Generalized Exponential Estimators of Finite Population Mean Using Transformed Auxiliary Variables

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Abstract The new generalized exponential ratio cum dual to ratio type estimators are suggested using the transformation of Srivenkataramana (Biometrika 67:199–204, 1980) to the auxiliary variables for estimating the finite population mean. The expressions for mean square errors and biases of the proposed estimators have been derived up to first order of approximation. The conditions have been obtained for which the proposed estimators are more efficient than existing estimators. Empirical study is presented to evaluate the performance of proposed estimators.

Keywords Exponential type estimators · Population mean · Transformed auxiliary variables · Mean square error · Bias

Introduction

The use of auxiliary information has been widely discussed in literature in order to improve the precision of the estimators of population parameters. The ratio method of estimation which has been proposed by [4] is used to estimate the population mean of the study variable Y based on linear positive correlation between study and auxiliary variable. Various types of estimators using auxiliary information are available in literature such as ratio, product and regression type estimators. [13] first time used the mean of the non sampled units of population in the ratio estimation and the new estimator was called dual to ratio estimator. [2] were the first to develop an exponential estimator using auxiliary information for estimating the finite population mean of the variable under study. [12] used information on auxiliary variable and developed ratio-cum-product estimator. [12] worked on ratio-cum-dual to ratio estimator for

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estimating finite population mean of the study variable. [6, 14] and [11] have proposed the exponential type estimators in various sampling designs.

The aim of this paper is to develop new improved estimators of population mean in simple random sampling without replacement (SRSWOR).

Let a population S of a finite size N consists of the units $S_1, S_2, \dots, S_N$. Let we denote the population mean of the study variate Y by $\bar{Y}$ and let $\bar{X}$ and $\bar{Z}$ be the population means of the supplementary variates X and Z respectively. A random sample is selected of size n with the help of a simple random sampling without replacement procedure. Let the study and the two supplementary variates are denoted by y, x and z with sample means $\bar{y}, \bar{x}$ and $\bar{z}$, respectively (1).

We also have following assumptions:

$$\left. \begin{aligned} \theta &= \frac{1}{n} - \frac{1}{N} = \frac{N-n}{Nn}, g = \frac{n}{N-n} \\ \bar{e}_y &= \frac{\bar{y}-\bar{Y}}{\bar{Y}}, \bar{e}_z = \frac{\bar{z}-\bar{Z}}{\bar{Z}}, \bar{e}_x = \frac{\bar{x}-\bar{X}}{\bar{X}}, \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} E(\bar{e}_x) &= 0, E(\bar{e}_y) = 0, E(\bar{e}_z) = 0, \\ E(\bar{e}_y^2) &= \theta C_y^2, E(\bar{e}_x^2) = \theta C_x^2, E(\bar{e}_z^2) = \theta C_z^2, \\ E(\bar{e}_x \bar{e}_y) &= \theta \rho_{xy} C_x C_y, \\ E(\bar{e}_x \bar{e}_z) &= \theta \rho_{xz} C_x C_z, \\ E(\bar{e}_y \bar{e}_z) &= \theta \rho_{yz} C_y C_z. \end{aligned} \right\} \quad (2)$$

$$C_y = \frac{S_y}{\bar{Y}}, \quad K_{ij} = \rho_{ij} \frac{C_i}{C_j}, \quad (3)$$

where $i = x, y, z$ and $j = x, y, z$. Also, let

$$x_{ti} = (1 + g) \bar{X} - g x_i, \quad (4)$$

and

$$z_{ti} = (1 + g) \bar{Z} - g z_i, \quad (5)$$

where $i = 1, 2, \dots, N$, then

$$\bar{x}_t = (1 + g) \bar{X} - g \bar{x}, \quad (6)$$

and

$$\bar{z}_t = (1 + g) \bar{Z} - g \bar{z}, \quad (7)$$

are also unbiased estimators of $\bar{X}$ and $\bar{Z}$ respectively.

(i) [4] suggested classical ratio estimator for population mean is given by

$$t_1 = \bar{y} \left[\frac{\bar{X}}{\bar{x}} \right]. \quad (8)$$

(ii) The product estimator of population mean, proposed by [9], is given by

$$t_2 = \bar{y} \left[\frac{\bar{x}}{\bar{X}} \right]. \quad (9)$$

(iii) [13] developed a dual-to-classical ratio type estimator is given by

$$t_3 = \bar{y} \left(\frac{\bar{x}_t}{\bar{X}} \right), \quad (10)$$

(iv) [3] developed a dual-to-product type estimator is given by

$$t_4 = \bar{y} \left(\frac{\bar{X}}{\bar{x}_t} \right), \quad (11)$$

(v) [2] provided an exponential ratio type estimator is given by

$$t_5 = \bar{y} \exp \left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right]. \quad (12)$$

(vi) [2] also suggested an exponential product type estimator is given by

$$t_6 = \bar{y} \exp \left[\frac{\bar{x} - \bar{X}}{\bar{X} + \bar{x}} \right]. \quad (13)$$

(vii) [10] suggested the following estimator

$$t_7 = \left(\frac{\bar{y}}{\bar{x}} \bar{X} \right) \left(\frac{\bar{Z}}{\bar{z}} \right). \quad (14)$$

(viii) [12] constructed a dual-to-exponential ratio type estimator as

$$t_8 = \bar{y} \exp \left[\frac{\bar{x}_t - \bar{X}}{\bar{x}_t + \bar{X}} \right]. \quad (15)$$

(ix) [12] also suggested dual-to-exponential product type estimator as

$$t_9 = \bar{y} \exp \left[\frac{\bar{X} - \bar{x}_t}{\bar{X} + \bar{x}_t} \right]. \quad (16)$$

(x) [15] developed a ratio-cum-dual to exponential ratio type estimator as

$$t_{10} = \bar{y} \left[\alpha \exp \left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right) + (1 - \alpha) \exp \left(\frac{\bar{x}_t - \bar{X}}{\bar{x}_t + \bar{X}} \right) \right]. \quad (17)$$

Proposed Estimators

The proposed estimators in the generalized form using the transformed auxiliary variables are given by

$$(i) \quad t_{nl1G} = \omega \bar{y} + (1 - \omega) \bar{y} \left(\frac{\bar{x}_t}{\bar{X}} \right) \exp \left[\alpha \frac{\bar{z}_t - \bar{Z}}{\bar{z}_t + \bar{Z}} \right], \quad (18)$$

$$(ii) \quad t_{nl2G} = \delta \bar{y} \left(\frac{\bar{X}}{\bar{x}} \right)^\beta \exp \left[\gamma \frac{\bar{Z} - \bar{z}}{\bar{z} + \bar{Z}} \right] + (1 - \delta) \bar{y} \left(\frac{\bar{x}_t}{\bar{X}} \right)^\beta \exp \left[\gamma \frac{\bar{z}_t - \bar{Z}}{\bar{z}_t + \bar{Z}} \right], \quad (19)$$

respectively.

where α , β and γ are generalized constants, ω and δ are optimization constants.

Using the notations in (1), the estimator (18) may be written as

$$t_{nl1G} = \omega \bar{Y} (1 + \bar{e}_y) + (1 - \omega) \bar{Y} (1 + \bar{e}_y) (1 - g \bar{e}_x) \exp \left[\alpha \frac{-\bar{Z} g \bar{e}_z}{2 \bar{Z} (1 - \frac{1}{2} g \bar{e}_z)} \right]. \quad (20)$$

Neglecting the terms with power two or greater, it is possible to re-write t_{nl1G} as

$$t_{nl1G} \approx \omega \bar{Y} + \omega \bar{Y} \bar{e}_y + (1 - \omega) \bar{Y} (1 + \bar{e}_y) (1 - g \bar{e}_x) \left(1 - \frac{\alpha}{2} g \bar{e}_z - \frac{\alpha}{8} g^2 \bar{e}_z^2 \right). \quad (21)$$

or

$$E(t_{nl1G} - \bar{Y}) \approx \bar{Y} \theta \left\{ -\frac{\alpha}{8} g^2 C_z^2 + \frac{\alpha}{2} g^2 C_z C_x \rho_{xz} - \frac{\alpha}{2} g C_y C_z \rho_{yz} - g C_y C_x \rho_{yx} \right. \\ \left. + \frac{\omega \alpha}{8} g^2 C_z^2 - \frac{\alpha}{2} \omega g^2 C_z C_x \rho_{zx} + \frac{\alpha}{2} \omega g C_y C_z \rho_{yz} + \omega g C_y C_x \rho_{yx} \right\}. \quad (22)$$

The bias of t_{nl1G} is given by

$$Bias(t_{nl1G}) \approx \bar{Y} \theta g^2 \frac{\alpha(\omega-1)}{2} \left[\frac{C_z^2}{4} - C_z C_x \rho_{xz} + \frac{C_y C_z \rho_{yz}}{g} + \frac{2 C_y C_x \rho_{yx}}{\alpha g} \right]. \quad (23)$$

In order to derive the mean square error of t_{nl1G} , again using (20) and after squaring and taking expectation on both sides, we have

$$E(t_{nl1G} - \bar{Y})^2 \approx \bar{Y}^2 E \left\{ \bar{e}_y^2 + \frac{\alpha^2}{4} g^2 \bar{e}_z^2 (\omega-1)^2 + (\omega-1)^2 g^2 \bar{e}_x^2 \right. \\ \left. + (\omega-1) \alpha g \bar{e}_y \bar{e}_z + (\omega-1)^2 \alpha g^2 \bar{e}_x \bar{e}_z + 2(\omega-1) g \bar{e}_y \bar{e}_x \right\}. \quad (24)$$

or

$$MSE(t_{nl1G}) \approx \bar{Y}^2 \theta \left\{ C_y^2 + \frac{\alpha^2}{4} g^2 C_z^2 (\omega-1)^2 + (\omega-1)^2 g^2 C_x^2 + (\omega-1) \alpha g C_y C_z \rho_{yz} \right. \\ \left. + (\omega-1)^2 \alpha g^2 C_x C_z \rho_{xz} + 2(\omega-1) \alpha g C_y C_x \rho_{yx} \right\}. \quad (25)$$

Differentiating partially with respect to ω , the optimal values of ω is obtained as

$$\omega = 1 - \frac{2(2C_y C_x \rho_{yx} + \alpha g C_y C_z \rho_{yz})}{g^2 (\alpha^2 C_z^2 + 4C_x^2 + 4\alpha C_x C_z \rho_{xz})} = \omega_o \quad (26)$$

when the optimal value (26) is replaced in (25), the minimum mean square error of t_{nl1G} , denoted by $MSE_{\min}(t_{nl1G})$, is given by

$$MSE_{\min}(t_{nl1G}) \approx \bar{Y}^2 \theta \left\{ C_y^2 + \frac{\alpha^2}{4} g^2 C_z^2 (\omega_o - 1)^2 + (\omega_o - 1)^2 g^2 C_x^2 \right. \\ \left. + (\omega_o - 1) \alpha g C_y C_z \rho_{yz} + (\omega_o - 1)^2 \alpha g^2 C_x C_z \rho_{xz} \right. \\ \left. + 2(\omega_o - 1) \alpha g C_y C_x \rho_{yx} \right\}. \quad (27)$$

Similarly, the bias and mean square error for the estimator (19) may be obtained as

$$Bias(t_{nl2G}) \approx \bar{Y} \theta \left\{ (\delta g - \delta - g) \left(\beta C_y C_x \rho_{yx} + \frac{\gamma}{2} C_y C_z \rho_{yz} \right) \right. \\ \left. + \frac{C_z^2}{8} (\delta \gamma (\gamma + 2) + \gamma (\gamma - 2) (1 - \delta) g^2) \right. \\ \left. + \frac{\beta}{2} C_z C_x \rho_{zx} (\delta \gamma + g^2 \gamma - \delta \gamma g^2) \right\}. \quad (28)$$

$$MSE(t_{nl2G}) \approx \bar{Y}^2 \theta \left[C_y^2 + \tau^2 \left\{ \beta^2 C_x^2 + \frac{\gamma^2}{4} C_z^2 + \beta \gamma C_z C_x \rho_{zx} \right\} \right. \\ \left. + \tau (2\beta C_y C_x \rho_{yx} + \gamma C_y C_z \rho_{yz}) \right]. \quad (29)$$

Table 1 special cases for estimator t_{nl1G}

Contant	ω	α	Estimators
	1		$\bar{y}$ mean per unit estimator
	0	0	$t_{10} = \bar{y} \left(\frac{\bar{y}_t}{\bar{X}} \right)$ [13]
	0	1	$t_{n1} = \bar{y} \left(\frac{\bar{y}_t}{\bar{X}} \right) \exp \left[\frac{\bar{z}_t - \bar{Z}}{\bar{Z} + \bar{z}_t} \right]$ dual-to-product cum exponential estimator
		1	$t_{n2} =$ $\omega \bar{y} + (1 - \omega) \bar{y} \left(\frac{\bar{y}_t}{\bar{X}} \right) \exp \left[\frac{\bar{z}_t - \bar{Z}}{\bar{z}_t + \bar{Z}} \right]$ dual-to-ratio type estimator

The minimum mean square error of t_{nl2G} , denoted by $MSE_{\min}(t_{nl2G})$ is given by

$$MSE_{\min}(t_{nl2G}) \approx \bar{Y}^2 \theta \left[C_y^2 + \tau_o^2 \left\{ \beta C_x^2 + \frac{\gamma^2}{4} C_z^2 + \beta \gamma C_z C_x \rho_{zx} \right\} + \tau_o (2\beta C_y C_x \rho_{yx} + \gamma C_y C_z \rho_{yz}) \right]. \quad (30)$$

$$\text{where, } \tau = \frac{-2\beta C_y C_x \rho_{yx} - \gamma C_y C_z \rho_{yz}}{2(C_x^2 + \frac{\gamma^2}{4} C_z^2 + \beta \gamma C_z C_x \rho_{zx})} = \tau_o$$

Some Special Cases For Generalized Estimators

In this section, the special cases of the proposed generalized estimators are discussed. Using the various values of the constants the special cases for estimator t_{nl1G} are obtained in Table 1.

The special cases for estimator t_{nl2G} are obtained in Table 2 using various values of constants.

Efficiency Comparisons

In this section, the theoretical comparison of the suggested estimators with some available estimators is presented. It is known that the variance of the sample mean estimator, $\bar{y}$, under SRSWOR is

$$V(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N} \right) S_y^2. \quad (31)$$

then

$$MSE(t_{n2}) \leq Var(\bar{y}).$$

$$\rho_{yx} \leq \frac{C_y C_z \rho_{yz} + (\omega_0 - 1) g (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})}{2 C_y C_x}. \quad (32)$$

$$MSE(t_{n2}) \leq MSE(t_1).$$

$$\rho_{yx} \leq \frac{(1 - \omega_0) C_y C_z \rho_{yz} - (1 - \omega_0)^2 g^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx}) + C_x^2}{C_y C_x (\omega_0 g - g + 2)}. \quad (33)$$

$$MSE(t_{n2}) \leq MSE(t_3).$$

Table 2 special cases for estimator t_{nl2G}

δ	β	γ	Estimators
0			$t_{n3} = \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right)^\beta \exp \left[\gamma \frac{\bar{z}_t - \bar{z}}{\bar{z}_t + \bar{z}} \right]$. dual-to-ratio type estimator
1			$t_{n4} = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right)^\beta \exp \left[\gamma \frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right]$. ratio type estimator
1	1	1	$t_{n5} = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right]$. ratio type exponential estimator
	1	1	$\bar{y}$ mean per unit estimator
1	-1	-1	$t_{n6} = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right]$ product type exponential estimator
0	1	1	$t_{n7} = \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right) \exp \left[\frac{\bar{z}_t - \bar{z}}{\bar{z}_t + \bar{z}} \right]$ ratio type exponential estimator
1	-1	-1	$t_{n8} = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right]$ product type exponential estimator
0	1	1	$t_{n9} = \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right) \exp \left[\frac{\bar{z}_t - \bar{z}}{\bar{z}_t + \bar{z}} \right]$ dual to exponential ratio type estimator
1	-1	-1	$t_{n10} = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right]$ Product type exponential estimator
	1	1	$t_{n11} = \delta \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right] + (1 - \delta) \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right) \exp \left[\frac{\bar{z}_t - \bar{z}}{\bar{z}_t + \bar{z}} \right]$ Ratio type estimator
	-1	-1	$t_{n12} = \delta \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right) \exp \left[\frac{\bar{z} - \bar{z}_t}{\bar{z} + \bar{z}_t} \right] + (1 - \delta) \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right) \exp \left[\frac{\bar{z}_t - \bar{z}}{\bar{z}_t + \bar{z}} \right]$
0	1	0	$t_3 = \bar{y} \left(\frac{\bar{x}_t}{\bar{x}} \right)$ [13] estimator
0	-1	0	$t_4 = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right)$ [3] product type estimator
1	1	0	[4] ratio estimator $t_1 = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right)$
1	-1	0	$t_2 = \bar{y} \left(\frac{\bar{x}}{\bar{x}_t} \right)$. [9] estimator

$$\rho_{yx} \leq \frac{(1 - \omega_0) C_y C_z \rho_{yz} - g(1 - \omega_0)^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx}) + g C_x^2}{(\omega_0 + 1) C_y C_x}. \quad (34)$$

$$MSE(t_{n2}) \leq MSE(t_5)$$

$$\rho_{yx} \leq \frac{(1 - \omega) g C_y C_z \rho_{yz} - (\omega - 1)^2 g^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})}{2 C_y C_x (\omega g - g + \frac{1}{2})}. \quad (35)$$

$$MSE(t_{n2}) \leq MSE(t_7).$$

$$\rho_{yx} \leq \frac{C_x C_z \rho_{zx} (g^2 (1 - \omega_0)^2 - 2) - (g^2 (1 - \omega_0)^2 - 1) (\frac{1}{4} C_z^2 + C_x^2)}{2 (1 + \omega_0 g - g) C_y C_z} + \frac{\rho_{yz}}{2}. \quad (36)$$

$$MSE(t_{n2}) \leq MSE(t_8).$$

$$\rho_{yx} \leq \frac{(\omega_0 - 1) g (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx}) - C_y C_z \rho_{yz}}{C_y C_x} + \frac{C_x}{4 C_y (\omega_0 - 1)}. \quad (37)$$

$$MSE(t_{n2}) \leq MSE(t_{10}).$$

$$\rho_{yx} \leq \frac{(1 - \omega_0) g C_z C_y \rho_{yz} - g^2 (1 - \omega_0)^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})}{C_y C_x (\alpha - 2g + \omega g)}$$

$$+ \frac{\alpha^2 C_x^2}{4}. \quad (38)$$

$$MSE(t_{n11}) \leq Var(\bar{y}).$$

$$\rho_{yx} \leq \frac{-[C_y C_z \rho_{yz} + \tau_0 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})]}{2 C_y C_x}. \quad (39)$$

$$MSE(t_{n11}) \leq MSE(t_1).$$

$$\rho_{yx} \leq \frac{[C_x^2 - \tau_0 C_y C_z \rho_{yz} - \tau_0^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})]}{2(\tau_0 + 1) C_y C_x}. \quad (40)$$

$$MSE(t_{n11}) \leq MSE(t_3).$$

$$\rho_{yx} \leq \frac{[g^2 C_x^2 - \tau_0 C_y C_z \rho_{yz} - \tau_0^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})]}{2(\tau_0 + g) C_y C_x}. \quad (41)$$

$$MSE(t_{n11}) \leq MSE(t_5).$$

$$\rho_{yx} \leq \frac{\frac{C_x^2}{4} - \tau_0 C_y C_z \rho_{yz} - \tau_0^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx})}{(2\tau_0 + 1) C_y C_x}. \quad (42)$$

$$MSE(t_{n11}) \leq MSE(t_7).$$

$$\rho_{yx} \leq \frac{(\tau_0 - 1) C_x}{2 C_y} + \frac{(1 - \frac{1}{4} \tau_0^2) C_z^2 - (\tau_0^2 - 2) C_z C_x \rho_{zx} - (\tau_0 + 2) C_y C_z \rho_{yz}}{2(\tau_0 + 1) C_y C_x}. \quad (43)$$

$$MSE(t_{n11}) \leq MSE(t_8).$$

$$\rho_{yx} \leq \frac{\frac{g^2 C_x^2}{4} - \tau_0^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx}) - \tau_0 C_y C_z \rho_{yz}}{(2\tau_0 + g) C_y C_x}. \quad (44)$$

$$MSE(t_{n11}) \leq MSE(t_{10}).$$

$$\rho_{yx} \leq \frac{\frac{\alpha^2 C_x^2}{4} - \tau_0^2 (C_x^2 + \frac{1}{4} C_z^2 + C_z C_x \rho_{zx}) - \tau_0 C_y C_z \rho_{yz}}{(2\tau_0 - \alpha) C_y C_x}. \quad (45)$$

$$MSE(t_{n12}) \leq MSE(\bar{y}).$$

$$\rho_{yx} \leq \frac{-\lambda_0 (C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}) - C_y C_z \rho_{yz}}{2 C_x C_y}. \quad (46)$$

$$MSE(t_{n12}) \leq MSE(t_4).$$

$$\rho_{yx} \leq \frac{g^2 C_z^2 - \lambda_0^2 (C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}) - C_y C_z \rho_{yz} (\lambda_0 + 2g)}{2\lambda_0 C_x C_y}. \quad (47)$$

$$MSE(t_{n12}) \leq MSE(t_6).$$

$$\rho_{yx} \leq \frac{\frac{C_z^2}{4} - \lambda_0^2 (C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}) - C_y C_z \rho_{yz} (\lambda_0 - 1)}{2\lambda_0 C_x C_y}. \quad (48)$$

$$MSE(t_{n12}) \leq MSE(t_9).$$

$$\rho_{yx} \leq \frac{\frac{g^2 C_z^2}{4} - \lambda_0^2 (C_x^2 + \frac{C_z^2}{4} + C_x C_z \rho_{xz}) - C_y C_z \rho_{yz} (\lambda_0 - g)}{2\lambda_0 C_x C_y}. \quad (49)$$

When the conditions (32), (33), (34), (35), (36), (37) and (38) are satisfied, our suggested estimator t_{n2} is more efficient as compared to $\bar{y}$, t_1 , t_3 , t_5 , t_7 , t_8 and t_{10} respectively.

When the condition (39), (40), (41), (42), (43), (44) and (45) are satisfied, our suggested estimator t_{n11} is more efficient as compare to $\bar{y}$, t_1 , t_3 , t_5 , t_7 , t_8 and t_{10} respectively. When the condition (46), (47), (48) and (49) are satisfied, our suggested estimator t_{n12} is more efficient as compare to $\bar{y}$, t_4 , t_6 and t_9 respectively.

Empirical Study

We have carried out an empirical study for the efficiency comparison of the new suggested estimators with some estimators available in literature with the help of the available data of some real populations. We have compared our proposed estimators with the classical ratio and product estimators, [2, 3, 10, 13] and [15] estimator. The description of populations is as follows:

Population I: [Source: [1]]

Y : Head length of second son.

X : Head length of first son.

Z : Head breadth of first son.

The required parameters of the population are:

$$\begin{aligned}\bar{Y} &= 183.84, & C_y &= 0.0546, & \rho_{yx} &= 0.7108, & N &= 25. \\ \bar{X} &= 185.72, & C_x &= 0.0526, & \rho_{yz} &= 0.6932, & n &= 5. \\ \bar{Z} &= 151.122, & C_z &= 0.0488, & \rho_{zx} &= 0.7346,\end{aligned}$$

Population II: [Source: [5]]

Y : Number of “placebo” children.

X : Number of paralytic polio cases in the “not inoculated” group

Z : Number of paralytic polio cases in the placebo group

The required parameters of the population are:

$$\begin{aligned}\bar{Y} &= 4.92, & C_y^2 &= 1.0248, & \rho_{yx} &= 0.6430, & N &= 34. \\ \bar{X} &= 2.59, & C_x^2 &= 1.1492, & \rho_{yz} &= 0.7326, & n &= 5. \\ \bar{Z} &= 2.91, & C_z^2 &= 1.5175, & \rho_{zx} &= 0.6837,\end{aligned}$$

Population III: [Source: Population census report of Multan district]

Y : Population matrix and above.

X : Primary but below matrix.

Z : Population of both sexes.

The required parameters of the population are:

$$\begin{aligned}\bar{Y} &= 646.215, & C_y^2 &= 2.277081, & \rho_{yx} &= 0.623, & N &= 424. \\ \bar{X} &= 4533.981, & C_x^2 &= 1.800964, & \rho_{yz} &= 0.907, & n &= 10. \\ \bar{Z} &= 325.0325, & C_z^2 &= 1.782225, & \rho_{zx} &= 0.682,\end{aligned}$$

Population IV: [Source: [7], p. 433]

Table 3 Percent relative efficiencies of ratio type estimators with respect to mean per unit estimator

Estimators	Population		
	1	2	3
$\bar{y}$	100.00	100.00	100.00
t_1	100.01	121.48	100.00
t_3	135.11	194.49	102.70
t_5	172.37	208.00	155.35
t_7	72.29	45.05	107.09
t_8	116.60	144.34	101.34
t_{10}	43.33	39.92	46.43
t_{n2}	225.37	234.96	249.43
t_{n11}	195.20	212.00	216.97

The bold values are the percentage relative efficiencies of proposed estimators with respect to mean per unit estimator

Y : Average miles per gallon

X : Top speed, miles per hour

Z : Cubic feet of cab space

The required parameters of the population are:

$$\bar{Y} = 33.83457, \quad C_y^2 = 0.088324, \quad \rho_{yx} = -0.69079, \quad N = 81.$$

$$\bar{X} = 112.4568, \quad C_x^2 = 0.015765, \quad \rho_{yz} = -0.36831, \quad n = 13.$$

$$\bar{Z} = 98.76543, \quad C_z^2 = 0.050987, \quad \rho_{zx} = -0.04265,$$

Population V: [Source: [8], p. 235]

Y : Negative logarithm of molefraction solubility

X : Dispersion partial solubility

Z : Dipolar partial solubility

The required parameters of the population are:

$$\bar{Y} = 0.272228, \quad C_y = 0.609188, \quad \rho_{yx} = -0.46706, \quad N = 26.$$

$$C_x = 0.654054, \quad \rho_{yz} = -0.80854, \quad n = 5.$$

$$C_z = 0.871866, \quad \rho_{zx} = 0.717483,$$

Results and Conclusion

The percentage relative efficiencies of the ratio type estimators (t_{n2} , t_{n11}) and product type estimator (t_{n12}), deduced from the proposed estimators is presented in Tables 3 and 4, respectively.

In this paper, we have suggested new generalized estimators for single phase sampling using two transformed auxiliary variables. The comparative study of t_{n2} , t_{n11} with the conventional ratio estimator, [2, 10, 13] and the [15] is given in Table 3. It is clearly indicated that the suggested estimators t_{n2} , t_{n11} are competent estimators as compared to well known estimators available in the literature.

Table 4 Percent relative efficiencies of product type estimators with respect to mean per unit estimator

	Estimators	Population	
		4	5
	$\bar{y}$	100.00	100.00
	t_2	98.27	95.08
	t_4	109.39	106.37
	t_6	115.30	107.95
	t_9	105.71	103.69
	t_{n12}	142.45	237.87

The bold values are the percentage relative efficiencies of proposed estimators with respect to mean per unit estimator

The population 4–5 in Table 4 have been utilized for the comparative study among the product type estimators as there is negative linear relationship between study variable and auxiliary variables. It is shown in Table 4 that t_{n12} is more efficient than the conventional product estimator, [2,3] and the [12] estimator.

According to this we may conclude that proposed generalized exponential ratio type estimators are efficient in comparison with all available other estimators. Similarly for populations 4 and 5 suggested generalized form of exponential product type estimator has shown better performance as compare to available product type estimators.

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