1}{4}\sqrt{-\Delta}\xi\right) \right) \right) \right\}, \quad (32)$$

$$u_1^{17}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \frac{\pm\sqrt{-\Delta}(F^2 - H^2) - F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + H} \right) \right\}, \quad (33)$$

$$u_1^{18}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{1}{2(C-1)} \left(-A + \frac{\pm\sqrt{-\Delta}(F^2 - H^2) + F\sqrt{-\Delta} \cos(\sqrt{-\Delta}\xi)}{F \sin(\sqrt{-\Delta}\xi) + H} \right) \right\}, \quad (34)$$

where F and H are arbitrary constants such that $F^2 - H^2 > 0$.

$$u_1^{19}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{2B \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta} \sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) + A \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)} \right\}, \quad (35)$$

$$u_1^{20}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta} \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) - A \sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)} \right\}, \quad (36)$$

$$u_1^{21}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{2B \cos(\sqrt{-\Delta}\xi)}{\sqrt{-\Delta} \sin(\sqrt{-\Delta}\xi) + A \cos(\sqrt{-\Delta}\xi) \pm \sqrt{-\Delta}} \right\}, \quad (37)$$

$$u_1^{22}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k + \frac{2B \sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta} \cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) - A \sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}} \right\}. \quad (38)$$

When $B = 0$ and $A(C - 1) \neq 0$,

$$u_1^{23}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{Ac_1}{(C-1)\{c_1 + \cosh(A\xi) - \sinh(A\xi)\}} \right\}, \quad (39)$$

$$u_1^{24}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{A\{\cosh(A\xi) + \sinh(A\xi)\}}{(C-1)\{c_1 + \cosh(A\xi) + \sinh(A\xi)\}} \right\}, \quad (40)$$

where c_1 is an arbitrary constant.

When $(C - 1) \neq 0$ and $A = B = 0$, the solution of Eq. (8) is

$$u_1^{25}(\xi) = \alpha_0 + \frac{6(1-C)}{a+b+c} \times \left\{ k - \frac{1}{(C-1)\xi + c_2} \right\}, \quad (41)$$

where c_2 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (15) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C - 1) \neq 0$ (or $B(C - 1) \neq 0$),

$$u_2^1(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a+b+c} \times \left(k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \tanh\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right) \right)^{-1}, \quad (42)$$

where $\xi = x + y + z - Vt$; α_0 , k , A , B and C are arbitrary constants.

$$u_2^2(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a+b+c} \times \left(k - \frac{1}{2(C-1)} \left(A + \sqrt{\Delta} \coth\left(\frac{1}{2}\sqrt{\Delta}\xi\right) \right) \right)^{-1}, \quad (43)$$

$$u_2^3(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a+b+c} \times \left[k - \frac{1}{2(C-1)} \left\{ A + \sqrt{\Delta} \left(\tanh(\sqrt{\Delta}\xi) \pm i \sec h(\sqrt{\Delta}\xi) \right) \right\} \right]^{-1}. \quad (44)$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C - 1) \neq 0$ (or $B(C - 1) \neq 0$),

$$u_2^{12}(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a + b + c} \times \left(k + \frac{1}{2(C - 1)} \left(-A + \sqrt{-\Delta} \tan \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right) \right)^{-1}, \quad (45)$$

$$u_2^{13}(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a + b + c} \times \left(k - \frac{1}{2(C - 1)} \left(A + \sqrt{-\Delta} \cot \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right) \right)^{-1}, \quad (46)$$

$$u_2^{14}(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a + b + c} \times \left\{ k + \frac{1}{2(C - 1)} \left(-A + \sqrt{-\Delta} \left(\tan(\sqrt{-\Delta} \xi) \pm \sec(\sqrt{-\Delta} \xi) \right) \right) \right\}^{-1}. \quad (47)$$

When $(C - 1) \neq 0$ and $A = B = 0$, the solution of Eq. (8) is

$$u_2^{25}(\xi) = \alpha_0 + \frac{6(B + k^2C - k^2 - kA)}{a + b + c} \times \left(k - \frac{1}{2(C - 1)\xi + c_2} \right)^{-1}, \quad (48)$$

where c_2 is an arbitrary constant.

We can write down the other families of exact solutions of Eq. (8) which are omitted for practicality.

Finally, substituting the solutions $G(\xi)$ of the Eq. (6) into Eq. (16) and simplifying, we obtain the following solutions:

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C - 1) \neq 0$ (or $B(C - 1) \neq 0$),

$$u_3^1(\xi) = \alpha_0 + \frac{3}{a + b + c} \times \left(\sqrt{\Delta} \tanh \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right) - \frac{3(4BC - A^2 - 4B)}{4(a + b + c)(1 - C)^2} \times \left(\sqrt{\Delta} \tanh \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right)^{-1}, \quad (49)$$

where $\xi = x + y + z - Vt$, α_0 , A , B and C are arbitrary constants.

$$u_3^2(\xi) = \alpha_0 + \frac{3}{a + b + c} \times \left(\sqrt{\Delta} \coth \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right) - \frac{3(4BC - A^2 - 4B)}{4(a + b + c)(1 - C)^2} \times \left(\sqrt{\Delta} \coth \left(\frac{1}{2} \sqrt{\Delta} \xi \right) \right)^{-1}, \quad (50)$$

$$u_3^3(\xi) = \alpha_0 + \frac{3}{a + b + c} \times \left\{ \sqrt{\Delta} \left(\tanh(\sqrt{\Delta} \xi) \pm i \sec h(\sqrt{\Delta} \xi) \right) \right\} - \frac{3(4BC - A^2 - 4B)}{4(a + b + c)(1 - C)^2} \times \left\{ \sqrt{\Delta} \left(\tanh(\sqrt{\Delta} \xi) \pm i \sec h(\sqrt{\Delta} \xi) \right) \right\}^{-1}, \quad (51)$$

Others families of exact solutions are omitted for the sake of simplicity.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C - 1) \neq 0$ (or $B(C - 1) \neq 0$),

$$u_3^{12}(\xi) = \alpha_0 - \frac{3}{a+b+c} \times \left(\sqrt{-\Delta} \tan \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right) + \frac{3(4BC - A^2 - 4B)}{4(a+b+c)(1-C)^2} \\ \times \left(\sqrt{-\Delta} \tan \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right)^{-1}, \quad (52)$$

$$u_3^{13}(\xi) = \alpha_0 + \frac{3}{a+b+c} \times \left(\sqrt{-\Delta} \cot \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right) - \frac{3(4BC - A^2 - 4B)}{4(a+b+c)(1-C)^2} \\ \times \left(\sqrt{-\Delta} \cot \left(\frac{1}{2} \sqrt{-\Delta} \xi \right) \right)^{-1}, \quad (53)$$

$$u_3^{14}(\xi) = \alpha_0 - \frac{3}{a+b+c} \times \left\{ \sqrt{-\Delta} \left(\tan(\sqrt{-\Delta} \xi) \pm \sec(\sqrt{-\Delta} \xi) \right) \right\} \\ + \frac{3(4BC - A^2 - 4B)}{4(a+b+c)(1-C)^2} \times \left\{ \sqrt{-\Delta} \left(\tan(\sqrt{-\Delta} \xi) \pm \sec(\sqrt{-\Delta} \xi) \right) \right\}^{-1}. \quad (54)$$

When $(C - 1) \neq 0$ and $A = B = 0$, the solution of Eq. (8) is

$$u_3^{25}(\xi) = \alpha_0 - \frac{6}{a+b+c} \times \left(\frac{A}{2} - \frac{1}{\xi + c_2} \right) + \frac{3(4BC - A^2 - 4B)}{2(a+b+c)(C-1)^2} \\ \times \left(\frac{A}{2} - \frac{1}{\xi + c_2} \right)^{-1}, \quad (55)$$

where c_2 is an arbitrary constant.

For convenience, the other families of exact solutions of Eq. (8) are omitted.

Graphical Presentation

Graph is a powerful tool for communication and describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given (Figs. 1, 2, 3 and 4). The graphs readily have shown the solitary wave form of the solutions.

Remark All the obtained solutions are checked by putting them back into the original equation and found correct.

Discussion If we replace A by $-A$ and B by $-B$ and put $C = 0$ in Eq. (6), then the novel (G'/G) -expansion method coincide with Akbar et al.'s generalized and improved (G'/G) -expansion method [29]. On the other hand if we put $k = 0$ in Eq. (5) and $C = 0$ in Eq. (6) then the method is identical to the improved (G'/G) -expansion method presented by Zhang et al. [31]. Again if we set $k = 0$, $C = 0$ and negative the exponents of (G'/G) are zero in Eq. (4), then the method turn out into the basic (G'/G) -expansion method introduced by Wang et al. [23]. At the end, if we put $C = 0$ in Eq. (6) and α_i ($i = 1, 2, 3, \dots, N$) are functions of x and t instead of constants then the method is transformed into the generalized the (G'/G) -expansion method developed by Zhang et al. [30]. Thus the methods presented in the Ref. [23, 29–31] are only special cases of the novel (G'/G) -expansion method.

Fig. 1 Soliton solution of Eq. (17) for different values of the parameters

Fig. 2 Soliton solution for Eq. (21) for different values of the parameters

Fig. 3 Soliton solution of Eq. (28) for different values of the parameters

Fig. 4 Soliton solution for Eq. (49) for different values of the parameters

Conclusion

The novel (G'/G) -expansion method is applied to the $(3 + 1)$ -dimensional Burgers equations and abundant exact traveling wave solutions are constructed of this equation. The obtained solutions are more general, and many known solutions are only a special case of them. Further, this study shows that the novel (G'/G) -expansion method is quite efficient and practically well suited to be used in finding exact solutions of NLEEs. We observe that, the basic (G'/G) -expansion method, the improve (G'/G) -expansion and the generalized and improved (G'/G) -expansion method are only special case of the novel (G'/G) -expansion method and thus the novel (G'/G) -expansion method would be a powerful mathematical tool for solving other NLEEs.

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