System of Differential Equation with Initial Value as Triangular Intuitionistic Fuzzy Number and its Application

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Abstract In this paper, we solve a system of differential equation of first order with initial value as triangular intuitionistic fuzzy number. Two different cases are discussed: (i) coefficient is positive crisp number, (ii) coefficient is negative crisp number. Examples are given. We apply these procedures in Arm Race Model. Also we valuation, ambiguities and rank of fuzzy solution and defuzzify the solution.

Keywords Fuzzy sets · Fuzzy differential equation · Initial value problem · Arm race model · Valuation, ambiguity, ranking and defuzzification of fuzzy number

Mathematics Subject Classification 03E72 · 34A07 · 34A12

Introduction

Zadeh [1] and Dubois and Parade [2] were the first who introduced the conception based on fuzzy number and fuzzy arithmetic. Generalizations of fuzzy sets theory [1] is considered to be one of Intuitionistic fuzzy set (IFS). Later on Atanassov generalized the concept of fuzzy set and introduced the idea of intuitionistic fuzzy set [3–5]. The fuzzy set considers only the degree of belongingness and non belongingness. fuzzy set theory does not incorporate the degree of hesitation (i.e., degree of non-determinacy defined as, $1 - \text{sum of membership function and non-membership function}$). To handle such situations, Atanassov [4] explored the concept of fuzzy set theory by intuitionistic fuzzy set (IFS) theory. The degree of acceptance in fuzzy sets is only considered, otherwise IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one [4].

Further improvement of IFS theory, together with intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, an intuitionistic fuzzy approach to artificial

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intelligence, and intuitionistic fuzzy generalized nets can be found in [6]. So then they are very necessary and powerful tool in modeling imprecision, valuable applications of IFSs have been flourished in many different fields, along pattern recognition [7], medical diagnosis [8], drug selection [9], microelectronic fault analysis [10], weight assessment [11], and decision-making problems [12, 13]

The ranking of fuzzy numbers is a significant issue in the discussion of fuzzy set theory. Methodically to rank fuzzy numbers, one fuzzy number needs to be compared with the others, but it is arduous to determine clearly which of them is larger or smaller. Many methods have been raised in literature to rank fuzzy numbers (for example, see, [14–17]). Noticeably, one thing is obvious that there exists no uniquely best method for comparing fuzzy numbers. The ranking of intuitionistic fuzzy numbers plays a main role in real life problems. Forthwith few methods for ranking IFNs has also been inaugurated [18–20].

It is seen that in recent years the topic of fuzzy differential equations (FDEs) has been rapidly grown. In the year 1987, the term “fuzzy differential equation (FDE)” was introduced by Kandel and Byatt [21]. There are different approaches to discuss the FDEs: (i) using the Hukuhara derivative of a fuzzy number valued function, (ii) Hüllermeier [22] and Diamond and Watson [23] suggested a different formulation for the fuzzy initial value problems (FIVP) based on a family of differential inclusions, (iii) In [24], Bede et al. defined generalized differentiability of the fuzzy number valued functions and studied FDE, (iv) applying a parametric representation of fuzzy numbers, Chen [25] established a new definition for the differentiation of a fuzzy valued function and use it in FDE.

First order system of fuzzy differential equations is important among all the fuzzy differential equation. There are many approaches to solve the SFDEs. Buckley et al. [26] solving the linear system of first order ordinary differential equations with fuzzy initial conditions by extension principle. Numerical solution of SFDE is developed by Fard [27]. The geometric approach is developed by Gasilova et al. in [28] and series solution is developed by Hashemi et al. [29]. There are other meny approaches to solve this SFDE (see [30–32]).

There are only few papers such as [33–37] in which intuitionistic fuzzy number are applied in differential equation.

Fuzzy differential equations play a significant role in the field of biology, engineering, physics as well as among other field of science. For example, in population models [38], civil engineering [39], bioinformatics and computational biology [40], economic model [41], quantum optics and gravity [42], modeling hydraulic [43], HIV model [44], decay model [45], predator–prey model [46], population dynamics model [47], friction model [48], growth model [49], bacteria culture model [50], bank account and drug concentration problem [51], barometric pressure problem [52], concentration problem [53], weight loss and oil production model [37]. System of fuzzy differential equation (SFDE) is the one of the most important differential equation for uncertainty modeling.

In spite of above mentioned developments, following lacunas still exists in the formulation and solution of inventory models of complementary items. Which are summarized below.

- (i) System of differential equation is solved with initial value as triangular intuitionistic fuzzy number which is not done previously.
- (ii) Finding the valuation and ambiguity and ranking of a solution of fuzzy differential equation which help us to compare between two fuzzy results.
- (iii) Application of intuitionistic fuzzy differential equation is done here. Very popular model namely Arm race model are solved intuitionistic fuzzy environment.
- (iv) Defuzzification of the corresponding solution is done here.

The paper is organized as follows: in “Preliminary Concepts” section preliminaries and basic concepts on intuitionistic fuzzy number, fuzzy derivative, valuation, ambiguity, ranking and defuzzification methods are given. In “System of Differential Equation with Initial Value as TIFN” section we the proposed system of differential equation with initial value as triangular intuitionistic fuzzy number. “Solution of System of Fuzzy Differential Equation” section contains the solution of said differential equation for two different cases. First case is coefficient is positive crisp number and second case is negative crisp number. In “Application” section an important application namely arm race model is illustrated. Valuation, ambiguity, ranking and defuzzification of the solution are find in the “Valuation, Ambiguity, Ranking and Defuzzification of Solution” section . Finally conclusions and future research scope of this article are drawn in last section, “Conclusion” section.

Preliminary Concepts

Definition *Fuzzy set* A fuzzy set $\tilde{A}$ is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0, 1]\}$. In the pair $(x, \mu_{\tilde{A}}(x))$ the first element x belong to the classical set A , the second element $\mu_{\tilde{A}}(x)$, belong to the interval $[0, 1]$, called membership function.

Definition *Height* The height $h(\tilde{A})$, of a fuzzy set $\tilde{A} = (x, \mu_{\tilde{A}}(x) : x \in X)$, is the largest membership grade obtained by any element in that set i.e. $h(\tilde{A}) = \sup \mu_{\tilde{A}}(x)$.

Definition *Support of fuzzy set* The support of fuzzy set $\tilde{A}$ is the set of all points x in X such that $\mu_{\tilde{A}}(x) > 0$ i.e., $\text{support}(\tilde{A}) = \{x | \mu_{\tilde{A}}(x) > 0\}$.

Definition *Convex fuzzy sets* A fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\} \subseteq X$ is called convex fuzzy set if all A_α for every $\alpha \in [0, 1]$ are convex sets i.e. for every element $x_1 \in A_\alpha$ and $x_2 \in A_\alpha$ and $\lambda x_1 + (1 - \lambda)x_2 \in A_\alpha \quad \forall \lambda \in [0, 1]$. Otherwise the fuzzy set is called non-convex fuzzy set.

Definition *α -cut of a fuzzy set* The α -level set (or interval of confidence at level α or α -cut) of the fuzzy set $\tilde{A}$ of universe X is a crisp set A_α that contains all the elements of X that have membership values in A greater than or equal to α i.e. $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\} \quad \forall \alpha \in [0, 1]$.

Definition *Strong α -cut of a fuzzy set* Strong α -cut is denoted by $A_\alpha^{\text{strong}} = \{x : \mu_{\tilde{A}}(x) > \alpha\} \quad \forall \alpha \in [0, 1]$

Definition *Fuzzy number* A fuzzy set $\tilde{A}$, defined on the universal set of real number R , is said to be a fuzzy number if its possess at least the following properties:

- (i) $\tilde{A}$ is convex.
- (ii) $\tilde{A}$ is normal i.e., $\exists x_0 \in R$ such that $\mu_{\tilde{A}}(x_0) = 1$.
- (iii) $\mu_{\tilde{A}}(x)$ is piecewise continuous.
- (iv) A_α must be closed interval for every $\alpha \in [0, 1]$.
- (v) The support of $\tilde{A}$, i.e., $\text{support}(\tilde{A})$ must be bounded.

Definition *Positive and negative fuzzy number* A fuzzy number $\tilde{A}$ is called positive (or negative), denoted by $\tilde{A} > 0$ (or $\tilde{A} < 0$), if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0, \forall x < 0 (x > 0)$.

Definition *Intuitionistic fuzzy set* Let a set X be fixed. An IFS $\tilde{A}^i$ in X is an object having the form $\tilde{A}^i = \{ \langle x, \mu_{\tilde{A}^i}(x), \nu_{\tilde{A}^i}(x) \rangle : x \in X \}$, where the $\mu_{\tilde{A}^i}(x) : X \rightarrow [0, 1]$ and

$\vartheta_{\tilde{A}^i}(x) : X \rightarrow [0, 1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set $\tilde{A}^i$, which is a subset of X , for every element of $x \in X$, $0 \leq \mu_{\tilde{A}^i}(x) + \vartheta_{\tilde{A}^i}(x) \leq 1$.

Definition *Intuitionistic fuzzy number* An IFN $\tilde{A}^i$ is defined as follows

- (i) an intuitionistic fuzzy subject of real line
- (ii) normal, i.e., there is any $x_0 \in R$ such that $\mu_{\tilde{A}^i}(x_0) = 1$ (so $\vartheta_{\tilde{A}^i}(x_0) = 0$)
- (iii) a convex set for the membership function $\mu_{\tilde{A}^i}(x)$, i.e.,

$$\mu_{\tilde{A}^i}(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_{\tilde{A}^i}(x_1), \mu_{\tilde{A}^i}(x_2)) \quad \forall x_1, x_2 \in R, \lambda \in [0, 1]$$

- (iv) a concave set for the non-membership function $\vartheta_{\tilde{A}^i}(x)$, i.e.,

$$\vartheta_{\tilde{A}^i}(\lambda x_1 + (1 - \lambda)x_2) \leq \max(\vartheta_{\tilde{A}^i}(x_1), \vartheta_{\tilde{A}^i}(x_2)) \quad \forall x_1, x_2 \in R, \lambda \in [0, 1].$$

Definition *Triangular intuitionistic fuzzy number* A TIFN $\tilde{A}^i$ is a subset of IFN in R with following membership function and non membership function as follows:

$$\mu_{\tilde{A}^i}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq a_2, \\ \frac{a_3-x}{a_3-a_2} & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \vartheta_{\tilde{A}^i}(x) = \begin{cases} \frac{a_2-x}{a_2-a'_1} & \text{for } a'_1 \leq x \leq a_2 \\ \frac{x-a'_2}{a'_3-a'_2} & \text{for } a_2 \leq x \leq a'_3 \\ 1 & \text{otherwise} \end{cases}$$

where $a'_1 \leq a_1 \leq a_2 \leq a_3 \leq a'_3$ and TIFN is denoted by $\tilde{A}_{TIFN}^i = (a_1, a_2, a_3; a'_1, a_2, a'_3)$

Note 1 Here $\mu_{\tilde{A}^i}(x)$ increases with constant rate for $x \in [a_1, a_2]$ and decreases with constant rate for $x \in [a_2, a_3]$ but $\vartheta_{\tilde{A}^i}(x_1)$ decreases with constant rate for $x \in [a'_1, a_2]$ and increases with constant rate for $x \in [a_2, a'_3]$.

Definition (α, β) -level interval or (α, β) -cuts of intuitionistic fuzzy number If $\tilde{A}^i$ is a IFN, then (α, β) -level interval or (α, β) -cuts is given by

$$A_{\alpha, \beta} = \begin{cases} [A_1(\alpha), A_2(\alpha)] & \text{for degree of acceptance } \alpha \in [0, 1] \\ [A'_1(\beta), A'_2(\beta)] & \text{for degree of acceptance } \beta \in [0, 1] \end{cases} \quad \text{with } \alpha + \beta \leq 1.$$

Here (i) $\frac{dA_1(\alpha)}{d\alpha} > 0$, $\frac{dA_2(\alpha)}{d\alpha} < 0 \forall \alpha \in [0, 1]$, $A_1(1) \leq A_2(1)$

and (ii) $\frac{dA'_1(\beta)}{d\beta} < 0$, $\frac{dA'_2(\beta)}{d\beta} > 0 \forall \beta \in [0, 1]$, $A'_1(0) \leq A'_2(0)$.

It is expressed as $A_{\alpha, \beta} = \{[A_1(\alpha), A_2(\alpha)]; [A'_1(\beta), A'_2(\beta)]\}$, $\alpha + \beta \leq 1$, $\alpha, \beta \in [0, 1]$.

For instance, if $\tilde{A}^i = (a_1, a_2, a_3; a'_1, a_2, a'_3)$ then (α, β) -level interval or (α, β) -cuts is given by

$$A_{\alpha, \beta} = \{[a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)]; [a_2 - \beta(a_2 - a'_1), a_2 + \beta(a'_3 - a_2)]\}, \alpha, \beta \in [0, 1]$$

where $\alpha + \beta \leq 1$, $\alpha, \beta \in [0, 1]$.

Definition [18] *Value and ambiguity of a triangular intuitionistic fuzzy number*

Let A_α^i and A_β^i be any α -cut and β -cut set of a triangular intuitionistic fuzzy number $\tilde{A}^i = (a_1, a_2, a_3; a'_1, a_2, a'_3)$ respectively. The values of the membership function $\mu_{\tilde{A}^i}(x)$

and the non-membership function $\vartheta_{\tilde{A}^i}(x)$ for the triangular intuitionistic fuzzy number $\tilde{A}^i$ are defined as follows:

$$V_{\mu}(\tilde{A}^i) = \int_0^1 \left(L_{\alpha}(A^i) + R_{\alpha}(A^i) \right) f(\alpha) d\alpha$$

and

$$V_{\vartheta}(\tilde{A}^i) = \int_0^1 \left(L_{\beta}(A^i) + R_{\beta}(A^i) \right) g(\beta) d\beta$$

The $f(\alpha)$ is a non-negative and non-decreasing function on the interval $[0, 1]$ with $f(0) = 0$ and $\int_0^1 f(\alpha) d\alpha = \frac{1}{2}$; The function $g(\beta)$ is a non-negative and non-increasing function on interval $[0, 1]$ with $g(1) = 0$ and $\int_0^1 g(\beta) d\beta = \frac{1}{2}$

If we choose $f(\alpha) = \alpha$ and $g(\beta) = 1 - \beta$ then the above assumption is correct.

Here $\alpha \in [0, w]$ and $\beta \in [v, 1]$

The ambiguities of membership function $\mu_{\tilde{A}^i}(x)$ and the non-membership function $\vartheta_{\tilde{A}^i}(x)$ are defined as

$$G_{\mu}(\tilde{A}^i) = \int_0^1 \left(R_{\alpha}(A^i) - L_{\alpha}(A^i) \right) f(\alpha) d\alpha$$

and

$$G_{\vartheta}(\tilde{A}^i) = \int_0^1 \left(R_{\beta}(A^i) - L_{\beta}(A^i) \right) g(\beta) d\beta$$

respectively.

Definition [20] *Ambiguity and value index and ranking of intuitionistic fuzzy number*

A value index and ambiguity index for triangular intuitionistic fuzzy number $\tilde{A}^i$ are defined as follows:

$$V(\tilde{A}^i, \lambda) = V_{\mu}(\tilde{A}^i) + \lambda \left(V_{\vartheta}(\tilde{A}^i) - V_{\mu}(\tilde{A}^i) \right)$$

and

$$G(\tilde{A}^i, \lambda) = G_{\mu}(\tilde{A}^i) + \lambda \left(G_{\vartheta}(\tilde{A}^i) - G_{\mu}(\tilde{A}^i) \right)$$

Respectively, where $\lambda \in [0, 1]$ is a weight which represents the decision maker's preference information. Limited to the above formulation, the choice $\lambda = \frac{1}{2}$ appears to be reasonable one. One can choose λ according to the suitability of the subject. $\lambda \in [0, \frac{1}{2})$ indicates decision maker's pessimistic attitude towards uncertainty while $\lambda \in (\frac{1}{2}, 1]$ indicates decision maker's optimistic attitude towards uncertainty.

With our choice $\lambda = \frac{1}{2}$, the value and ambiguity indices for triangular intuitionistic fuzzy number reduces to:

$$V\left(\tilde{A}^i, \frac{1}{2}\right) = \frac{V_{\mu}(\tilde{A}^i) + V_{\vartheta}(\tilde{A}^i)}{2}$$

and

$$G\left(\tilde{A}^i, \frac{1}{2}\right) = \frac{G_\mu(\tilde{A}^i) + G_\vartheta(\tilde{A}^i)}{2}$$

Hence the rank is

$$R\left(\tilde{A}^i\right) = V\left(\tilde{A}^i, \frac{1}{2}\right) + G\left(\tilde{A}^i, \frac{1}{2}\right)$$

Definition Let I be a real interval. A mapping $\tilde{u} : I \rightarrow E$ is called a fuzzy process. We denote α -level set by:

$$[\tilde{u}(t)]^{\alpha, \beta} = \left[u_1^\alpha(t), u_2^\alpha(t); u_1^\beta(t), u_2^\beta(t) \right], \quad t \in I, \alpha, \beta \in [0, 1].$$

The Seikkala derivative $\tilde{x}'(t)$ of a fuzzy process $\tilde{x}$ is defined by

$$\left[\frac{d\tilde{u}(t)}{dt} \right]^{\alpha, \beta} = \left[\frac{du_1^\alpha(t)}{dt}, \frac{du_2^\alpha(t)}{dt}; \frac{du_1^\beta(t)}{dt}, \frac{du_2^\beta(t)}{dt} \right], \quad \alpha, \beta \in [0, 1].$$

Provided that is a equation defines a fuzzy number $\frac{d\tilde{u}(t)}{dt} \in E$.

Definition The fuzzy integral of fuzzy process $\tilde{u}(t)$, $\int_a^b \tilde{u}(t)dt$ for $a, b \in I$, is defined by

$$\left[\int_a^b \tilde{u}(t)dt \right]^\alpha = \left[\int_a^b u_1^\alpha(t)dt, \int_a^b u_2^\alpha(t)dt; \int_a^b u_1^\beta(t)dt, \int_a^b u_2^\beta(t)dt \right]$$

provided that the Lebesgue integrals on the right exist.

Definition Let $f : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0 (Bede–Gal differential) if there exists an element $f'(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) -^h f(x_0)$, $\exists f(x_0) -^h f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) -^h f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0 - h)}{h} = f'(x_0)$$

or

(ii) for all $h < 0$ sufficiently small, $\exists f(x_0) -^h f(x_0 + h)$, $\exists f(x_0 - h) -^h f(x_0)$ and the limits (in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0 + h)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) -^h f(x_0)}{h} = f'(x_0)$$

Definition If $[x_1(t, \alpha), x_2(t, \alpha); x_1'(t, \beta), x_2'(t, \beta)]$ be the solution of the intuitionistic fuzzy differential equation then the solution is written as intuitionistic fuzzy number as below

$$(x_1(t, \alpha = 0), x_1(t, \alpha = 1), x_2(t, \alpha = 0); x_1'(t, \beta = 1), x_1'(t, \beta = 0), x_2'(t, \beta = 1))$$

Definition Crispification of a TFN($\tilde{A} = (a_1, a_2, a_3)$) based on the centre of area operator method is defined as

$$\hat{A}_{TFN} = \frac{\int_{\text{supp}(\tilde{A})} x \mu_{\tilde{A}}(x) dx}{\int_{\text{supp}(\tilde{A})} \mu_{\tilde{A}}(x) dx} = \frac{a_1 + a_2 + a_3}{3}$$

Definition [54] Crispification of a TIFN $\tilde{A}^i = (a_1, a_2, a_3; a'_1, a_2, a'_3)$ is defined by

$$\begin{aligned} \hat{A}_{TIFN} &= \frac{\int_{\text{supp}(\tilde{A})} x |\mu_{\tilde{A}^i}(x) - \vartheta_{\tilde{A}^i}(x)| dx}{\int_{\text{supp}(\tilde{A})} |\mu_{\tilde{A}^i}(x) - \vartheta_{\tilde{A}^i}(x)| dx} \\ &= \frac{1}{3} \left[\frac{(a'_3 - a'_1)(a_2 - 2a'_3 - 2a'_1) + (a_3 - a_1)(a_1 + a_2 + a_3) + 3(a_3'^2 - a_1'^2)}{a'_3 - a'_1 + a_3 - a_1} \right] \end{aligned}$$

System of Differential Equation with Initial Value as TIFN

Consider the first order linear homogeneous system of Intuitionistic fuzzy ordinary differential equation $\frac{dx}{dt} = Ay$, $\frac{dy}{dt} = Bx$ (A and B positive or negative) with $x(t_0) = x_0$ and $y(t_0) = y_0$. Here x_0 and y_0 are triangular Intuitionistic fuzzy number.

Let the solution of the above SFDE be $\tilde{x}(t)$ and its (α, β) -cut be

$$x(t, \alpha, \beta) = [x_1(t, \alpha), x_2(t, \alpha); x'_1(t, \beta), x'_2(t, \beta)]$$

and

$$y(t, \alpha, \beta) = [y_1(t, \alpha), y_2(t, \alpha); y'_1(t, \beta), y'_2(t, \beta)]$$

The solution is a strong solution if

- (i) $\frac{dx_1(t, \alpha)}{d\alpha} > 0$, $\frac{dx_2(t, \alpha)}{d\alpha} < 0$, $\frac{dy_1(t, \alpha)}{d\alpha} > 0$, $\frac{dy_2(t, \alpha)}{d\alpha} < 0$ $\forall \alpha \in [0, 1], \forall \beta \in [0, 1]$,
 $x_1(t, 1) \leq x_2(t, 1)$, $y_1(t, 1) \leq y_2(t, 1)$
- (ii) $\frac{dx'_1(t, \beta)}{d\alpha} < 0$, $\frac{dx'_2(t, \beta)}{d\alpha} > 0$, $\frac{dy'_1(t, \beta)}{d\beta} < 0$, $\frac{dy'_2(t, \beta)}{d\beta} > 0$ $\forall \alpha \in [0, 1], \forall \beta \in [0, 1]$,
 $x'_1(t, 0) \leq x'_2(t, 0)$, $y'_1(t, 0) \leq y'_2(t, 0)$

Otherwise the solution is weak solution.

Solution of System of Fuzzy Differential Equation

Consider the system of fuzzy differential equation

$$\frac{dx(t)}{dt} = Ay(t) \quad (1)$$

and

$$\frac{dy(t)}{dt} = Bx(t) \quad (2)$$

with $x(0) = \tilde{x}_0 = (a_1, a_2, a_3; a'_1, a_2, a'_3)$ and $y(0) = \tilde{y}_0 = (b_1, b_2, b_3; b'_1, b_2, b'_3)$

Here two cases arise: case I: $A, B > 0$ and case II: $A, B < 0$

Solution of Case I

$A, B > 0$ i.e., coefficient is positive crisp number

Taking (α, β) -cut of Eq. (1) we get

$$\begin{aligned} \frac{d}{dt} ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \\ = A ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \end{aligned} \quad (3)$$

Taking (α, β) -cut of Eq. (2) we get

$$\begin{aligned} \frac{d}{dt} ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \\ = B ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \end{aligned} \quad (4)$$

With initial condition

$$x(t_0; \alpha, \beta) = ([a_1(\alpha), a_2(\alpha)]; [a'_1(\beta), a'_2(\beta)], \alpha + \beta \leq 1, \alpha, \beta \in [0, 1])$$

and

$$y(t_0; \alpha, \beta) = ([b_1(\alpha), b_2(\alpha)]; [b'_1(\beta), b'_2(\beta)], \alpha + \beta \leq 1, \alpha, \beta \in [0, 1])$$

From (3) we get

$$\frac{dx_1(t, \alpha)}{dt} = Ay_1(t, \alpha) \quad (5)$$

$$\frac{dx_2(t, \alpha)}{dt} = Ay_2(t, \alpha) \quad (6)$$

$$\frac{dx'_1(t, \beta)}{dt} = Ay'_1(t, \beta) \quad (7)$$

$$\frac{dx'_2(t, \beta)}{dt} = Ay'_2(t, \beta) \quad (8)$$

From (4) we get

$$\frac{dy_1(t, \alpha)}{dt} = Bx_1(t, \alpha) \quad (9)$$

$$\frac{dy_2(t, \alpha)}{dt} = Bx_2(t, \alpha) \quad (10)$$

$$\frac{dy'_1(t, \beta)}{dt} = Bx'_1(t, \beta) \quad (11)$$

$$\frac{dy'_2(t, \beta)}{dt} = Bx'_2(t, \beta) \quad (12)$$

with initial condition

$$x_1(t_0, \alpha) = a_1(\alpha) \quad (13)$$

$$x_2(t_0, \alpha) = a_2(\alpha) \quad (14)$$

$$x'_1(t_0, \beta) = a'_1(\beta) \quad (15)$$

$$x'_2(t_0, \beta) = a'_2(\beta) \quad (16)$$

$$y_1(t_0, \alpha) = b_1(\alpha) \quad (17)$$

$$y_2(t_0, \alpha) = b_2(\alpha) \quad (18)$$

$$y_1'(t_0, \beta) = b_1'(\beta) \quad (19)$$

$$y_2'(t_0, \beta) = b_2'(\beta) \quad (20)$$

From (5) and (9) we get

$$\frac{dx_1(t, \alpha)}{dt} = Ay_1(t, \alpha) \quad (21)$$

and

$$\frac{dy_1(t, \alpha)}{dt} = Bx_1(t, \alpha) \quad (22)$$

We have $\frac{d^2x_1(t, \alpha)}{dt^2} = A \frac{dy_1(t, \alpha)}{dt} = ABx_1(t, \alpha)$

The solution is

$$x_1(t, \alpha) = c_1 e^{\sqrt{AB}t} + c_2 e^{-\sqrt{AB}t} \quad (23)$$

From (5) we get

$$c_1 e^{\sqrt{AB}t} - c_2 e^{-\sqrt{AB}t} = \sqrt{\frac{A}{B}} y_1(t, \alpha) \quad (24)$$

Using initial condition we get

$$c_1 e^{\sqrt{AB}t_0} + c_2 e^{-\sqrt{AB}t_0} = a_1(\alpha) \quad \text{and} \quad c_1 e^{\sqrt{AB}t_0} - c_2 e^{-\sqrt{AB}t_0} = \sqrt{\frac{A}{B}} b_1(\alpha)$$

Therefore, $c_1 = \frac{1}{2} \left(a_1(\alpha) + \sqrt{\frac{A}{B}} b_1(\alpha) \right) e^{-\sqrt{AB}t_0}$ and $c_2 = \frac{1}{2} \left(a_1(\alpha) - \sqrt{\frac{A}{B}} b_1(\alpha) \right) e^{\sqrt{AB}t_0}$

Therefore,

$$x_1(t, \alpha) = \frac{1}{2} \left\{ a_1(\alpha) + \sqrt{\frac{A}{B}} b_1(\alpha) \right\} e^{\sqrt{AB}(t-t_0)} + \frac{1}{2} \left\{ a_1(\alpha) - \sqrt{\frac{A}{B}} b_1(\alpha) \right\} e^{-\sqrt{AB}(t-t_0)}$$

$$y_1(t, \alpha) = \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_1(\alpha) + \sqrt{\frac{A}{B}} b_1(\alpha) \right\} e^{\sqrt{AB}(t-t_0)} - \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_1(\alpha) - \sqrt{\frac{A}{B}} b_1(\alpha) \right\} e^{-\sqrt{AB}(t-t_0)}$$

Similarly we get

$$x_2(t, \alpha) = \frac{1}{2} \left\{ a_2(\alpha) + \sqrt{\frac{A}{B}} b_2(\alpha) \right\} e^{\sqrt{AB}(t-t_0)} + \frac{1}{2} \left\{ a_2(\alpha) - \sqrt{\frac{A}{B}} b_2(\alpha) \right\} e^{-\sqrt{AB}(t-t_0)}$$

$$y_2(t, \alpha) = \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_2(\alpha) + \sqrt{\frac{A}{B}} b_2(\alpha) \right\} e^{\sqrt{AB}(t-t_0)} - \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_2(\alpha) - \sqrt{\frac{A}{B}} b_2(\alpha) \right\} e^{-\sqrt{AB}(t-t_0)}$$

and

$$x_1'(t, \beta) = \frac{1}{2} \left\{ a_1'(\beta) + \sqrt{\frac{A}{B}} b_1'(\beta) \right\} e^{\sqrt{AB}(t-t_0)} + \frac{1}{2} \left\{ a_1'(\beta) - \sqrt{\frac{A}{B}} b_1'(\beta) \right\} e^{-\sqrt{AB}(t-t_0)}$$

Graph 1 Graph of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x_1'(t, \beta)$ and $x_2'(t, \beta)$ for $t = 4$

$$\begin{aligned}
 y_1'(t, \beta) &= \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_1'(\beta) + \sqrt{\frac{A}{B}} b_1'(\beta) \right\} e^{\sqrt{AB}(t-t_0)} \\
 &\quad - \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_1'(\beta) - \sqrt{\frac{A}{B}} b_1'(\beta) \right\} e^{-\sqrt{AB}(t-t_0)} \\
 x_2'(t, \beta) &= \frac{1}{2} \left\{ a_2'(\beta) + \sqrt{\frac{A}{B}} b_2'(\beta) \right\} e^{\sqrt{AB}(t-t_0)} + \frac{1}{2} \left\{ a_2'(\beta) - \sqrt{\frac{A}{B}} b_2'(\beta) \right\} e^{-\sqrt{AB}(t-t_0)} \\
 y_2'(t, \beta) &= \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_2'(\beta) + \sqrt{\frac{A}{B}} b_2'(\beta) \right\} e^{\sqrt{AB}(t-t_0)} \\
 &\quad - \frac{1}{2} \sqrt{\frac{B}{A}} \left\{ a_2'(\beta) - \sqrt{\frac{A}{B}} b_2'(\beta) \right\} e^{-\sqrt{AB}(t-t_0)}
 \end{aligned}$$

Numerical Example Consider a system of differential equation (Graphs 1, 2)

$$\begin{aligned}
 \frac{dx(t)}{dt} &= 2y(t) \\
 \frac{dy(t)}{dt} &= 3x(t)
 \end{aligned}$$

With initial condition $x(0) = \tilde{x}_0 = (25, 27, 28; 26, 27, 29)$ and $y(0) = \tilde{y}_0 = (14, 16, 19; 13, 16, 18)$

Solution Using the results of “Solution of Case I” section of case 1 we get (Tables 1 and 2)

$$\begin{aligned}
 x_1(t, \alpha) &= \frac{1}{2} \left\{ (25+2\alpha) + \sqrt{\frac{2}{3}} (14+2\alpha) \right\} e^{\sqrt{6}t} + \frac{1}{2} \left\{ (25+2\alpha) - \sqrt{\frac{2}{3}} (14+2\alpha) \right\} e^{-\sqrt{6}t} \\
 y_1(t, \alpha) &= \frac{1}{2} \sqrt{\frac{3}{2}} \left\{ (25+2\alpha) + \sqrt{\frac{2}{3}} (14+2\alpha) \right\} e^{\sqrt{6}t}
 \end{aligned}$$

Graph 2 Graph of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y_1'(t, \beta)$ and $y_2'(t, \beta)$ for $t = 4$

Table 1 Value of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x_1'(t, \beta)$ and $x_2'(t, \beta)$ at $t = 4$ for different α, β

α, β	$x_1(t, \alpha)$	$x_2(t, \alpha)$	$x_1'(t, \beta)$	$x_2'(t, \beta)$
0	327, 823.4690	391, 555.1032	360, 514.9127	360, 514.9127
0.1	331, 092.6134	388, 451.0841	357, 410.8937	363, 784.0571
0.2	334, 361.7578	385, 347.0651	354, 306.8747	367, 053.2015
0.3	337, 630.9022	382, 243.0460	351, 202.8556	370, 322.3458
0.4	340, 900.0465	379, 139.0270	348, 098.8366	373, 591.4902
0.5	344, 169.1909	376, 035.0079	344, 994.8175	376, 860.6346
0.6	347, 438.3353	372, 930.9889	341, 890.7985	380, 129.7790
0.7	350, 707.4796	369, 826.9699	338, 786.7795	383, 398.9233
0.8	353, 976.6240	366, 722.9508	335, 682.7604	386, 668.0677
0.9	357, 245.7684	363, 618.9318	332, 578.7414	389, 937.2121
1	360, 514.9127	360, 514.9127	329, 474.7223	393, 206.3564

$$\begin{aligned}
 & -\frac{1}{2}\sqrt{\frac{3}{2}}\left\{(25+2\alpha)-\sqrt{\frac{2}{3}}(14+2\alpha)\right\}e^{-\sqrt{6}t} \\
 x_2(t, \alpha) &= \frac{1}{2}\left\{(28-\alpha)+\sqrt{\frac{2}{3}}(19-3\alpha)\right\}e^{\sqrt{6}t}+\frac{1}{2}\left\{(28-\alpha)-\sqrt{\frac{2}{3}}(19-3\alpha)\right\}e^{-\sqrt{6}t} \\
 y_2(t, \alpha) &= \frac{1}{2}\sqrt{\frac{3}{2}}\left\{(28-\alpha)+\sqrt{\frac{2}{3}}(19-3\alpha)\right\}e^{\sqrt{6}t}-\frac{1}{2}\sqrt{\frac{3}{2}}\left\{(28-\alpha)-\sqrt{\frac{2}{3}}(19-3\alpha)\right\}e^{-\sqrt{6}t}
 \end{aligned}$$

and

$$x_1'(t, \beta) = \frac{1}{2}\left\{(27-\beta)+\sqrt{\frac{2}{3}}(16-3\beta)\right\}e^{\sqrt{6}t}+\frac{1}{2}\left\{(27-\beta)-\sqrt{\frac{2}{3}}(16-3\beta)\right\}e^{-\sqrt{6}t}$$

Table 2 Value of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y'_1(t, \beta)$ and $y'_2(t, \beta)$ at $t = 4$ for different α, β

α, β	$y_1(t, \alpha)$	$y_2(t, \alpha)$	$y'_1(t, \beta)$	$y'_2(t, \beta)$
0	401,500.1115	479,555.1036	441,538.7895	441,538.7895
0.1	405,503.9793	475,753.4722	437,737.1581	445,542.6573
0.2	409,507.8471	471,951.8408	433,935.5267	449,546.5251
0.3	413,511.7149	468,150.2094	430,133.8953	453,550.3929
0.4	417,515.5827	464,348.5780	426,332.2638	457,554.2607
0.5	421,519.4505	460,546.9465	422,530.6324	461,558.1285
0.6	425,523.3183	456,745.3151	418,729.0010	465,561.9963
0.7	429,527.1861	452,943.6837	414,927.3696	469,565.8641
0.8	433,531.0539	449,142.0523	411,125.7382	473,569.7319
0.9	437,534.9217	445,340.4209	407,324.1068	477,573.5997
1	441,538.7895	441,538.7895	403,522.4754	481,577.4675

$$y'_1(t, \beta) = \frac{1}{2}\sqrt{\frac{3}{2}} \left\{ (27-\beta) + \sqrt{\frac{2}{3}}(16-3\beta) \right\} e^{\sqrt{6}t} - \frac{1}{2}\sqrt{\frac{3}{2}} \left\{ (27-\beta) - \sqrt{\frac{2}{3}}(16-3\beta) \right\} e^{-\sqrt{6}t}$$

$$x'_2(t, \beta) = \frac{1}{2} \left\{ (27+2\beta) + \sqrt{\frac{2}{3}}(16+2\beta) \right\} e^{\sqrt{6}t} + \frac{1}{2} \left\{ (27+2\beta) - \sqrt{\frac{2}{3}}(16+2\beta) \right\} e^{-\sqrt{6}t}$$

$$y'_2(t, \beta) = \frac{1}{2}\sqrt{\frac{3}{2}} \left\{ (27+2\beta) + \sqrt{\frac{2}{3}}(16+2\beta) \right\} e^{\sqrt{6}t} - \frac{1}{2}\sqrt{\frac{3}{2}} \left\{ (27+2\beta) - \sqrt{\frac{2}{3}}(16+2\beta) \right\} e^{-\sqrt{6}t}$$

Note From above tables and graphs and from “System of Differential Equation with Initial Value as TIFN” section we conclude that the solution is a strong solution.

Solution of Case II

$A, B < 0$ i.e., the coefficient is negative crisp number

Let $A = -C$ and $B = -D$

Taking (α, β) -cut of Eq. (1) we get

$$\begin{aligned} \frac{d}{dt} ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \\ = -C ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \end{aligned} \quad (25)$$

Taking (α, β) -cut of Eq. (2) we get

$$\begin{aligned} \frac{d}{dt} ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \\ = -D ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \end{aligned} \quad (26)$$

with initial condition

$$x(t_0; \alpha, \beta) = ([a_1(\alpha), a_2(\alpha)]; [a'_1(\beta), a'_2(\beta)], \alpha + \beta \leq 1, \alpha, \beta \in [0, 1])$$

and

$$y(t_0; \alpha, \beta) = ([b_1(\alpha), b_2(\alpha)]; [b'_1(\beta), b'_2(\beta)], \alpha + \beta \leq 1, \alpha, \beta \in [0, 1])$$

From (25) we get

$$\frac{dx_1(t, \alpha)}{dt} = -Cy_2(t, \alpha) \quad (27)$$

$$\frac{dx_2(t, \alpha)}{dt} = -Cy_1(t, \alpha) \quad (28)$$

$$\frac{dx'_1(t, \beta)}{dt} = -Cy'_2(t, \beta) \quad (29)$$

$$\frac{dx'_2(t, \beta)}{dt} = -Cy'_1(t, \beta) \quad (30)$$

From (26) we get

$$\frac{dy_1(t, \alpha)}{dt} = -Dx_2(t, \alpha) \quad (31)$$

$$\frac{dy_2(t, \alpha)}{dt} = -Dx_1(t, \alpha) \quad (32)$$

$$\frac{dy'_1(t, \beta)}{dt} = -Dx'_2(t, \beta) \quad (33)$$

$$\frac{dy'_1(t, \beta)}{dt} = -Dx'_1(t, \beta) \quad (34)$$

with initial condition

$$x_1(t_0, \alpha) = a_1(\alpha) \quad (35)$$

$$x_2(t_0, \alpha) = a_2(\alpha) \quad (36)$$

$$x'_1(t_0, \beta) = a'_1(\beta) \quad (37)$$

$$x'_2(t_0, \beta) = a'_2(\beta) \quad (38)$$

$$y_1(t_0, \alpha) = b_1(\alpha) \quad (39)$$

$$y_2(t_0, \alpha) = b_2(\alpha) \quad (40)$$

$$y'_1(t_0, \beta) = b'_1(\beta) \quad (41)$$

$$y'_2(t_0, \beta) = b'_2(\beta) \quad (42)$$

From Eqs. (27) and (32) we have

$$\frac{dx_1(t, \alpha)}{dt} = -Cy_2(t, \alpha) \quad (43)$$

and

$$\frac{dy_2(t, \alpha)}{dt} = -Dx_1(t, \alpha) \quad (44)$$

Therefore, $\frac{d^2x_1(t, \alpha)}{dt^2} = CDx_1(t, \alpha)$, $CD > 0$

The solution is,

$$x_1(t, \alpha) = c_1e^{\sqrt{CD}t} + c_2e^{-\sqrt{CD}t} \quad (45)$$

and now from (27) we get

$$\sqrt{\frac{C}{D}}y_2(t, \alpha) = -c_1e^{\sqrt{CD}t} + c_2e^{-\sqrt{CD}t} \quad (46)$$

Using initial condition we have

$$c_1 e^{\sqrt{CD}t_0} + c_2 e^{-\sqrt{CD}t_0} = a_1(\alpha) \quad (47)$$

and

$$-c_1 e^{\sqrt{CD}t_0} + c_2 e^{-\sqrt{CD}t_0} = \sqrt{\frac{C}{D}} b_2(\alpha) \quad (48)$$

adding (47) and (48) we get

$$c_2 = \frac{1}{2} \left(a_1(\alpha) + \sqrt{\frac{C}{D}} b_2(\alpha) \right) e^{\sqrt{CD}t_0} \quad \text{and} \quad c_1 = \frac{1}{2} \left(a_1(\alpha) - \sqrt{\frac{C}{D}} b_2(\alpha) \right) e^{-\sqrt{CD}t_0}$$

Now from (45) we get

$$\begin{aligned} x_1(t, \alpha) &= \frac{1}{2} \left\{ a_1(\alpha) - \sqrt{\frac{C}{D}} b_2(\alpha) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \left\{ a_1(\alpha) + \sqrt{\frac{C}{D}} b_2(\alpha) \right\} e^{-\sqrt{CD}(t-t_0)} \\ y_2(t, \alpha) &= -\frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a_1(\alpha) - \sqrt{\frac{C}{D}} b_2(\alpha) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a_1(\alpha) + \sqrt{\frac{C}{D}} b_2(\alpha) \right\} \\ &\quad \times e^{-\sqrt{CD}(t-t_0)} \end{aligned}$$

Similarly we get from (28) and (31) we have

$$\frac{d^2 x_2(t, \alpha)}{dt^2} = CD x_2(t, \alpha)$$

Solution is given by

$$x_2(t, \alpha) = c_1 e^{\sqrt{CD}t} + c_2 e^{-\sqrt{CD}t} \quad (49)$$

From (28) we get

$$\sqrt{\frac{C}{D}} y_1(t, \alpha) = -c_1 e^{\sqrt{CD}t} + c_2 e^{-\sqrt{CD}t} \quad (50)$$

Using initial condition we have

$$c_1 e^{\sqrt{CD}t_0} + c_2 e^{-\sqrt{CD}t_0} = a_2(\alpha) \quad (51)$$

and

$$-c_1 e^{\sqrt{CD}t_0} + c_2 e^{-\sqrt{CD}t_0} = \sqrt{\frac{C}{D}} b_1(\alpha) \quad (52)$$

Solving we get

$$c_2 = \frac{1}{2} \left(a_2(\alpha) + \sqrt{\frac{C}{D}} b_1(\alpha) \right) e^{\sqrt{CD}t_0} \quad \text{and} \quad c_1 = \frac{1}{2} \left(a_2(\alpha) - \sqrt{\frac{C}{D}} b_1(\alpha) \right) e^{-\sqrt{CD}t_0}$$

Now from (49) and (50) we get

$$\begin{aligned} x_2(t, \alpha) &= \frac{1}{2} \left\{ a_2(\alpha) - \sqrt{\frac{C}{D}} b_1(\alpha) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \left\{ a_2(\alpha) + \sqrt{\frac{C}{D}} b_1(\alpha) \right\} e^{-\sqrt{CD}(t-t_0)} \\ y_1(t, \alpha) &= -\frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a_2(\alpha) - \sqrt{\frac{C}{D}} b_1(\alpha) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a_2(\alpha) + \sqrt{\frac{C}{D}} b_1(\alpha) \right\} \end{aligned}$$

Graph 3 Graph of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x'_1(t, \beta)$ and $x'_2(t, \beta)$ for $t = 0.05$

$$\times e^{-\sqrt{CD}(t-t_0)}$$

We have also

$$\begin{aligned} x'_1(t, \beta) &= \frac{1}{2} \left\{ a'_1(\beta) - \sqrt{\frac{C}{D}} b'_2(\beta) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \left\{ a'_1(\beta) + \sqrt{\frac{C}{D}} b'_2(\beta) \right\} e^{-\sqrt{CD}(t-t_0)} \\ y'_2(t, \beta) &= -\frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a'_1(\eta) - \sqrt{\frac{C}{D}} b'_2(\beta) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a'_1(\beta) + \sqrt{\frac{C}{D}} b'_2(\beta) \right\} \\ &\quad \times e^{-\sqrt{CD}(t-t_0)} \\ x'_2(t, \beta) &= \frac{1}{2} \left\{ a'_2(\beta) - \sqrt{\frac{C}{D}} b'_1(\beta) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \left\{ a'_2(\beta) + \sqrt{\frac{C}{D}} b'_1(\beta) \right\} e^{-\sqrt{CD}(t-t_0)} \\ y'_1(t, \beta) &= -\frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a'_2(\beta) - \sqrt{\frac{C}{D}} b'_1(\beta) \right\} e^{\sqrt{CD}(t-t_0)} + \frac{1}{2} \sqrt{\frac{D}{C}} \left\{ a'_2(\beta) + \sqrt{\frac{C}{D}} b'_1(\beta) \right\} \\ &\quad \times e^{-\sqrt{CD}(t-t_0)} \end{aligned}$$

Numerical Example Consider the system of differential equation (Graphs 3, 4)

$$\begin{aligned} \frac{dx(t)}{dt} &= -6y(t) \\ \frac{dy(t)}{dt} &= -9x(t) \end{aligned}$$

with initial condition $x(0) = \tilde{x}_0 = (52, 55, 57; 53, 55, 58)$ and $y(0) = \tilde{y}_0 = (64, 66, 69; 63, 66, 68)$

Graph 4 Graph of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y_1'(t, \beta)$ and $y_2'(t, \beta)$ for $t = 0.05$

Table 3 Value of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x_1'(t, \beta)$ and $x_2'(t, \beta)$ at $t = 0.05$ for different α, β

α, β	$x_1(t, \alpha)$	$x_2(t, \alpha)$	$x_1'(t, \beta)$	$x_2'(t, \beta)$
0	34.3808	41.2561	38.5059	38.5059
0.1	34.7933	40.9810	38.2309	38.9185
0.2	35.2058	40.7060	37.9559	39.3310
0.3	35.6183	40.4310	37.6809	39.7435
0.4	36.0308	40.1560	37.4059	40.1560
0.5	36.4433	39.8810	37.1309	40.5685
0.6	36.8559	39.6060	36.8559	40.9810
0.7	37.2684	39.3310	36.5809	41.3936
0.8	37.6809	39.0560	36.3058	41.8061
0.9	38.0934	38.7809	36.0308	42.2186
1	38.5059	38.5059	35.7558	42.6311

The solution is given by the equations (Tables 3, 4)

$$\begin{aligned}
 x_1(t, \alpha) &= \frac{1}{2} \left\{ (52 + 3\alpha) - \sqrt{\frac{6}{9}} (69 - 3\alpha) \right\} e^{\sqrt{54}t} \\
 &\quad + \frac{1}{2} \left\{ (52 + 3\alpha) + \sqrt{\frac{6}{9}} (69 - 3\alpha) \right\} e^{-\sqrt{54}t} \\
 y_1(t, \alpha) &= -\frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (57 - 2\alpha) - \sqrt{\frac{6}{9}} (64 + 2\alpha) \right\} e^{\sqrt{54}t}
 \end{aligned}$$

Table 4 Value of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y'_1(t, \beta)$ and $y'_2(t, \beta)$ at $t = 0.05$ for different α, β

α, β	$y_1(t, \alpha)$	$y_2(t, \alpha)$	$y'_1(t, \beta)$	$y'_2(t, \beta)$
0	42.1378	49.7801	45.1947	45.1947
0.1	42.4435	49.3215	44.7362	45.5004
0.2	42.7492	48.8630	44.2776	45.8061
0.3	43.0549	48.4045	43.8191	46.1118
0.4	43.3606	47.9459	43.3606	46.4175
0.5	43.6662	47.4874	42.9020	46.7232
0.6	43.9719	47.0288	42.4435	47.0288
0.7	44.2776	46.5703	41.9849	47.3345
0.8	44.5833	46.1118	41.5264	47.6402
0.9	44.8890	45.6532	41.0679	47.9459
1	45.1947	45.1947	40.6093	48.2516

$$\begin{aligned}
& + \frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (57 - 2\alpha) + \sqrt{\frac{6}{9}} (64 + 2\alpha) \right\} e^{-\sqrt{54}t} \\
x_2(t, \alpha) &= \frac{1}{2} \left\{ (57 - 2\alpha) - \sqrt{\frac{6}{9}} (64 + 2\alpha) \right\} e^{\sqrt{54}t} \\
& + \frac{1}{2} \left\{ (57 - 2\alpha) + \sqrt{\frac{6}{9}} (64 + 2\alpha) \right\} e^{-\sqrt{54}t} \\
y_2(t, \alpha) &= -\frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (52 + 3\alpha) - \sqrt{\frac{6}{9}} (69 - 3\alpha) \right\} e^{\sqrt{54}t} \\
& + \frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (52 + 3\alpha) + \sqrt{\frac{6}{9}} (69 - 3\alpha) \right\} e^{-\sqrt{54}t}
\end{aligned}$$

and

$$\begin{aligned}
x'_1(t, \beta) &= \frac{1}{2} \left\{ (55 - 2\beta) - \sqrt{\frac{6}{9}} (66 + 2\beta) \right\} e^{\sqrt{54}t} \\
& + \frac{1}{2} \left\{ (55 - 2\beta) + \sqrt{\frac{6}{9}} (66 + 2\beta) \right\} e^{-\sqrt{54}t} \\
y'_1(t, \beta) &= -\frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (55 + 3\beta) - \sqrt{\frac{6}{9}} (66 - 3\beta) \right\} e^{\sqrt{54}t} \\
& + \frac{1}{2} \sqrt{\frac{9}{6}} \left\{ (55 + 3\beta) + \sqrt{\frac{6}{9}} (66 - 3\beta) \right\} e^{-\sqrt{54}t} \\
x'_2(t, \beta) &= \frac{1}{2} \left\{ (55 + 3\beta) - \sqrt{\frac{6}{9}} (66 - 3\beta) \right\} e^{\sqrt{54}t} \\
& + \frac{1}{2} \left\{ (55 + 3\beta) + \sqrt{\frac{6}{9}} (66 - 3\beta) \right\} e^{-\sqrt{54}t}
\end{aligned}$$

$$y_2'(t, \beta) = -\frac{1}{2}\sqrt{\frac{9}{6}} \left\{ (55 - 2\beta) - \sqrt{\frac{6}{9}} (66 + 2\beta) \right\} e^{\sqrt{54}t} \\ + \frac{1}{2}\sqrt{\frac{9}{6}} \left\{ (55 - 2\beta) + \sqrt{\frac{6}{9}} (66 + 2\beta) \right\} e^{-\sqrt{54}t}$$

Note From above tables and graphs and from section 3 we conclude that the solution is a strong solution.

Application

Crisp Model [55]

Arm race model Let $x(t)$ be the armaments of nation X ; and $y(t)$ be the armaments of nation Y at time t . The rate of change of the armaments on one side depends on the number of armaments on the opposing side, because if one nation increases its armaments, the other will follow suit. That is, $\frac{dx}{dt}$ (or $\frac{dy}{dt}$) is proportional to y (or x): We assign constants of proportionality k and l to x and y , respectively, which represent the efficiency of increasing armaments.

Hence we can establish a system of differential equations in the following form:

$$\frac{dx}{dt} = ky \\ \frac{dy}{dt} = lx$$

This system can be used to describe the relationship between two nations or alliances, each of which decides to defend itself against possible attack by the other.

In the more realistic model we shall provide a more detailed analysis with an example. Here we assume that the efficiency of increasing armaments for each nation is equal, and we take

$$k = l = 0.9$$

Let us consider the initial conditions $x(0) = 20$, $y(0) = 0$.

Fuzzy Model

In previous problem if the initial condition is triangular Intuitionistic fuzzy number then the problem becomes

$$\frac{dx}{dt} = ky \\ \frac{dy}{dt} = lx$$

with initial condition $x(0) = (18, 20, 23; 17, 20, 22)$, $y(0) = 0$ and $k = l = 0.9$

Solution Here system of differential is

$$\frac{dx}{dt} = 0.9y \quad (53)$$

$$\frac{dy}{dt} = 0.9x \quad (54)$$

Graph 5 Graph of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x'_1(t, \beta)$ and $x'_2(t, \beta)$ for $t = 5$

with initial condition $x(0) = (18, 20, 23; 17, 20, 22)$, $y(0) = 0$.

Using (i)-gH differentiable system we have from (1) we get (Graphs 5, 6)

$$\begin{aligned} \frac{d}{dt} ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \\ = 0.9 ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \end{aligned} \quad (55)$$

and

$$\begin{aligned} \frac{d}{dt} ([y_1(t, \alpha), y_2(t, \alpha)]; [y'_1(t, \beta), y'_2(t, \beta)]) \\ = 0.9 ([x_1(t, \alpha), x_2(t, \alpha)]; [x'_1(t, \beta), x'_2(t, \beta)]) \end{aligned} \quad (56)$$

i.e.,

$$\begin{aligned} \frac{dx_1(t, \alpha)}{dt} &= 0.9y_1(t, \alpha) \\ \frac{dx_2(t, \alpha)}{dt} &= 0.9y_2(t, \alpha) \\ \frac{dx'_1(t, \beta)}{dt} &= 0.9y'_1(t, \beta) \\ \frac{dx'_2(t, \beta)}{dt} &= 0.9y'_2(t, \beta) \end{aligned}$$

and

$$\frac{dy_1(t, \alpha)}{dt} = 0.9x_1(t, \alpha)$$

Graph 6 Graph of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y'_1(t, \beta)$ and $y'_2(t, \beta)$ for $t = 5$

$$\frac{dy_2(t, \alpha)}{dt} = 0.9x_2(t, \alpha)$$

$$\frac{dy'_1(t, \beta)}{dt} = 0.9x'_1(t, \beta)$$

$$\frac{dy'_2(t, \beta)}{dt} = 0.9x'_2(t, \beta)$$

Here, $(x(0))_{\alpha, \beta} = (18 + 2\alpha, 23 - 3\alpha; 20 - 3\beta, 20 + 2\beta)$
and

$$(y(0))_{\alpha, \beta} = (0, 0; 0, 0)$$

Therefore the solution is (Tables 5, 6)

$$x_1(t, \alpha) = \frac{1}{2}(18 + 2\alpha)e^{0.9t} + \frac{1}{2}(18 + 2\alpha)e^{-0.9t} = (9 + \alpha)(e^{0.9t} + e^{-0.9t})$$

$$x_2(t, \alpha) = \frac{1}{2}(23 - 3\alpha)e^{0.9t} + \frac{1}{2}(23 - 3\alpha)e^{-0.9t} = (11.5 - 1.5\alpha)(e^{0.9t} + e^{-0.9t})$$

$$x'_1(t, \beta) = \frac{1}{2}(20 - 3\beta)e^{0.9t} + \frac{1}{2}(20 - 3\beta)e^{-0.9t} = (10 - 1.5\beta)(e^{0.9t} + e^{-0.9t})$$

$$x'_2(t, \beta) = \frac{1}{2}(20 + 2\beta)e^{0.9t} + \frac{1}{2}(20 + 2\beta)e^{-0.9t} = (10 + \beta)(e^{0.9t} + e^{-0.9t})$$

and

$$y_1(t, \alpha) = \frac{1}{2}(18 + 2\alpha)e^{0.9t} - \frac{1}{2}(18 + 2\alpha)e^{-0.9t} = (9 + \alpha)(e^{0.9t} - e^{-0.9t})$$

Table 5 Value of $x_1(t, \alpha)$, $x_2(t, \alpha)$, $x'_1(t, \beta)$ and $x'_2(t, \beta)$ at $t = 5$ for different α, β

α, β	$x_1(t, \alpha)$	$x_2(t, \alpha)$	$x'_1(t, \beta)$	$x'_2(t, \beta)$
0	819.1542	1,046.6970	910.1713	910.1713
0.1	827.4727	1,032.0676	895.6710	918.4037
0.2	835.8421	1,017.5469	881.2683	926.6945
0.3	844.2588	1,003.1246	866.9539	935.0393
0.4	852.7192	988.7914	852.7192	943.4340
0.5	861.2202	974.5387	838.5565	951.8750
0.6	869.7588	960.3587	824.4589	960.3587
0.7	878.3323	946.2446	810.4200	968.8820
0.8	886.9381	932.1900	796.4342	977.4419
0.9	895.5737	918.1892	782.4962	986.0357
1	904.2370	904.2370	768.6015	994.6607

Table 6 Value of $y_1(t, \alpha)$, $y_2(t, \alpha)$, $y'_1(t, \beta)$ and $y'_2(t, \beta)$ at $t = 5$ for different α, β

α, β	$y_1(t, \alpha)$	$y_2(t, \alpha)$	$y'_1(t, \beta)$	$y'_2(t, \beta)$
0	801.1542	1,023.6970	890.1713	890.1713
0.1	810.8391	1,011.3213	877.6665	899.9423
0.2	820.4731	998.8368	865.0641	909.6550
0.3	830.0599	986.2540	852.3733	919.3136
0.4	839.6029	973.5821	839.6029	928.9223
0.5	849.1053	960.8297	826.7604	938.4848
0.6	858.5701	948.0045	813.8529	948.0045
0.7	868.0000	935.1134	800.8866	957.4846
0.8	877.3977	922.1629	787.8673	966.9281
0.9	886.7655	909.1586	774.8002	976.3378
1	896.1056	896.1056	761.6898	985.7162

$$\begin{aligned}
 y_2(t, \alpha) &= \frac{1}{2}(23 - 3\alpha)e^{0.9t} - \frac{1}{2}(23 - 3\alpha)e^{-0.9t} = (11.5 - 1.5\alpha)(e^{0.9t} - e^{-0.9t}) \\
 y'_1(t, \beta) &= \frac{1}{2}(20 - 3\beta)e^{0.9t} - \frac{1}{2}(20 - 3\beta)e^{-0.9t} = (10 - 1.5\beta)(e^{0.9t} - e^{-0.9t}) \\
 y'_2(t, \beta) &= \frac{1}{2}(20 + 2\beta)e^{0.9t} - \frac{1}{2}(20 + 2\beta)e^{-0.9t} = (10 + \beta)(e^{0.9t} - e^{-0.9t})
 \end{aligned}$$

Remarks From above tables and graphs we see that the conditions for strong solution hold. Hence the solution is a strong solution.

Therefore, the solution is written as,

$$\begin{aligned}
 \tilde{x}(t) &= (9, 10, 11.5; 8.5, 10, 11)(e^{0.9t} + e^{-0.9t}) \quad \text{and} \quad \tilde{y}(t) \\
 &= (9, 10, 11.5; 8.5, 10, 11)(e^{0.9t} - e^{-0.9t})
 \end{aligned}$$

Valuation, Ambiguity, Ranking and Defuzzification of Solution

Finding the Valuation

$$\begin{aligned}
 V_{\mu}(\tilde{x}(t)) &= \frac{1}{2} \int_0^1 (x_1(t, \alpha) + x_2(t, \alpha)) f(\alpha) d\alpha \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (41 - \alpha) e^{0.9t} + \frac{1}{2} (41 - \alpha) e^{-0.9t} \right] \alpha d\alpha \\
 &= \frac{121}{24} (e^{0.9t} + e^{-0.9t}) \\
 V_{\mu}(\tilde{x}(t=1)) &= 14.4503 \\
 V_{\vartheta}(\tilde{x}(t)) &= \frac{1}{2} \int_0^1 (x'_1(t, \beta) + x'_2(t, \beta)) f(\beta) d\beta \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (40 - \beta) e^{0.9t} + \frac{1}{2} (40 - \beta) e^{-0.9t} \right] (1 - \beta) d\beta \\
 &= \frac{119}{24} (e^{0.9t} + e^{-0.9t}) \\
 V_{\vartheta}(\tilde{x}(t=1)) &= 14.2114 \\
 V_{\mu}(\tilde{y}(t)) &= \frac{1}{2} \int_0^1 (y_1(t, \alpha) + y_2(t, \alpha)) f(\alpha) d\alpha \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (41 - \alpha) e^{0.9t} - \frac{1}{2} (41 - \alpha) e^{-0.9t} \right] \alpha d\alpha \\
 &= \frac{121}{24} (e^{0.9t} - e^{-0.9t}) \\
 V_{\mu}(\tilde{y}(t=1)) &= 10.3507 \\
 V_{\vartheta}(\tilde{y}(t)) &= \frac{1}{2} \int_0^1 (y'_1(t, \beta) + y'_2(t, \beta)) g(\beta) d\beta \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (40 - \beta) e^{0.9t} - \frac{1}{2} (40 - \beta) e^{-0.9t} \right] \beta d\beta \\
 &= \frac{119}{24} (e^{0.9t} - e^{-0.9t}) \\
 V_{\vartheta}(\tilde{y}(t=1)) &= 10.1796
 \end{aligned}$$

Finding the Ambiguities

$$\begin{aligned}
 G_{\mu}(\tilde{x}(t)) &= \frac{1}{2} \int_0^1 (x_2(t, \alpha) - x_1(t, \alpha)) f(\alpha) d\alpha \\
 &= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (5 - 5\alpha) e^{0.9t} - \frac{1}{2} (5 - 5\alpha) e^{-0.9t} \right] \alpha d\alpha \\
 &= \frac{5}{24} (e^{0.9t} + e^{-0.9t}) \\
 G_{\mu}(\tilde{x}(t=1)) &= 0.5971
 \end{aligned}$$

$$\begin{aligned}
G_{\vartheta}(\tilde{x}(t)) &= \frac{1}{2} \int_0^1 (x'_2(t, \beta) - x'_1(t, \beta)) g(\beta) d\beta \\
&= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (5\beta) e^{0.9t} + \frac{1}{2} (5\beta) e^{-0.9t} \right] \beta d\beta \\
&= \frac{5}{12} (e^{0.9t} + e^{-0.9t})
\end{aligned}$$

$$G_{\vartheta}(\tilde{x}(t=1)) = 1.1942$$

$$\begin{aligned}
G_{\mu}(\tilde{y}(t)) &= \frac{1}{2} \int_0^1 (y_2(t, \alpha) - y_1(t, \alpha)) f(\alpha) d\alpha \\
&= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (5 - 5\alpha) e^{0.9t} - \frac{1}{2} (5 - 5\alpha) e^{-0.9t} \right] \alpha d\alpha \\
&= \frac{5}{24} (e^{0.9t} - e^{-0.9t})
\end{aligned}$$

$$G_{\mu}(\tilde{y}(t=1)) = 0.4277$$

$$\begin{aligned}
G_{\vartheta}(\tilde{y}(t)) &= \frac{1}{2} \int_0^1 (y'_2(t, \beta) - y'_1(t, \beta)) g(\beta) d\beta \\
&= \frac{1}{2} \int_0^1 \left[\frac{1}{2} (5\beta) e^{0.9t} - \frac{1}{2} (5\beta) e^{-0.9t} \right] \beta d\beta \\
&= \frac{5}{12} (e^{0.9t} - e^{-0.9t})
\end{aligned}$$

$$G_{\vartheta}(\tilde{y}(t=1)) = 0.8554$$

Now we find the valuation, ambiguity index and ranking of solutions.

$$V\left(\tilde{x}(t=1), \frac{1}{2}\right) = \frac{V_{\mu}(\tilde{x}(t=1)) + V_{\vartheta}(\tilde{x}(t=1))}{2} = 14.3309$$

and

$$G\left(\tilde{x}(t=1), \frac{1}{2}\right) = \frac{G_{\mu}(\tilde{x}(t=1)) + G_{\vartheta}(\tilde{x}(t=1))}{2} = 0.8957$$

Ranking

Hence the rank of $\tilde{x}$ at $t=1$ is

$$R(\tilde{x}(t=1)) = V\left(\tilde{x}(t=1), \frac{1}{2}\right) + G\left(\tilde{x}(t=1), \frac{1}{2}\right) = 15.2266$$

and

$$V\left(\tilde{y}(t=1), \frac{1}{2}\right) = \frac{V_{\mu}(\tilde{y}(t=1)) + V_{\vartheta}(\tilde{y}(t=1))}{2} = 10.2652$$

and

$$G\left(\tilde{y}(t=1), \frac{1}{2}\right) = \frac{G_{\mu}(\tilde{y}(t=1)) + G_{\vartheta}(\tilde{y}(t=1))}{2} = 0.6416$$

Hence the rank of $\tilde{y}$ at $t=1$ is

$$R(\tilde{y}(t=1)) = V\left(\tilde{y}(t=1), \frac{1}{2}\right) + G\left(\tilde{y}(t=1), \frac{1}{2}\right) = 10.9068$$

We conclude that after one year the armaments of nation X is greater than nation Y .

Defuzzification

The solution can also be written as a fuzzy number as

$$\begin{aligned}\tilde{x}(t) &= (9, 10, 11.5; 8.5, 10, 11) (e^{0.9t} + e^{-0.9t}) \quad \text{and} \quad \tilde{y}(t) \\ &= (9, 10, 11.5; 8.5, 10, 11) (e^{0.9t} - e^{-0.9t})\end{aligned}$$

Using the definition of defuzzification of intuitionistic fuzzy number we have,

$$\hat{x}(t) = \frac{30.05}{3} (e^{0.9t} + e^{-0.9t}) \quad \text{and} \quad \hat{y}(t) = \frac{30.05}{3} (e^{0.9t} - e^{-0.9t})$$

Now at $t = 1$ $\hat{x}(1) = 28.7095$ and $\hat{y}(1) = 20.5646$

Hence after one year the armaments of nation X and Y is 29 and 21 respectively.

Conclusion

In this paper we solved first order system of differential equation initial value as triangular Intuitionistic fuzzy number. We use (α, β) -cut method for solving this system of fuzzy differential equation. i.e., first take (α, β) -cut of the fuzzy differential equation. Then the system of fuzzy differential equation converted to another system of crisp differential equations, and then solve the differential equation. An Arm race model is solved in intuitionistic fuzzy environment. We find valuation, ambiguity and ranking for comparing this two fuzzy solutions. Also find the defuzzification value of the solution, and it helps who are not known fuzzy set theory. This process is also a promising method to solve other similar models in intuitionistic fuzzy environment. So, future research will be concern with system of higher order differential equation and system of nonlinear first order (or higher order) differential equation and its application with Intuitionistic fuzzy environment.

References

1. Zadeh, L.A.: Fuzzy sets. *Inf. Control* **8**, 338–353 (1965)
2. Dubois, D., Parade, H.: Operation on fuzzy number. *Int. J. Fuzzy Syst.* **9**, 613–626 (1978)
3. Atanassov, K.: Intuitionistic fuzzy sets. VII ITKR's session, Sofia (deposited in Central Science and Technical Library of the Bulgarian Academy of Sciences 1697/84) (1983)
4. Atanassov, K.: Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **20**, 87–96 (1986)
5. Atanassov, K.: Operators over interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **64**(2), 159–174 (1994)
6. Nikolova, M., Nikolov, N., Cornelis, C., Deschrijver, G.: Survey of the research on intuitionistic fuzzy sets. *Adv. Stud. Contemp. Math.* **4**(2), 127–157 (2002)
7. Li, D.F., Cheng, C.T.: New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognit. Lett.* **23**(1–3), 221–225 (2002)
8. De, S.K., Biswas, R., Roy, A.R.: An application of intuitionistic fuzzy sets in medical diagnosis. *Fuzzy Sets Syst.* **117**(2), 209–213 (2001)
9. Kharal, A.: Homeopathic drug selection using intuitionistic fuzzy sets. *Homeopathy* **98**(1), 35–39 (2009)
10. Shu, M.H., Cheng, C.H., Chang, J.R.: Using intuitionistic fuzzy sets for fault-tree analysis on printed circuit board assembly. *Microelectron. Reliab.* **46**(12), 2139–2148 (2006)
11. Wang, Z., Li, K.W., Wang, W.: An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights. *Inf. Sci.* **179**(17), 3026–3040 (2009)
12. Li, D.F.: Multiattribute decision making models and methods using intuitionistic fuzzy sets. *J. Comput. Syst. Sci.* **70**, 73–85 (2005)

13. Ye, J.: Multicriteria fuzzy decision-making method based on a novel accuracy function under interval-valued intuitionistic fuzzy environment. *Expert Syst. Applicat.* **36**, 6899–6902 (2009)
14. Abbasbandy, S.: Ranking of fuzzy numbers, some recent and new formulas. *IFSA-EUSFLAT* **2009**, 642–646 (2009)
15. Chu, T.C., Tsao, C.T.: Ranking fuzzy numbers with an area between the centroid point and original point. *Comput. Math. Applicat.* **43**, 111–117 (2002)
16. Jimenez, M.: Ranking fuzzy numbers through the comparison of its expected intervals. *Int. J. Uncertain. Fuzz. Knowl.-Based Syst.* **4**, 379–388 (1996)
17. Maleki, H.R.: Ranking functions and their applications to fuzzy linear programming. *Far East J. Math. Sci.* **4**, 283–301 (2002)
18. Rezvani, Salim: Ranking method of trapezoidal intuitionistic fuzzy numbers. *Ann. Fuzzy Math. Inf.* **5**(3), 515–523 (2013)
19. Dubey, D., Mehra, A.: Linear programming with Triangular Intuitionistic Fuzzy Number, *EUSFLAT-LFA 2011*, Aix-les-Bains, France, July 2011
20. De, P.K., Das, D.: A study on ranking of trapezoidal intuitionistic fuzzy numbers. *Int. J. Comput. Inf. Syst. Industr. Manag. Applicat.* **6**, 437–444 (2014). (ISSN 2150–7988)
21. Kandel, A., Byatt, W.J.: Fuzzy differential equations. In: *Proceedings of International Conference Cybernetics and Society*, Tokyo, pp. 1213–1216 (1978)
22. Hüllermeier, E.: An approach to modelling and simulation of uncertain dynamical systems. *Int. J. Uncertain. Fuzz. Knowl.-Based Syst.* **5**, 117–137 (1997)
23. Diamond, P.: Time-dependent differential inclusions, cocycle attractors and fuzzy differential equations. *IEEE Trans. Fuzzy Syst.* **7**, 734–740 (1999)
24. Bede, B., Gal, S.G.: Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Set Syst.* **151**, 581–599 (2005)
25. Chen, M., Wu, C., Xue, X., Liu, G.: On fuzzy boundary value problems. *Inf. Sci.* **178**, 1877–1892 (2008)
26. Buckley, J.J., Feuring, T., Hayashi, Y.: Linear system of first order ordinary differential equations: fuzzy initial conditions. *Soft Comput.* **6**, 415–421 (2002)
27. Fard, O.S.: An iterative scheme for the solution of generalized system of linear fuzzy differential equations. *World Appl. Sci. J.* **7**(12), 1597–11604 (2009). (ISSN 1818–4952)
28. Gasilova, N., Amrahov, S.E., GolayogluFatullayev, C.: A geometric approach to solve fuzzy linear systems of differential equations. *Appl. Math. Inf. Sci.* **5**(3), 484–499 (2011)
29. Hashemi, M.S., Malekinagad, J., Marasi, H.R.: Series solution of the system of fuzzy differential equation. *Adv. Fuzzy Syst.* **2012**, 1–16 (2012)
30. Buckley, J.J., Reilly, K.D., Jowers, L.J.: Simulating continuous fuzzy system:I. *Iran. J. Fuzzy Syst.* **2**(1), 1–17 (2005)
31. Otadi, M., Abbasbandy, S., Mosleh, M.: System of linear fuzzy differential equations. In: *First Joint Congress on Fuzzy and Intelligent Systems*. Ferdowsi university of Mashhad, Iran (2007)
32. Sadeghi, A., Ahmad, I.M., Jameel, A.F.: Solving system of fuzzy differential equations. *Int. Math. Forum* **6**(42), 2087–2100 (2011)
33. Lata, S., Kumar, A.: A new method to solve time-dependent intuitionistic fuzzy differential equations and its application to analyze the intuitionistic fuzzy reliability of industrial systems. *Concurr Eng.* **20**(3), 177–184 (2012)
34. Melliani, S., Chadli, L.S.: Introduction to intuitionistic fuzzy partial differential equations. *Notes IFS* **7**(3), 39–42 (2001)
35. Said Abbasbandy, T., Viranloo, Allah: Numerical solution of fuzzy differential equation by Runge-Kutta method and the intuitionistic treatment. *Notes IFS* **8**(3), 45–53 (2002)
36. Melliani, S., Chadli, L.S.: Intuitionistic fuzzy differential equation. *Notes IFS* **6**(2), 37–41 (2000)
37. Mondal, Sankar Prasad, Roy, Tapan Kumar: First order homogeneous ordinary differential equation with initial value as triangular intuitionistic fuzzy number. *J. Uncertain. Math. Sci.* (accepted)
38. Buckley, J.J., Feuring, T., Hayashi, Y.: Linear systems of first order ordinary differential equations: fuzzy initial conditions. *Soft Comput.* **6**, 415–421 (2002)
39. Oberguggenberger, M., Pittschmann, S.: Differential equations with fuzzy parameters. *Math. Mod. Syst.* **5**, 181–202 (1999)
40. Casasnovas, J., Rossell, F.: Averaging fuzzy biopolymers. *Fuzzy Sets Syst.* **152**, 139–158 (2005)
41. Mondal, S.P., Roy, T.K.: Application of first order non homogeneous fuzzy differential equation. *Adv. Fuzzy Sets Syst.* **16**(1), 1–29 (2013). Pushpa Publishing House
42. El Naschie, M.S.: From experimental quantum optics to quantum gravity via a fuzzy Khler manifold. *Chaos Solitons Fractals* **25**, 969–977 (2005)

43. Bencsik, A., Bede, B., Tar, J., Fodor, J.: Fuzzy differential equations in modeling hydraulic differential servo cylinders. In: Third Romanian-Hungarian joint symposium on applied computational intelligence (SACI), Timisoara, Romania (2006)
44. Hassan, Z., Kamyad, A.V., Heydari, A.A.: Fuzzy modeling and control of hiv infection. *Comput. Math. Methods Med.* (2012). doi:[10.1155/2012/893474](https://doi.org/10.1155/2012/893474)
45. Diniz, G.L., Fernandes, J.F.R., Meyer, J.F.C.A., Barros, L.C.: A fuzzy Cauchy problem modeling the decay of the biochemical oxygen demand in water. In: Proceedings of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference, IEEE (2001)
46. Ahmad, M.Z., De Baets, B.: A predator–prey model with fuzzy initial populations. In: Proceedings of the 13th IFSA World Congress and 6th European Society of Fuzzy Logic and Technology Conference, IFSA-EUSFLAT (2009)
47. Barros, L.C., Bassanezi, R.C., Tonelli, P.A.: Fuzzy modelling in population dynamics. *Ecol. Model.* **128**, 27–33 (2000)
48. Bede, Barnabas, Rudas, Imre J., Fodor, Janos: Friction model by fuzzy differential equations. In: Melin, P., Castillo, O., Aguilar, L.T., Kacprzyk, J., Pedrycz, W. (eds.) *Foundations of Fuzzy Logic and Soft Computing. Lecture Notes in Computer Science*, vol. 4529, pp. 23–32. Springer, Berlin (2007)
49. Mondal, S.P., Banerjee, S., Roy, T.K.: First order linear homogeneous ordinary differential equation in fuzzy environment. *Int. J. Pure Appl. Sci. Technol.* **14**(1), 16–26 (2013)
50. Mondal, S.P., Roy, T.K.: First order linear non homogeneous ordinary differential equation in fuzzy environment. *Math. Theory Model.* **3**(1), 85–95 (2013)
51. Mondal, S.P., Roy, T.K.: First order linear homogeneous ordinary differential equation in fuzzy environment based on Laplace transform. *J. Fuzzy Set Valued Anal.* **2013**, 1–18 (2013)
52. Mondal, S.P., Roy, T.K.: First order linear homogeneous fuzzy ordinary differential equation based on lagrange multiplier method. *J. Soft Comput. Applicat.* **2013**, 1–17 (2013)
53. Mondal, S.P., Roy, T.K.: First order linear non homogeneous ordinary differential equation in fuzzy environment based on Laplace transform. *J. Math. Comput. Sci.* (accepted)
54. Varghese, A., Kuriakose, S.: Centroid of an intuitionistic fuzzy number. *Notes Intuitionistic Fuzzy Sets* **18**(1), 19–24 (2012)
55. Yuan, Y., Joubert, S.V., Gai, Y.: Real life applications of ODES for undergraduates